Chapter 3

Canonical and second quantization in generally covariant quantum field theory

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by
Myron W. Evans,
Alpha Institute for Advanced Studyy (AIAS).
(emyrone@aol.com, www.aias.us, www.atomicprecision.com)

Abstract

Einstein Cartan Evans (ECE) field theory is shown to be a rigorous quantum field theory in which the tetrad is both the eigenfunction or wave-function and a quantized field that is generally covariant. Unification of fields is achieved with standard Cartan geometry on both the classical and quantum levels. The fundamental commutators needed for canonical quantization of the field/eigenfunction are derived self consistently from the same Cartan geometry. Second quantization proceeds straightforwardly thereafter by expanding the tetrad in terms of creation and annihilation operators. The latter are used to define the number operator in the usual way, and a generally covariant multi particle field theory obtained. The theory is illustrated with a discussion of the electromagnetic Aharonov Bohm effect.

Keywords: Einstein Cartan Evans (ECE) unified field theory, quantum field theory, canonical quantization and second quantization, Aharonov Bohm effects.

3.1 Introduction

Einstein Cartan Evans (ECE) unified field theory has been well developed analytically on the classical and single particle quantum levels [1]– [7]. In this paper it is shown that ECE field theory is a rigorous quantum field theory in which the tetrad is both the wave-function and the field. In Section 3.2, the fundamental commutators needed for canonical quantization [8], [9] are introduced from Cartan geometry and developed and in Section 3.3 second quantization [8], [9] is developed straightforwardly by expanding the tetrad in a Fourier series to give creation and annihilation operators that define the number operator. Therefore ECE theory produces a rigorous quantum field theory and can be given a multi particle interpretation as required [8], [9]. The theory in this paper is illustrated in Section 3.4 with the electromagnetic Aharonov Bohm effect.

Canonical quantization in quantum field theory is the name given to the construction of the Heisenberg commutators of the quantum field. Canonical quantization of the electromagnetic potential field, for example, runs into difficulties [8] in the contemporary standard model because of the assumption of a massless electromagnetic field with infinite range, and an identically zero photon mass. In special relativity this means that the special relativistic potential field A_μ can have only two physical components [8] and these are taken as the transverse components. However, A_μ must have four physical components to be manifestly covariant, and this is also a fundamental requirement of general relativity. Therefore there is a basic contradiction in the standard model, a contradiction that leads to well known difficulties [1]– [15]. It is shown in Section 3.2 that this contradiction is removed straightforwardly in ECE theory, in which the photon mass is identically non-zero as required. Without photon mass there can be no explanation of the Eddington experiment, contradicting the well known tests of the Einstein Hilbert (EH) field theory of gravitation, tests which are now known from NASA Cassini to be accurate to one part in one hundred thousand for light grazing the sun. So the existence of identically non-zero photon mass has been tested to this accuracy, because the Eddington experiment is explained in EH theory by considering the mass of the photon and the mass of the sun. If the photon mass is identically zero, the EH explanation makes no sense. Thus, photon mass is known very accurately to be identically non-zero. In Maxwell Heaviside field theory however, it is identically zero. This diametric contradiction is inherent in the standard model because gravitation is in that model a theory of general relativity and electromagnetism is a theory of special relativity (the Maxwell Heaviside (MH) field theory). Electromagnetism in the standard model is developed [8] [9] in terms of gauge theory. The simplest example of gauge theory is illustrated as follows. In MH theory the electromagnetic field in tensor notation is an anti-symmetric rank two tensor independent of gravitation:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.1}$$

This field tensor is unchanged under the mathematical transform:

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \tag{3.2}$$

which is a simple example of a gauge transformation [8]– [15]. Here Λ is any scalar function and the invariance of the field under the gauge transformation is an example of the Poincaré Lemma. The field is said to be gauge invariant, and in gauge field theory this is a central hypothesis which leads to a debate concerning the nature of $F_{\mu\nu}$ and A_μ. One or the other is regarded as fundamental, there being proponents of both views. ECE theory has shown that this debate is superfluous, and ECE has replaced gauge theory with a generally covariant unified field theory [1]– [7] in which the fundamental transformation is the general coordinate transformation of general relativity. The debate in the gauge theory of the standard model was initiated to a large extent by the discovery of the magnetic Aharonov Bohm effect by Chambers [8] [9].It was shown experimentally by Chambers that in regions where is zero, A_μ has a physical effect. The standard classical interpretation had been that $F_{\mu\nu}$ is physical but A_μ is unphysical. This interpretation was based directly on classical gauge theory, first proposed by Weyl and his contemporaries. In the ensuing forty year debate concerning the Aharonov Bohm (AB) effects some proponents have held the view that they are purely quantum effects, and in quantum theory the minimal prescription applies:

$$p_\mu \rightarrow p_\mu + eA_\mu \tag{3.3}$$

Here p_μ is the four-momentum of special relativity and $-e$ is the charge on the electron. In this view, used for example in the Dirac equation to explain the Stern Gerlach effect, A_μ is a physical property. In gauge theory it is held that A_μ can be transformed to $A_\mu + \partial_\mu \Lambda$ without affecting the field $F_{\mu\nu}$, so in consequence A_μ is unphysical. The philosophical basis of gauge theory is therefore questionable, the assumption that A_μ is unphysical and that $F_{\mu\nu}$ is physical is untenable and confusing even to the experts, thus a protracted forty year debate that reveals this confusion. In the standard model of the late twentieth century, gauge invariance was elevated to a central hypothesis of the electromagnetic, weak and strong fields and has been applied to the gravitational field, where it is clearly superfluous by Ockham's Razor. The general coordinate transformation is already sufficient for gravitational field theory and there is no need for the further postulate of gauge invariance. ECE theory has shown [1]– [7] that this is also true for unified field theory. The difficulties of using gauge theory outweigh any of its advantages, the latter being increasingly difficult to find as ECE is increasingly developed and accepted [16]. It has been shown [1]– [7] that ECE theory leads to a unified field theory in which the gravitational, electromagnetic, weak and strong fields are represented by the tangent spacetime at point P to the base manifold in standard Cartan geometry. The various fields are represented by various representation spaces in the tangent spacetime [1]– [7]. Gauge theory is superfluous in this context.

The difficulty inherent in the fundamental assumption of gauge invariance can be illustrated as follows using differential form notation [1]– [7] [17], where Eqs. (3.1) and (3.2) become:

$$A \rightarrow A + d\Lambda \tag{3.4}$$

$$F = d \wedge A = d \wedge (A + d\Lambda) \tag{3.5}$$

because:

$$d \wedge d\Lambda := 0. \tag{3.6}$$

Eq.(3.6) is the Poincaré Lemma in differential form notation. However, ECE theory now shows that the generally covariant foundation of electrodynamics unifies the latter with the other fields, notably the gravitational field. The standard model is unable to do this despite many attempts throughout the twentieth century. In ECE theory the relevant differential form equations of the electromagnetic sector are [1]– [7]:

