ANALYTICAL SOLUTION OF THE N PARTICLE GRAVITATIONAL PROBLEM:
STOKES’ THEOREM AND ORBITAL CIRCULATION.

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ABSTRACT

The N particle gravitational problem is solved analytically in the pairwise additive approximation. The solution is illustrated for three interacting particles, and Stokes’ Theorem used to introduce the concept of orbital circulation. It is shown numerically that the orbital circulation is non-zero for all conical sections and precessing conical sections with the exception of circular orbits. The orbital circulation may therefore be used to characterize all orbits known in cosmology.

Keywords: Classical limit of ECE theory, solution of the N particle gravitational problem, orbital circulation.
1. INTRODUCTION

Recently in this series of papers {1 - 10} on the applications of ECE theory the Einsteinian general relativity (EGR) has been refuted in many ways and replaced by a relativity based on the constrained Minkowski metric. In the classical limit of this new relativity it has been shown that all the features of planetary precession can be explained straightforwardly with the equation of the precessing conical section, and a myriad of new properties discovered in terms of the precession constant \(x\). All known orbits in cosmology have been shown to be explicable in terms of \(x\) and the equation of the precessing conical section. In immediately preceding papers the investigation of the classical limit of the new relativity has been extended to the multi particle gravitational problem, thought to have no known analytical solution, and in the preceding paper a new form of Kepler’s third law inferred for a precessing orbit.

In Section 2 of this paper it is shown that the \(N\) particle gravitational problem can be solved analytically in a relatively straightforward way given the usual form of the starting lagrangian for the problem. This is a four hundred year old problem in cosmology, and up to now it has been thought to have only specialized solutions discovered by Euler, Lagrange, Poincare and others. Stokes’ Theorem is used to develop the solution for three interacting particles, and a new concept developed of orbital swirl or circulation. In Section 3 some of the results are illustrated graphically.

2. SOLUTION AND CONCEPT OF ORBITAL CIRCULATION

Consider three masses interacting simultaneously with the following lagrangian...
This is the standard format of the lagrangian for what is known as the three particle gravitational problem in Newtonian dynamics. Here \( m \) are the masses, \( G \) is Newton’s constant, and the coordinates are defined in Fig. (1).

\[
L = \frac{1}{2} \left( m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + m_3 \dot{r}_3^2 \right) + \frac{m_1 m_2 G}{|r_1 - r_2|} + \frac{m_1 m_3 G}{|r_1 - r_3|} + \frac{m_2 m_3 G}{|r_2 - r_3|} .
\]

Therefore the lagrangian is:

\[
m_1 \dot{r}_1 + m_2 \dot{r}_2 + m_3 \dot{r}_3 = 0 .
\]

As in Fig. (1):

\[
\begin{align*}
R_1 &= r_1 - r_2 \\
R_2 &= r_2 - r_3 \\
R_3 &= r_3 - r_1
\end{align*}
\]
\[ L = \frac{1}{2} \left( m_1 |\dot{s}_1|^2 + m_2 |\dot{s}_2|^2 + m_3 |\dot{s}_3|^2 \right) - 6 \]
\[ + \frac{m_1 m_2 \dot{\theta}}{R_1} + \frac{m_1 m_3 \dot{\theta}}{R_2} + \frac{m_2 m_3 \dot{\theta}}{R_3} \]

From Eqs. (2) and (3):
\[ m_1 \ddot{s}_1 + m_2 \left( \ddot{s}_1 - \ddot{R}_1 \right) + m_3 \ddot{s}_2 = 0 \quad (7) \]

i.e.
\[ \ddot{s} = -\left( \frac{m_1 \ddot{R}_1 + m_3 \ddot{s}_3}{m_1 + m_2} \right) \quad (8) \]

Similarly:
\[ m_1 \ddot{s}_1 + m_2 \left( \ddot{s}_1 - \ddot{R}_1 \right) + m_2 \ddot{s}_3 = 0 \quad (9) \]

i.e.:
\[ \ddot{s}_1 = \frac{m_2 \ddot{R}_1 - m_3 \ddot{s}_3}{m_1 + m_2} \quad (10) \]