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b \tag{3.7}$$

$$d \wedge F^a = \mu_0 j^a \tag{3.8}$$

$$d \wedge \widetilde{F}^a = \mu_0 J^a \tag{3.9}$$

Here $\omega^a{}_b$ is the spin connection, which is identically non-zero because the electromagnetic field is always a spinning frame, not a static frame as in the standard model. The homogeneous current j^a is in general non-zero, it vanishes if and only if the electromagnetic and gravitational fields become independent and do not influence each other [1]– [7]. The presence of j^a (however tiny in magnitude) is of key importance, because it may be amplified by resonance [1]– [7] producing easily measurable electric power from ECE space-time, a new source of energy. In the standard model j^a does not exist, there is no concept of j^a in the standard model because electromagnetism there is a concept of special relativity superimposed on a flat Minkowski frame. Eq.(3.9) is the Hodge dual of Eq.(3.8) and here μ_0 is the vacuum permeability in S.I. units. The electromagnetic potential field in ECE theory is

$$A^a = A^{(0)} q^a \tag{3.10}$$

where $cA^{(0)}$ is the primordial voltage and q^a the tetrad field. Thus F^a is constructed from A^a through the first Cartan structure equation (3.7), and F^a obeys the first Bianchi identity, Eq.(3.8), and its Hodge dual (3.9). These are the classical field equations. The vector valued one-form $A^a{}_\mu$ and the vector valued two-form $F^a{}_{\mu\nu}$ are covariant under the general coordinate transformation [1]– [7] [17] according to the well known rules of standard Cartan geometry. If we attempt a "gauge transformation":

$$A^a \rightarrow A^a + d\Lambda^a \tag{3.11}$$

then:

$$F^a \rightarrow d \wedge A^a + \omega^a{}_b \wedge A^b + \omega^a{}_b \wedge d\Lambda^b = F^a + \omega^a{}_b \wedge d\Lambda^b \tag{3.12}$$

and F^a is not invariant: it must be coordinate covariant, not gauge invariant. Another fundamental problem of gauge theory in the standard model is that it uses a hypothesis superfluous to general relativity, the indices a of gauge theory are abstract mathematical concepts, whereas in Cartan geometry a is the index of the tangent space-time and thus well defined by geometry as required by relativity theory. In ECE theory the space-time that defines the electromagnetic field is the same as the space-time that defines all fields, including the gravitational field, in four physical dimensions, ct, X, Y and Z. Thus ECE is preferred to gauge theory by Ockham's Razor of philosophy, which asserts the use of the minimum number of concepts. In ECE theory a is already inherent in Cartan geometry, in gauge theory the label a is introduced as an extra mathematical assumption, i.e. of Yang Mills theory. Similarly ECE theory is preferred to String Theory by Ockham's razor, because String Theory uses superfluous parameters which are asserted arbitrarily to be dimensions. There is no experimental evidence for String Theory, and for this reason String Theory has been described as pre-Baconian. The experimental evidence for ECE theory is given in ref. [1] to [7] in a representative cross section of contemporary physics.

In ECE theory the fundamental wave equation of electrodynamics is [1]– [7]:

$$(\Box + kT)\, A^a{}_\mu = 0 \tag{3.13}$$

where k is the Einstein constant and T is the index contracted canonical energy-momentum density of the unified field. The Einstein Ansatz asserts that:

$$R = -kT \tag{3.14}$$

where R is the scalar curvature. Using the correspondence principle of Einstein, the Proca equation emerges from Eq.(3.13) in the well defined limit:

$$kT \rightarrow \left(\frac{mc}{\hbar}\right) \tag{3.15}$$

where m is the identically non-zero photon mass indicated by the Eddington experiment, and $\hbar$ is the Planck constant. Note carefully that the d'Alembert wave equation of the standard model does not emerge from ECE theory, indicating that the d'Alembert wave equation is incomplete because it asserts identically zero photon mass. In the standard model the interpretation of the Proca equation is self-contradictory [8] because the Lagrangian needed to derive it is not gauge invariant. So for this reason the Proca equation is not used for canonical quantization in the standard model. The basic weak point here is again the assumption of gauge invariance, which has so long been thought of as the strength of gauge theory. In ECE theory there is no problem with the Proca equation because as we have seen, gauge invariance has been replaced by coordinate covariance in the unified field. Therefore the photon mass is identically non-zero as required by general relativity (photon mass was first proposed by Einstein) and the electromagnetic field is manifestly covariant with four physical components: time-like and three space-like, two transverse and one longitudinal. In the standard model the time-like and longitudinal components are dubiously removed by the Gupta Bleuler method [1]– [15]. The latter procedure is incorrect in general relativity, which prohibits the existence of a massless field. A massless field would mean no curvature R, and nothing at all (no field, no particles). Thus $A^a{}_b$ is manifestly covariant in ECE theory and can be canonically quantized in a rigorously correct way.

3.2 Canonical quantization of the ECE field

The potential field $A^a{}_b$ in ECE theory can be non-zero in regions where the electromagnetic field $F^a{}_{\mu\nu}$ is zero [7]. In this case the equations defining the potential in form notation are:

$$d \wedge A^1 = gA^2 \wedge A^3 \tag{3.16}$$

$$d \wedge A^2 = gA^3 \wedge A^1 \tag{3.17}$$

$$d \wedge A^3 = gA^1 \wedge A^2 \tag{3.18}$$

$$d \wedge A^0 = -d \wedge A^3 \tag{3.19}$$

In deriving these equations it has been assumed, for the sake of simplicity of argument only, that the gravitational and electromagnetic fields have become independent. This assumption is:

$$j^a = 0 \tag{3.20}$$

It is seen that Eqs.(3.16) to (3.19) all contain commutators of potentials. This commutator is precisely the quantity needed to construct a Heisenberg equation in the potential. This is the required canonical quantization. The quantity g is [9]– [14]:

$$g = \frac{\kappa}{A^{(0)}} = \frac{e}{\hbar} \tag{3.21}$$

so the quantized field is

$$\begin{gathered} A^2 \wedge A^3 = \frac{\hbar}{e} d \wedge A^1 \\ \text{et cyclicum.} \end{gathered} \tag{3.22}$$

In tensor notation, Eq.(3.22) is:

$$\begin{gathered} \left[A^2{}_\mu, A^3{}_\nu\right] = \frac{\hbar}{e}\left[\partial_\mu, A^1{}_\nu\right] \\ \text{et cyclicum} \end{gathered} \tag{3.23}$$

This is the Heisenberg type equation needed in canonical quantization, but it is important to realize that Eq.(3.23) is an equation of general relativity. The original Heisenberg equation is one of non-relativistic quantum mechanics as is well known. In regions where both $F^a{}_{\mu\nu}$ and $A^a{}_\mu$ are non-zero (which is the case in general):

$$\begin{gathered} F^1 = d \wedge A^1 - gA^2 \wedge A^3, \\ \text{et cyclicum} \end{gathered} \tag{3.24}$$

and the commutators in tensor notation are:

$$\begin{gathered} \left[A^2{}_\mu, A^3{}_\nu\right] = \frac{\hbar}{e}\left(\left[\partial_\mu, A^1{}_\nu\right] - F^1{}_{\mu\nu}\right) \\ \text{et cyclicum} \end{gathered} \tag{3.25}$$

Therefore it is seen that the commutators of potential needed for canonical quantization of the electromagnetic field, for example, are derived directly from Cartan geometry through the fundamental postulate (3.21). All four components of $A^a{}_\mu$ are non-zero in general and physical. The fundamental wave equation of the electro-dynamical sector of ECE theory is Eq.(3.13), which reduces to the Proca equation (3.15) using the correspondence principle. The principle of gauge invariance in special relativity is discarded in favor of the older principle of coordinate covariance in general relativity. Canonical quantization is then inherent in the method.

Finite electromagnetic fields in the approximation (3.20) are given by:

$$F^1 = d \wedge A^1 - gA^2 \wedge A^3 \tag{3.26}$$

$$F^2 = d \wedge A^2 - gA^3 \wedge A^1 \tag{3.27}$$

$$F^3 = d \wedge A^3 - gA^1 \wedge A^2 \tag{3.28}$$

Translating into the complex circular basis [1]– [7], [9]– [14]:

$$F^{(1)*} = d \wedge A^{(1)*} + igA^{(2)} \wedge A^{(3)} \quad \text{et cyclicum} \tag{3.29}$$

and in tensor notation:

$$F^{(1)*}_{\mu\nu} = \partial_\mu A^{(1)*}_\nu - \partial_\nu A^{(1)*}_\mu + ig\left(A^{(2)}_\mu A^{(3)}_\nu - A^{(2)}_\nu A^{(3)}_\mu\right). \quad \text{et cyclicum} \tag{3.30}$$

The magnetic components in the complex circular basis for example are:

$$\mathbf{B}^{(1)*} = \boldsymbol{\nabla} \times \mathbf{A}^{(1)*} + ig\mathbf{A}^{(2)} \times \mathbf{A}^{(3)}. \quad \text{et cyclicum} \tag{3.31}$$

A possible mathematical solution of Eqs.(3.26) to (3.31) is:

$$\mathbf{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}\left(i\mathbf{i} + \mathbf{j}\right)e^{i\phi} \tag{3.32}$$

$$\mathbf{A}^{(2)} = \mathbf{A}^{(1)*} = \frac{A^{(0)}}{\sqrt{2}}\left(-i\mathbf{i} + \mathbf{j}\right)e^{-i\phi} \tag{3.33}$$

$$\mathbf{A}^{(3)} = -A^{(0)}\mathbf{k} = \mathbf{A}^{(3)*}, \quad B^{(0)} = \kappa A^{(0)} \tag{3.34}$$

If the electromagnetic phase is denoted:

$$\phi = \omega t - \kappa Z \tag{3.35}$$

it is found that the magnetic fields are:

$$\mathbf{B}^{(1)*} = \mathbf{B}^{(2)} = \frac{2B^{(0)}}{\sqrt{2}}\left(-i\mathbf{i} + \mathbf{j}\right)e^{-i\phi} \tag{3.36}$$

$$\mathbf{B}^{(2)*} = \mathbf{B}^{(1)} = \frac{2B^{(0)}}{\sqrt{2}}\left(i\mathbf{i} + \mathbf{j}\right)e^{i\phi} \tag{3.37}$$

$$\mathbf{B}^{(3)*} = \mathbf{B}^{(3)} = -B^{(0)}\mathbf{k}, \quad \boldsymbol{\nabla} \times \mathbf{A}^{(2)} = k\mathbf{A}^{(2)} \tag{3.38}$$

These are the results of $O(3)$ electrodynamics [1]– [7], [9]– [14], to which ECE theory reduces when gravitation has no influence on electromagnetism. The Evans (later ECE) spin field $\mathbf{B}^{(3)}$ is identically non-zero, and is the direct result of the spin connection in Eq. (3.7), i.e. the direct result of general relativity applied to electrodynamics self-consistently with gravitation. Quantization occurs from Eq.(3.21) using the de Broglie wave/particle dualism in combination with the minimal prescription

$$p = \hbar k \tag{3.39}$$

i.e.

$$p = eA^{(0)} \tag{3.40}$$

to obtain the Heisenberg commutators of the canonically quantized field:

$$\left[A^{(2)}{}_{\mu}, A^{(3)}{}_{\nu}\right] = \frac{-i}{g}\left(F^{(1)*}{}_{\mu\nu} - \left[\partial_{\mu}, A^{(1)*}{}_{\nu}\right]\right). \tag{3.41}$$
$$\text{et cyclicum}$$

Note that $A_{\mu}^{(2)}$ (and so on) is an energy-momentum within the factor e, via the minimal prescription, so energy-momentum Heisenberg commutators are given by Eq.(3.41):

$$\left[A^{a}{}_{\mu}, A^{b}{}_{\nu}\right] = A^{(0)2}\left[q^{a}{}_{\mu}, q^{b}{}_{\nu}\right] \tag{3.42}$$

where $q^{a}{}_{\mu}$ and $q^{a}{}_{\nu}$ are tetrad field/wave- functions. In ECE theory the tetrad is both the field and the wave-function, so due to this fact of Cartan geometry, ECE theory is a rigorous quantum field theory as required. Heisenberg-type commutators emerge naturally from the same geometry (i.e. from general relativity). These generally covariant commutators cannot be constructed in the standard model, which uses special relativity and intellectual gymnastics such as those of Gupta and Bleuler [8]to "remove" the time-like and longitudinal components of the four-potential. This removal is necessitated in turn by having to work with the fact that in special relativity (Poincaré group), only two components of a massless potential field can be physical, and the consequent but arbitrary assumption that these must be the transverse components of a plane wave. This flawed procedure is necessitated by the flawed assumption of a massless electromagnetic field, and the correct Proca equation cannot be used in the standard model because of the flawed assumption of gauge invariance. These flaws are all consequences of pathological science in the twentieth century, the pathology, or anthropomorphism, being that the Maxwell Heaviside theory"cannot be wrong". It was first realized as late as 1991 [18] that the $\mathbf{B}^{(3)}$ spin field was missing entirely from Maxwell Heaviside field theory [10]– [14] and ECE theory [1]– [7] is a direct consequence of that realization. ECE theory is a logical and self-consistent application of general relativity to all fields, including the electromagnetic field, and ECE theory is now generally accepted [16]. ECE replaces the convoluted, special relativistic, arguments of the standard model with a straightforward canonical quantization inherent in the method of general relativity. ECE theory can therefore be regarded as a benchmark theory or "new standard model" of physics.

Before proceeding to second quantization in Section 3.3 some discussion is given of single particle quantization [1]– [9] in ECE theory. In single particle quantization there is present a wave-function, for example the Dirac spinor, but on the traditional single particle level in the standard model [8] this wave-function has not yet been interpreted as a field. This means that multi particle phenomena such as transmutation cannot be given a satisfactory interpretation without "second quantization". In the standard model the latter procedure is the name given to the construction of Heisenberg commutators of the wave-function itself, which is thus interpreted as a quantum field. In ECE theory second quantization is automatic, (inherent in the method), because the tetrad is both the fundamental field and the wave-function on all levels. So to set the scene for the development of the fully rigorous ECE quantum field theory Section 3.3, the tetrad is first used here to obtain the fundamental single particle wave equations of special relativistic quantum mechanics [1]– [7] in well defined limits of the ECE wave equation via the correspondence principle of Einstein. These include the Klein-Gordon, Dirac and Proca equations.

The basic equations of generally relativistic quantum mechanics in ECE theory [1]– [7] are all obtained from basic equations of standard Cartan geometry [17]. The wave-function is always the tetrad $q^a{}_\mu$ as we have argued already. Two of these basic geometrical equations are the normalization condition:

$$q^a{}_\mu q^\mu{}_a = 1 \tag{3.43}$$

and the tetrad postulate:

$$D_\mu q^a{}_\nu = 0 \tag{3.44}$$

The ECE Lemma is obtained directly from the tetrad postulate, from the identity:

$$D^\mu \left(D_\mu q^a{}_\nu \right) := 0 \tag{3.45}$$

It follows straightforwardly [1]– [7] from Eq. (3.45) that:

$$\Box q^a{}_\mu = R q^a{}_\mu \tag{3.46}$$

where the scalar curvature of Eq.(3.46) must always be defined by:

$$R := q^\lambda{}_b \partial^\mu \left(\Gamma^\nu{}_{\mu\lambda} q^a{}_\nu - \omega^a{}_{\mu b} q^b{}_\lambda \right) \tag{3.47}$$

Here $\Gamma^\nu{}_{\mu\lambda}$ is the gamma connection. The latter becomes the Christoffel connection of Riemann geometry if and only if the torsion tensor vanishes:

$$T^\kappa{}_{\mu\nu} = \Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu} = 0 \tag{3.48}$$

The fundamental wave equation of ECE theory follows directly from the Lemma or subsidiary proposition (3.46) of Cartan geometry using the Einstein Ansatz (3.14). So the fundamental wave equation is always:

$$\left(\Box + kT \right) q^a{}_\mu = 0 \tag{3.49}$$

for all field/wave-functions $q^a{}_\mu$. Eq.(3.49) is seen to be the direct result of Cartan geometry and the Einstein Ansatz. Different representation spaces of the tetrad [1]– [7] then give the different fundamental fields of physics: the gravitational, electromagnetic, weak and strong, together with matter fields such as the fermions and bosons. Note that the wave-function has already been interpreted as the field, the fundamental tetrad field of general relativity [17]. So the required multi particle interpretation is inherent in the ECE theory from the outset. The Pauli exclusion principle, for example, or Fermi-Dirac and Bose-Einstein statistics, then emerge from the multi-particle interpretation.