Therefore:
\[ \ddot{s}_1 = \frac{m_2 \ddot{R}_1 + m_3 \ddot{s}_3 - 2 m_2 m_3 \ddot{R}_1 \cdot \ddot{s}_3}{(m_1 + m_2)^3} \quad (11) \]

and
\[ \ddot{s}_2 = \frac{m_1 \ddot{R}_1 + m_3 \ddot{s}_3 + 2 m_1 m_3 \ddot{R}_1 \cdot \ddot{s}_3}{(m_1 + m_2)^3} \quad (12) \]

Solving Eqs. (11) and (12) gives the reduced Lagrangian:
\[ L = \frac{1}{2} \mu \left( \dot{R}_1^2 + m_3 \left( 1 + \frac{m_3}{m_1 + m_2} \right) \dot{\theta}_1^2 \right) + \frac{m_1 m_2 G}{R_1} + \frac{m_1 m_3 G}{R_2} + \frac{m_2 m_3 G}{R_3} \quad - (13) \]

where the reduced mass is:
\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \quad - (14) \]

In plane polar coordinates:
\[ \dot{R}_1^2 = R_1^2 + R_2^2 \dot{\theta}_1^2 \quad - (15) \]

Now consider the Euler Lagrange equations:
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{R}_1} \right) - \frac{\partial L}{\partial R_1} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} \quad - (16) \]

with the lagrangian (13). The solution of Eqs. (16) and (17) is well known (11) to be the elliptical orbit:
\[ R_1 = \frac{d_1}{1 + \varepsilon_1 \cos \theta_1} \quad - (18) \]

where \( d_1 \) is the semi right latitude and \( \varepsilon_1 \) is the eccentricity. Similarly:
\[ R_2 = \frac{d_2}{1 + \varepsilon_2 \cos \theta_2} \quad - (19) \]
and

\[ R_3 = \frac{\lambda_3}{1 + \xi_3 \cos \theta_3} \quad -(20) \]

where:

\[ R_2 = R_1 + R_3 \quad -(21) \]

This is the same result as in previous work \{1 - 10\} and obtained with a different method.

Therefore it follows that the general solution of the N particle problem is:

\[ R_i = \frac{\lambda_i}{1 + \xi_i \cos \theta_i} \quad -(22) \]

given the lagrangian \( (B) \) extended to N particles. For N masses in a plane the constraint \( -(21) \) is extended to N masses. For precessing orbits \{1 - 10\} the general solution is

\[ R_i = \frac{\lambda_i}{1 + \xi_i \cos (x_i \cdot \theta_i)} \quad -(23) \]

where \( x_i \) is the precession factor for each orbit. Therefore the N particle gravitational problem has been solved for the first time.

Straightforward application of Stokes' Theorem gives:

\[ \oint R_i \cdot dR_i = \oint S_i \cdot n \, dA_i \quad -(24) \]

where \( S_i \) is the orbital circulation:
From Eq. (22) in the Newtonian limit:

\[
\frac{dR_i}{d\theta_i} = \frac{E_i}{\kappa} R_i \sin \theta_i \quad -(26)
\]

so:

\[
S_i = -\frac{E_i \sin \theta_i}{1 + E_i \cos \theta_i} \quad -(27)
\]

The orbital circulation is a new concept in cosmology and some examples are graphed in Section 3. For precessing orbits:

\[
S_i = -\frac{E_i x_i \sin (x_i \theta_i)}{1 + E_i \cos (x_i \theta_i)} \quad -(28)
\]

in which the gravitational potential is the new universal law of gravitation:

\[
U(R) = -\frac{mM6\pi^2 x^2}{R} - \frac{L^2}{2mR^2} (1 - x^2) \quad -(29)
\]

in which the total angular momentum is the constant:

\[
L = \frac{mM}{m + M} R^2 \frac{d\theta_i}{dt} \quad -(30)
\]

In general for any curve in a plane \{11\}:
\[ dA_i = \frac{1}{2} R_i^2 \, d\theta_i \quad -(31) \]

so:
\[ \oint R_i \cdot dR_i = \frac{1}{2} \oint S_i \cdot k \cdot n \cdot R_i^2 \, d\theta_i \quad -(32) \]