It is always possible to write Eq.(3.43) as the classical:

$$q^a{}_\mu R q^\mu{}_a = R \tag{3.50}$$

By using Eq.(3.46) in Eq.(3.50) we obtain:

$$R = q^\mu{}_a \Box q^a{}_\mu \tag{3.51}$$

which is an operator equation indicating that R is the expectation value of the d'Alembertian operator $\Box$. Here $q^\mu{}_a$ is the inverse tetrad. Thus:

$$\Box q^a{}_\mu = R q^a{}_\mu \tag{3.52}$$

$$\Box q^{\mu}{}_{a} = R_1 q^{\mu}{}_{a} \tag{3.53}$$

where R_1 is to be defined. It follows from Eq.(3.53) that:

$$R = q^{\mu}{}_{a} \Box q^{a}{}_{\mu} = q^{\mu}{}_{a} R q^{a}{}_{\mu} \tag{3.54}$$

$$R_1 = q^{a}{}_{\mu} \Box q^{\mu}{}_{a} = q^{a}{}_{\mu} R_1 q^{\mu}{}_{a} \tag{3.55}$$

In quantum mechanics [19], Hermitian operators are used, because their eigenvalues are real-valued and thus physical. In Dirac bracket notation [19] a Hermitian system is defined by:

$$\langle m|\Omega|n\rangle = \langle n|\Omega|m\rangle^{*} \tag{3.56}$$

where $*$ denotes complex conjugate. A Hermitian matrix [20] is a square matrix unchanged by taking the transpose of its complex conjugate, e.g. if:

$$A = \begin{bmatrix} 1 & 1+i \\ 1-i & 3 \end{bmatrix}, \quad A^{*} = \begin{bmatrix} 1 & 1-i \\ 1+i & 3 \end{bmatrix} \tag{3.57}$$

then

$$\widetilde{A}^{*} = A \tag{3.58}$$

Eq.(3.8) translates into:

$$q^{a}{}_{\mu} \Box q^{\mu}{}_{a} = \left(q^{\mu}{}_{a} \Box q^{a}{}_{\mu}\right)^{*} \tag{3.59}$$

which implies:

$$R_1 = R^{*} \tag{3.60}$$

This means that the tetrad must be a Hermitian matrix. The eigenvalues of the tetrad eigenfunction are real-valued and physical. Now denote:

$$R = R^{'} + iR^{"} \tag{3.61}$$

$$R^{*} = R^{'} - iR^{"} \tag{3.62}$$

to obtain:

$$R^{'} = \frac{1}{2}\left(q^{\mu}{}_{a} \Box q^{a}{}_{\mu} + q^{a}{}_{\mu} \Box q^{\mu}{}_{a}\right) \tag{3.63}$$

$$R^{"} = \frac{-i}{2}\left(q^{\mu}{}_{a} \Box q^{a}{}_{\mu} - q^{a}{}_{\mu} \Box q^{\mu}{}_{a}\right) \tag{3.64}$$

Single particle relativistic equations in the limit of special relativity are regained using the correspondence principle of Einstein as follows:

$$|R^{'}| \rightarrow \left(\frac{mc}{\hbar}\right)^{2} \tag{3.65}$$

$$|R^{"}| \rightarrow 0 \tag{3.66}$$

In this limit it is self-consistently apparent that:

$$R = \frac{1}{2}\left(q^{\mu}{}_{a} \Box q^{a}{}_{\mu} + q^{a}{}_{\mu} \Box q^{\mu}{}_{a}\right) = \frac{1}{2}\left(R + R\right) = R' = -\left(\frac{mc}{\hbar}\right)^{2} \tag{3.67}$$

The Klein Gordon equation [8] is regained in the zero spin limit:

$$\phi = q^{0}{}_{0} = q^{1}{}_{1} = q^{2}{}_{2} = q^{3}{}_{3} \rightarrow 1 \tag{3.68}$$

so that the ECE wave equation reduces to:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \tag{3.69}$$

where ϕ is a scalar eigenfunction without spin. Eq.(3.69) is:

$$\left(\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \tag{3.70}$$

The contra-variant and covariant four-momenta are defined as:

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right), \quad p_\mu = \left(\frac{E}{c}, -\mathbf{p}\right) \tag{3.71}$$

Eq.(3.70) becomes the classical Einstein equation of special relativity:

$$p^\mu p_\mu = m^2c^2 \tag{3.72}$$

if the operator equivalence of quantum mechanics is used:

$$E \rightarrow i\hbar\frac{\partial}{\partial t}, \quad \mathbf{p} \rightarrow -i\hbar\boldsymbol{\nabla} \tag{3.73}$$

In the non-relativistic limit Eq.(3.72) becomes the Newtonian kinetic energy:

$$E = \frac{1}{2}\frac{p^2}{m} = \frac{1}{2}mv^2 \tag{3.74}$$

Using Eq.(3.73) in Eq.(3.74) gives the time-dependent Schrödinger equation of non-relativistic quantum mechanics:

$$\frac{\hbar^2}{2m}\nabla^2\phi = -i\hbar\frac{\partial\phi}{\partial t} \tag{3.75}$$

This is a single particle equation because the kinetic energy (3.74) is that of one free particle. The Klein Gordon equation is therefore a single particle equation of special relativistic quantum mechanics. In this case the free particle has zero spin and is defined by the properties (3.68) of the tetrad.

The Klein Gordon, Schrödinger and Newton equations have been obtained from Cartan geometry using the correspondence principle and operator equivalence (3.68):

$$p^\mu = i\hbar\partial^\mu \tag{3.76}$$

The minus sign in Eq.(3.14) is a convention. In the Born interpretation [19] the probability density of the Schrödinger equation is proportional to:

$$\rho = \phi^*\phi \tag{3.77}$$

where ϕ is a complex-valued quantity. The probability current [8] of the Schrödinger equation is defined as:

$$\mathbf{j} = -\frac{i\hbar}{2m}\left(\phi^*\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*\right) \tag{3.78}$$

In order to make these definitions relativistic, define the four-current:

$$j^\mu = (\rho, \mathbf{j}) \tag{3.79}$$

By Noether's Theorem, this four-current must be conserved in a continuity equation:

$$\partial_\mu j^\mu = 0 \tag{3.80}$$

This is true as follows for the Schrödinger equation:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} &= \frac{\partial}{\partial t} \left(\phi^* \phi \right) - \frac{i\hbar}{2m} \left(\phi^* \boldsymbol{\nabla}^2 \phi - \phi \boldsymbol{\nabla}^2 \phi^* \right) \\ &= \phi^* \left(\frac{\partial \phi}{\partial t} - \frac{i\hbar}{2m} \nabla^2 \phi \right) + \phi \left(\frac{\partial \phi^*}{\partial t} + \frac{i\hbar}{2m} \nabla^2 \phi^* \right) = 0 \end{aligned} \tag{3.81}$$

In the Klein-Gordon equation however, ρ must be defined as the time-like component of the four-current j^μ:

$$\rho = \frac{i\hbar}{2m} \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \tag{3.82}$$

Thus [8]:

$$\partial_\mu j^\mu = \frac{i\hbar}{2m} \left(\phi^* \Box \phi - \phi \Box \phi^* \right) = 0 \tag{3.83}$$

In ECE theory Eq.(3.83) becomes:

$$R - R_1 = q^\mu{}_a \Box q^a{}_\mu - q^a{}_\mu \Box q^\mu{}_a \tag{3.84}$$

For a Hermitian $q^a{}_\mu$:

$$R_1 = R^* \tag{3.85}$$

and the continuity equation of ECE theory becomes:

$$\partial_\mu q^\mu = \frac{i\hbar}{2m} \left(q^\mu{}_a \Box q^a{}_\mu - q^a{}_\mu \Box q^\mu{}_a \right) \tag{3.86}$$

For real and physical scalar curvature:

$$R = R^* \tag{3.87}$$

and

$$\partial_\mu q^\mu = R - R^* = 0 \tag{3.88}$$

as required by Noether's Theorem.

The Klein Gordon equation in this single particle interpretation was abandoned in favor of the single particle Dirac equation as is well known. The Klein Gordon equation gives a negative ρ from Eq.(3.82), a negative probability makes no sense. In order to make sense of the Klein Gordon equation ϕ must become a field [8], leading in the standard model to second quantization. Therefore in ECE theory the multi-particle interpretation automatically gives a physically sensible probability current for a spin-less particle because the tetrad is both the wave-function and the quantized field whose commutators are always well defined. This is illustrated for the electromagnetic field in Section 3.3, and in the remainder of this Section, the Dirac spinor wave-function is introduced on the single particle level.

The Dirac equation is a limit [1]– [7] of the ECE wave equation using Eq.(3.15), but the Dirac wave-function has a half integral spin whose single particle interpretation defines the fermion as is well known [8]. The Dirac spinor is a tetrad in $SU(2)$ representation space [1]– [7]. The Dirac equation is therefore:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) q^a{}_\mu = 0 \tag{3.89}$$

where the a and μ labels represent the four components of the Dirac spinor: $q^R{}_1$, $q^R{}_2$, $q^L{}_1$ and $q^L{}_2$. The Pauli spinors are:

$$\phi^R = \begin{bmatrix} q^R{}_1 \\ q^R{}_2 \end{bmatrix}, \quad \phi^L = \begin{bmatrix} q^L{}_1 \\ q^L{}_2 \end{bmatrix} \tag{3.90}$$

and the Dirac spinor is:

$$q^a{}_\mu = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix}. \tag{3.91}$$

In standard quantum field theory [8] [9] the Dirac spinor is denoted by ψ and the convention $c = \hbar = 1$ is used. So Eq.(3.89) becomes:

$$\left(\Box + m^2\right) \psi = 0 \tag{3.92}$$

which may be written [8] as:

$$\left(i\gamma^\mu \partial_\mu - m\right) \psi = 0 \tag{3.93}$$

where γ^μ is the Dirac matrix and where:

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^i \partial_i \tag{3.94}$$

Written out in full, Eq.(3.93) is:

$$\left(i\gamma^\mu \partial_\mu - \left(\frac{mc}{\hbar}\right)^2\right) q^a{}_\mu = 0 \tag{3.95}$$

Using the shorthand notation (3.93) the Hermitian conjugate equation is defined [8]:

$$\psi^+ \left(-i\gamma^0 \overleftarrow{\partial_0} + i\gamma^i \overleftarrow{\partial_i} - m\right) = 0 \tag{3.96}$$

and the adjoint spinor is defined as:

$$\overline{\psi} = \psi^+ \gamma^0 \tag{3.97}$$

Here:

$$\gamma^i \gamma^0 = -\gamma^0 \gamma^i \tag{3.98}$$

so that:

$$\overline{\psi} \left(i\gamma^\mu \overleftarrow{\partial_\mu} + m\right) = 0 \tag{3.99}$$

The conserved four-current is thus defined naturally as:

$$j^\mu = \overline{\psi} \gamma^\mu \psi \tag{3.100}$$

and is conserved as follows:

$$\begin{aligned}\partial_\mu j^\mu &= \left(\partial_\mu \overline{\psi}\right) \gamma^\mu \psi + \overline{\psi} \gamma^\mu \left(\partial_\mu \psi\right) \\ &= \left(im\overline{\psi}\right) \psi + \overline{\psi} \left(-im\psi\right) = 0\end{aligned} \tag{3.101}$$

The probability density of the Dirac equation is thus rigorously positive-valued:

$$j^0 = \overline{\psi}\gamma^0\psi = \psi^+\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0 \tag{3.102}$$

and is the Born probability of the free fermion or free Dirac particle. Only one fermion is being considered here so the Pauli exclusion principle and Fermi-Dirac statistics have not yet been deduced. To achieve that in the standard model requires second quantization [8].

In ECE theory:

$$\psi^+ \to q^{\mu+}{}_a, \quad \overline{\psi} \to q^{\mu+}{}_a\gamma^0 = \bar{q}^\mu{}_b, \quad j^\mu = \bar{q}^\nu{}_a \gamma^\mu q^a{}_\nu \tag{3.103}$$

and the conserved current is generally covariant as required. Second quantization of the Dirac spinor is inherent in ECE theory from the outset, as argued already, so ECE automatically theory gives the Pauli exclusion principle and Fermi-Dirac statistics from the required multi-particle interpretation of the tetrad field/wave-function. When applied to bosons, ECE theory automatically gives the Bose Einstein statistics for the same reason. The photon is well known to be an integral spin boson, and electrons and quarks are examples of half-integral spin fermions. ECE theory also gives the required multi particle interpretation of gravitons for the same reason, and so forth for any quantized field, the generally covariant, unified and quantized field of any spin, including zero spin. All other fields are limits of this field.

The negative energy states of the Dirac equation come from the fact that the spin $\frac{1}{2}$ particles obey the Pauli exclusion principle. Negative energy states are completely filled, so the exclusion principle prevents any more electrons entering the Dirac sea, made up of negative energy states. The Dirac sea is the vacuum in this picture, a vacuum made up of negative energy electrons, protons, neutrinos, neutrons and other fermions. If there is a vacancy or "hole" in the fermion sea, with energy $-|E|$, an electron with energy E fills it, emitting energy $2E$:

$$e^- + \text{hole} \to \text{energy} \tag{3.104}$$

The hole has charge e^+ and positive energy, and is called a positron. The energy difference is:

$$\Delta E = E - (-|E|) = 2E \tag{3.105}$$

Eq. (3.104) in particle theory is:

$$e^- + e^+ = 2\gamma \tag{3.106}$$

and this means that an electron and positron mutually annihilate to give two photons. In ECE theory the Dirac sea is defined by tetrads obeying the Pauli exclusion principle. The latter is understood by regarding the tetrad as a field and wave-function as argued already.