Assuming:
\[ k \cdot n = 1 \quad -(33) \]

then:
\[ \oint R_i \cdot dR_i = -\frac{1}{2} \oint S_i \cdot R_i \, d\theta_i = \xi_i \, d\xi_i \int \frac{\sin \theta_i \, d\theta_i}{(1 + \xi_i \cos \theta_i)^{\frac{3}{2}}} \quad -(34) \]

This integral was evaluated numerically and gives the result:
\[ \oint R_i \cdot dR_i = \frac{1}{2} R_i^2 \quad -(35) \]

For precessing orbits:
\[ \oint R_i \cdot dR_i = \xi_i \, d\xi_i \int \frac{\sin (x_i \theta_i) \, d\theta_i}{(1 + \xi_i \cos (x_i \theta_i))^3} \quad -(36) \]

In the three particle problem:
\[ \oint R_2 \cdot dR_2 = \oint (R_1 + R_3) \cdot dR_2 \quad -(37) \]
so the circulation vectors are additive as follows:

\[
\mathbf{s}_2 = \mathbf{s}_1 + \mathbf{s}_2 - (38)
\]

providing the constraint on the solution (27):

\[
\frac{\xi_2 \sin \theta_2}{1 + \xi_2 \cos \theta_2} = \frac{\xi_1 \sin \theta_1}{1 + \xi_1 \cos \theta_1} + \frac{\xi_3 \sin \theta_3}{1 + \xi_3 \cos \theta_3}
\]

(39)

For each orbit in the Newtonian limit the elapsed times are:

\[
t_i = \left(1 - \xi_i^2\right)^{3/2} \left(\frac{\tau_i}{2\pi}\right) \sqrt{\frac{\mu \xi_i}{1 + \xi_i \cos \theta_i}} - (40)
\]

where the eccentricities are:

\[
\xi_i = \left(1 - \left(\frac{a_i}{b_i}\right)^2\right)^{1/2}, - (41)
\]

and where \(\tau_i\) are the times taken for each orbit to transcribe 2\(\pi\) radians.

The orbital circulation seems to be a new concept of cosmology, and may be used to characterize every orbit. It is non-zero for every orbit in general, with the exception of a circular orbit. For a circular orbit:

\[
\frac{\mathbf{R}}{d\mathbf{R}} = \mathbf{X} \cdot \mathbf{Y} - (42)
\]

\[
\frac{d\mathbf{R}}{d\mathbf{R}} = d\mathbf{X} \cdot \mathbf{Y} - (43)
\]

therefore:

\[
\oint \mathbf{R} \cdot d\mathbf{R} = \int_{0}^{2\pi} (xdx + ydy) - (44)
\]
In the case of the circle:

\[ x = R \cos \theta, \quad y = R \sin \theta \quad - (45) \]

\[ dx = -R \sin \theta \, d\theta \quad - (46) \]

\[ dy = R \cos \theta \, d\theta \quad - (47) \]

Therefore:

\[ \oint \left( x \, dx + y \, dy \right) = \int_0^{2\pi} R \cos \theta \left( -R \sin \theta \right) \, d\theta + R \sin \theta \left( R \cos \theta \right) \, d\theta \]

\[ = 0 \quad - (48) \]

The contour integral around the circumference of a circle is zero. The curl is:

\[ \nabla \times \mathbf{R} = \left( \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \hat{k} = 0 \quad - (49) \]

Also, Stokes' Theorem is verified. In cylindrical polar coordinates:

\[ \mathbf{R} = R \mathbf{e}_r \quad - (50) \]

Where:

\[ \mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \quad - (51) \]

And:

\[ \nabla \times \mathbf{R} = -\frac{1}{R} \frac{\partial R}{\partial \theta} \hat{k} \quad - (52) \]

The circle is defined by the conical section equation:
\[ R = \frac{L}{1 + \epsilon \cos \theta} \quad -(53) \]

with:

\[ \epsilon = 0. \quad -(54) \]

So:

\[ R = d = \text{constant} \quad -(55) \]

and

\[ \frac{\partial R}{\partial \theta} = 0, \quad -(56) \]

d therefore

\[ \nabla \times R = 0 \quad -(57) \]

\[ \text{Q.D.D.} \]

3. GRAPHICAL ILLUSTRATIONS.

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