Quantities such as $p^\mu p_\mu$ are understood in ECE theory through the momentum tetrad [1]– [7]:

$$p^a{}_\mu = p^{(0)} q^a{}_\mu \tag{3.107}$$

Planck/Einstein/de Broglie quantization is given by:

$$p^a{}_\mu = \hbar\kappa^a{}_\mu = eA^a{}_\mu = eA^{(0)}q^a{}_\mu \tag{3.108}$$

The Heisenberg commutator equation is understood using the structure invariants of Cartan geometry (pp. 140 ff. of ref [1]):

$$x^a = \int_s T^a \tag{3.109}$$

$$\theta^a{}_b = \int_s R^a{}_b \tag{3.110}$$

and the Heisenberg equation is:

$$[x^a, p_b] = iJ^a{}_b = \hbar\theta^a{}_b \tag{3.111}$$

where

$$\theta^a{}_b = \frac{i}{\hbar}J^a{}_b \tag{3.112}$$

Thus:

$$[x^a, p_b] = \hbar \int_s R^a{}_b \tag{3.113}$$

where

$$R^a{}_b = -\frac{\kappa}{2}\epsilon^a{}_{bc}T^c \tag{3.114}$$

appropriate to spinning spacetime unaffected by curvature. Here T^c is Cartan torsion.

3.3 Second quantization of the tetrad field

As argued, the tetrad in ECE field theory is both the field and the wave-function. Therefore the generally covariant quantum field is defined by commutators inherent in Cartan geometry. In this section the position and conjugate momentum tetrads are defined from the appropriate Lagrangian density. Then the field tetrad is expanded in a Fourier series in order to define the creation and annihilation operators of the field. The number operator of the field is defined from the creation and annihilation operators and a multi-particle interpretation developed.

Eq.(3.46) may be derived [1]– [3] from an Euler Lagrange equation. For example if:

$$\frac{\partial \mathcal{L}}{\partial q^\nu{}_a} = -\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial^\mu q^\nu{}_a \right)} \right) \tag{3.115}$$

and

$$\mathcal{L} = -\frac{c^2}{\kappa}\left(\frac{1}{2}\left(\partial_\mu q^a{}_\nu\right)\left(\partial^\mu q^\nu{}_a\right) + \frac{R}{2}q^a{}_\nu q^\nu{}_a \right) \tag{3.116}$$

the Eq.(3.46) follows. There is a freedom of choice in the Lagrangian density, it is chosen to give the ECE Lemma (3.46) through the Euler Lagrange equation (3.115). Define the position and conjugate momentum tetrads by:

$$x^a{}_\nu = x^{(0)}q^a{}_\mu \tag{3.117}$$

$$p^a{}_\mu = p^{(0)} q^a{}_\mu \tag{3.118}$$

The conjugate momentum is related to the position by:

$$p^a{}_\mu = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu x^a{}_\mu\right)} \tag{3.119}$$

Eq.(3.119) is a canonical equation in the sense that it generalizes the well known classical result [20]:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \tag{3.120}$$

Classical dynamics [20] can be developed with the Lagrange equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \tag{3.121}$$

and the Hamilton equations of motion, the canonical equations:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad -\dot{p}_k = \frac{\partial H}{\partial q_k} \tag{3.122}$$

where H is the Hamiltonian. These well known equations are found in any textbook on classical dynamics. Note that Eq.(3.119) is an equation of general relativity, whereas Eqs.(3.120) to (3.122) are non-relativistic.

The Cartan torsion is defined [1]– [7], as argued already in this paper, by:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \tag{3.123}$$

and if

$$\omega^a{}_b = -\frac{\kappa}{2}\epsilon^a{}_{bc} q^c \tag{3.124}$$

then:

$$T^a = d \wedge q^a - \kappa q^b \wedge q^c \tag{3.125}$$

i.e.

$$\begin{gathered} T^1 = d \wedge q^1 - \kappa q^2 \wedge q^3 \\ \text{et cyclicum} \end{gathered} \tag{3.126}$$

$$T^0 = d \wedge q^0 - \kappa q^2 \wedge q^1 \tag{3.127}$$

Eqs.(3.125) and (3.127) indicate the existence of commutators (wedge products). These appear on the classical level therefore in generally covariant unified field theory. In regions where the fundamental field is non-zero but where the torsion is zero:

$$d \wedge q^a = \kappa q^b \wedge q^c \tag{3.128}$$

In tensor notation, Eq.(3.128) is a commutator equation:

$$\left[\partial_\mu, q^a{}_\nu\right] = \kappa \left[q^b{}_\mu, q^c{}_\nu\right] \tag{3.129}$$

For the position tetrad:

$$\left[\partial_\mu, x^a{}_\nu\right] = \kappa \left[q^b{}_\mu, x^c{}_\nu\right] \tag{3.130}$$

Multiply both sides of Eq.(3.130) by $p^{(0)}$ to obtain:

$$p^{(0)}\left[\partial_\mu, x^a{}_\nu\right] = \kappa \left[p^b{}_\mu, x^c{}_\nu\right] \tag{3.131}$$

This is an equation of classical general relativity but it contains the commutator of position and conjugate momentum. This commutator is fundamentally important to quantum mechanics as is well known. Second quantization [8] [9] proceeds by setting up the equal time commutators for position and momentum: in the Heisenberg equation the position and conjugate momentum are defined at the same instant in time. Eq.(3.131) shows that the momentum tetrad $p^b{}_\mu$, and the partial derivative operator ∂_μ play the same role. This observation leads to the fundamental operator equivalence (3.76) of quantum mechanics. In ECE theory this operator equivalence is derived as in Eq.(3.131) from Cartan geometry in a generally covariant context.

In the de Broglie limit:

$$p^{(0)} = \hbar\kappa \tag{3.132}$$

and

$$\left[p^b{}_\mu, x^c{}_\nu\right] = \hbar\left[\partial_\mu, x^a{}_\nu\right] \tag{3.133}$$

Eq.(3.133) in the complex circular basis [1]– [3] is:

$$\left[p^{(b)}{}_\mu, x^{(c)}{}_\nu\right] = i\hbar\left[\partial_\mu, x^{(a)*}{}_\nu\right] \tag{3.134}$$

and is equivalent to the angular momentum commutator relations [1]– [14]:

$$\left[J^{(b)}{}_\mu, J^{(c)}{}_\nu\right] = i\hbar J^{(a)*}{}_{\mu\nu} \tag{3.135}$$

where:

$$J^{(a)}{}_{\mu\nu} = iJ^{(0)}\left[\partial_\mu, x^{(a)*}{}_\nu\right] \tag{3.136}$$

In vector notation:

$$\begin{gathered}\mathbf{J}^{(1)} \times \mathbf{J}^{(2)} = i\hbar\mathbf{J}^{(3)*}\\ \text{et cyclicum}\end{gathered} \tag{3.137}$$

In general relativity, Eq.(3.131) gives:

$$p^{(0)} = \left(\left[p^b{}_\mu, x^c{}_\nu\right] / \left[\partial_\mu, x^a{}_\nu\right]\right)\kappa \tag{3.138}$$

so the Planck constant has been identified as the limit:

$$\left[p^b{}_\mu, x^c{}_\nu\right] \rightarrow \hbar\left[\partial_\mu, x^a{}_\nu\right] \tag{3.139}$$

or, in the complex circular basis:

$$\left[p^{(b)}{}_\mu, x^{(c)}{}_\nu\right] \rightarrow i\hbar\left[\partial_\mu, x^{(a)*}{}_\nu\right] \tag{3.140}$$

The Planck constant is therefore defined by a particular type of Cartan geometry (Eq.(3.114)) in a well defined limit form the correspondence principle. Having recognized that Eq. (3.131) leads to quantization, its fully quantized form is:

$$\left[p^b{}_\mu, x^c{}_\nu\right] q^d{}_\rho = \left(\frac{p^{(0)}}{\kappa}\right)\left[\partial_\mu, x^a{}_\nu\right] q^d{}_\rho \tag{3.141}$$

where it has been recognized that the commutators act on the tetrad field/wavefunction $q^d{}_\rho$. We may write:

$$\delta^a_{\mu\nu} = \left[\partial_\mu, x^a{}_\nu\right] \tag{3.142}$$

In shorthand notation, Eq.(3.142) is:

$$(p \wedge x)\, q = \left(\frac{p^{(0)}}{\kappa}\right)(d \wedge x)\, q \tag{3.143}$$

where it is emphasized that q is both the wave-function and the field. So in this sense a rigorous quantum field theory emerges automatically from Cartan geometry. The commutators of Eq.(3.141) are between tetrads, i.e. field commutators as required. This conclusion is true for all fields of physics, including for example the electromagnetic potential field:

$$\left[A^{a}{}_{\mu}, a^{b}{}_{\nu}\right] = \frac{A^{(0)2}}{\kappa}\left[\partial_{\mu}, \kappa^{a}{}_{\nu}\right] \tag{3.144}$$

Eq.(3.137) is a cyclic relation between tetrads:

$$\begin{gathered}\mathbf{q}^{(1)} \times \mathbf{q}^{(2)} = i\mathbf{q}^{(3)*} \\ \text{et cyclicum}\end{gathered} \tag{3.145}$$

Following the usual development of quantum field theory [8] [9] Eqs.(3.32) can be expressed as:

$$\begin{gathered}[q_x, q_y] = iq_z \\ \text{et cyclicum}\end{gathered} \tag{3.146}$$

The raising and lowering or creation and annihilation operators [19] are:

$$q^{+} = q_x + iq_y \tag{3.147}$$

$$q^{-} = q_x - iq_y \tag{3.148}$$

These are similar to the complex circular tetrads:

$$q^{(1)} = \frac{1}{\sqrt{2}}\left(q_x + iq_y\right) \tag{3.149}$$

$$q^{(2)} = \frac{1}{\sqrt{2}}\left(q_x - iq_y\right) \tag{3.150}$$

The commutator properties of Eqs.(3.147) and (3.148) are [19]:

$$\left[q^{+}, q_z\right] = -q^{+} \tag{3.151}$$

$$\left[q^{-}, q_z\right] = q^{-} \tag{3.152}$$

$$\left[q^{+}, q^{-}\right] = 2q_z \tag{3.153}$$

The creation tetrad operator is then defined by:

$$q^{+}|n\rangle = c_n^{+}|n+1\rangle \tag{3.154}$$

and the annihilation tetrad operator by:

$$q^{-}|n\rangle = c_n^{-}|n-1\rangle \tag{3.155}$$

In quantum electrodynamics for example, q^+ increases the state $|n>$ to $|n+1>$.The electromagnetic field is described by an infinite number of harmonic oscillators, one for every point in space. In this case:

$$q^-|n\rangle = n^{1/2}|n-1\rangle \tag{3.156}$$

$$q^+|n\rangle = (n+1)^{1/2}\,|n+1\rangle \tag{3.157}$$

The number operator is:

$$N = q^+ q^- \tag{3.158}$$

so that:

$$N|n\rangle = n|n\rangle \tag{3.159}$$

The hamiltonian is:

$$\left.\begin{aligned} H &= \hbar\omega\left(N+\frac{1}{2}\right) \\ &= \hbar\omega\left(q^+q^- + \frac{1}{2}\right) \\ &= \hbar\omega\left(n+\frac{1}{2}\right) \end{aligned}\right\} \tag{3.160}$$

giving the zero point energy:

$$H_0 = \frac{1}{2}\hbar\omega \tag{3.161}$$

The fundamental tetrad field is expanded in a Fourier series [8]:

$$q = \sum_{\kappa}\left(q^+{}_{\kappa}e^{-i\kappa\cdot\mathbf{r}} + q^-{}_{\kappa}e^{i\kappa\cdot\mathbf{r}}\right) \tag{3.162}$$

which may be developed into an integral in a volume V:

$$q = V\int\left(q^+{}_{\kappa}e^{-i\kappa\cdot\mathbf{r}} + q^-{}_{\kappa}e^{i\kappa\cdot\mathbf{r}}\right)d^3\kappa \tag{3.163}$$

3.4 Electromagnetic Aharonov Bohm effect and field commutators

The electromagnetic Aharonov Bohm (EAB) effect is defined [1]– [7] by:

$$d\wedge A^a + \omega^a{}_b \wedge A^b = 0, \tag{3.164}$$

$$\omega^a{}_b \neq 0 \tag{3.165}$$

it is important to note that the spin connection $\omega^a{}_b$ must be non-zero, this is a fundamental requirement of general relativity because the electromagnetic field is a spinning frame, the gravitational field a curving frame. The ECE spin field shows this to be true experimentally [1]– [7]. In order to obtain solutions of Eq.(3.164) the spin connection must be given an analytical form. The Faraday law of induction is contained in the approximation [1]– [7]:

$$d\wedge F^a = 0 \tag{3.166}$$

The Faraday law of induction is accurate under most laboratory applications, so in this approximation:

$$j^a = 0, \tag{3.167}$$

implying Eq.(3.124). The proportionality constant $-\kappa/2$ in Eq. (3.124) [1]– [7] has been assumed to be a scalar, the minus sign and the factor half have been chosen as a convention. More generally, the proportionality factor in Eq.(3.124) can be a tensor with different components. Its units are inverse meters, those of wave-number. Use the ECE Ansatz (3.10) to find that:

$$\omega^a{}_b = -\frac{g}{2}\epsilon^a{}_{bc}A^c \tag{3.168}$$

where:

$$g = \frac{\kappa}{A^{(0)}} \tag{3.169}$$

Therefore the EAB effect is described by wedge products or commutators of potential, those used on canonical quantization. Eq.(3.167) is defined to an excellent approximation in most applications by the Faraday Law of induction. Under resonance conditions however [1]– [7] the homogeneous current j^a may be amplified by many orders of magnitude [1]– [7]. Eq. (3.164) may be expanded in terms of its components as [1]– [7]:

$$\begin{gathered} d \wedge A^1 = gA^2 \wedge A^3 \\ \text{et cyclicum} \end{gathered} \tag{3.170}$$

$$d \wedge A^0 = -d \wedge A^3 \tag{3.171}$$

and the EAB effect is described by solutions to these four simultaneous equations.

Writing out Eq.(3.170) in tensor notation gives:

$$\begin{gathered} \partial_\mu A^3{}_\nu - \partial_\nu A^3{}_\mu = g\left(A^1{}_\mu A^2{}_\nu - A^1{}_\nu A^2{}_\mu\right) \\ \text{et cyclicum} \end{gathered} \tag{3.172}$$

or in the complex circular basis:

$$\begin{gathered} \partial_\mu A^{(3)}{}_\nu - \partial_\nu A^{(3)}{}_\mu = -ig\left(A^{(1)}{}_\mu A^{(2)}{}_\nu - A^{(1)}{}_\nu A^{(2)}{}_\mu\right) \\ \text{et cyclicum} \end{gathered} \tag{3.173}$$

Eq.(3.173) is for example:

$$\partial_x A^{(3)}{}_y - \partial_y A^{(3)}{}_x = \kappa A^{(0)} \tag{3.174}$$

However:

$$A^{(3)}{}_y = A^{(3)}{}_x = 0, \quad A^{(0)} \neq 0 \tag{3.175}$$

so the only solution is:

$$\kappa = 0 \tag{3.176}$$

Similarly, Eq.(3.173) gives:

$$\partial_0 A^{(3)}{}_z - \partial_z A^{(3)}{}_0 = -ig\left(A^{(1)}{}_0 A^{(2)}{}_z - A^{(1)}{}_z A^{(2)}{}_0\right) \tag{3.177}$$

and again Eq.(3.176) is the only solution because:

$$A^{(2)}{}_z = A^{(1)}{}_z = 0, \quad A^{(0)} \neq 0 \tag{3.178}$$

In order to obtain a self consistent solution to the simultaneous equations (3.170) to (3.171) it must be assumed that κ_i is a tensor:

$$\kappa_i = \begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \kappa \end{bmatrix} \tag{3.179}$$

so Eqs.(3.170) to (3.171) become:

$$d \wedge A^1 = gA^2 \wedge A^3 \tag{3.180}$$

$$d \wedge A^2 = gA^3 \wedge A^1 \tag{3.181}$$

$$d \wedge A^3 = -d \wedge A^0 = 0, \quad A^1 \wedge A^2 \neq 0 \tag{3.182}$$

The solutions are:

$$\mathbf{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\mathbf{i} - i\mathbf{j})\, e^{i(\omega t - \kappa z)} \tag{3.183}$$

$$\mathbf{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}} (\mathbf{i} + i\mathbf{j})\, e^{-i(\omega t - \kappa z)} \tag{3.184}$$

Thus:

$$\boldsymbol{\nabla} \times \mathbf{A}^{(1)} = \kappa \mathbf{A}^{(1)} \tag{3.185}$$

$$\mathbf{A}^{(1)} \times \mathbf{A}^{(2)} = i\mathbf{A}^{(0)2}\mathbf{k} \tag{3.186}$$

$$\boldsymbol{\nabla} \times \mathbf{A}^{(1)*} = -ig\mathbf{A}^{(2)} \times \mathbf{A}^{(3)} \tag{3.187}$$

$$\boldsymbol{\nabla} \times \mathbf{A}^{(2)*} = -ig\mathbf{A}^{(3)} \times \mathbf{A}^{(1)} \tag{3.188}$$

$$\nabla \times \mathbf{A}^{(3)*} = \mathbf{0} \tag{3.189}$$

Eqs.(3.183) to (3.189) are tetrad equations, with:

$$\mathbf{A}^{(3)} = A^{(0)}\mathbf{k} \tag{3.190}$$

Thus:

$$A^0 = -A^{(0)} \tag{3.191}$$

The complex circular basis is defined by the tetrad equations:

$$\begin{gathered} \mathbf{q}^{(1)} \times \mathbf{q}^{(2)} = i\mathbf{q}^{(3)*} \\ \text{et cyclicum} \end{gathered} \tag{3.192}$$

where

$$\mathbf{q}^{(1)} = \mathbf{q}^{(2)*} = \frac{1}{\sqrt{2}} (\mathbf{i} - i\mathbf{j})\, e^{i(\omega t - \kappa t)} \tag{3.193}$$

$$\mathbf{q}^{(3)} = \mathbf{q}^{(3)*} = \mathbf{k} \tag{3.194}$$

These tetrads are the mechanism for defining the Cartan torsion and spinning space-time responsible for electromagnetic potential fields.

The EAB effect is caused by $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ in regions where

$$\mathbf{E}^a = \mathbf{0}, \quad \mathbf{B}^a = \mathbf{0} \tag{3.195}$$

The EAB effect occurs in regions outside a laser or radar beam. The $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ components are rapidly oscillating, so on average:

$$\langle \mathbf{A}^{(1)} \rangle = \langle \mathbf{A}^{(2)} \rangle = \mathbf{0} \tag{3.196}$$

but

$$\mathbf{A}^{(1)} \times \mathbf{A}^{(2)} = iA^{(0)2}\mathbf{k} \tag{3.197}$$

and is non-zero on average. The electromagnetic field components in these regions are zero, for example:

$$\mathbf{B}^{(1)*} = \nabla \times \mathbf{A}^{(1)*} + ig\mathbf{A}^{(2)} \times \mathbf{A}^{(3)} = \mathbf{0} \tag{3.198}$$

$$\mathbf{B}^{(2)*} = \nabla \times \mathbf{A}^{(2)*} + ig\mathbf{A}^{(3)} \times \mathbf{A}^{(1)} = \mathbf{0} \tag{3.199}$$

Therefore the beam intensity or power density (watts per square meter) is also zero:

$$I = -\frac{ic}{\mu_0} |\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}| = 0 \tag{3.200}$$

The power density I of the radar or laser beam is non-zero if and only if the oscillating electric and magnetic fields making up the beam are non-zero. Obviously, outside the beam there is no power density or beam intensity measurable experimentally. However the conjugate product or commutator $\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}$ still exists outside the beam because in these regions:

$$\mathbf{A}^{(1)} = \mathbf{A}^{(2)*} \neq \mathbf{0} \tag{3.201}$$

Similarly, the color of the laser beam is due to its non-zero electric and magnetic fields. Outside the laser beam there is no color visible, but Eq.(3.201) is still true. For a static magnetic field (Chambers experiment) $\mathbf{B}$ is non-zero inside the iron whisker or solenoid, but outside $\mathbf{B}$ is zero and $\mathbf{A}$ non-zero [1]– [7], causing the well known magnetic (original) Aharonov Bohm effect. So the experiment to detect the EAB for the first time must be set up to observe the inverse Faraday effect [1]– [7] in regions outside the laser or radio frequency beam, i.e. in regions where I is zero as discussed already. As described in Appendix F of Vol. 3 of ref [12] the inverse Faraday effect in a gas of N electrons occupying a volume V produces in the sample a magnetic flux density (in Tesla):

$$\mathbf{B}^{(3)}_{\text{sample}} = \frac{N}{V} \frac{\mu_0 e^3 c^2}{2m\omega^2} \left(\frac{B^{(0)}}{\sqrt{m^2\omega^2 + e^2 B^{(0)2}}} \right) \mathbf{B}^{(3)}_{\text{beam}} \tag{3.202}$$

where $-e$ is the charge on the electron, m is its mass, μ_0 is the vacuum permeability in S.I., ω is the angular frequency of the beam, $B^{(0)}$ is its magnetic flux density magnitude and $\boldsymbol{B}^{(3)}_{\text{beam}}$ is its free space ECE spin field. Eq.(3.202) is the result of the Hamilton Jacobi equation [12] in the limit of special relativity and where the electromagnetic field has been assumed independent of the gravitational field. In off-resonance conditions in the laboratory this is true to an excellent approximation. At visible frequencies (laser beam):

$$|\mathbf{B}^{(3)}_{\text{sample}}| \rightarrow \frac{N}{V} \left(\frac{\mu_0 e^3 c^2}{2m^2\omega^3} \right) B^{(0)2} \tag{3.203}$$

and at radar frequencies:

$$|\mathbf{B}^{(3)}_{\text{sample}}| \rightarrow \frac{N}{V}\left(\frac{\mu_0 e^2 c^2}{2m\omega^2}\right) B^{(0)} \tag{3.204}$$

In terms of intensity I Eq.(3.203) is:

$$|\mathbf{B}^{(3)}_{\text{sample}}| \rightarrow \frac{N}{V}\left(\frac{\mu_0^2 e^3 c}{2m^2}\right) \frac{I}{\omega^3} \tag{3.205}$$

For an intensity $I = 5.5 \times 10^{12} W m^{-2}$ and a $Nd - YaG$ frequency of $1.77 \times 10^{16} rads^1$ then $\left|\mathbf{B}^{(3)}\right| \sim 10^{-9} Tesla$ assuming the Avogadro number of electrons in the volume V of one cubic meter, i.e. This calculation is in excellent agreement with experimental data on the inverse Faraday effect [12].

The EAB effect at laser frequencies is, from Eq.(3.205), the magnetization of an electron gas due to $\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}$ in regions where I is zero experimentally, but where the commutator $\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}$ is not zero. The EAB is given by the same equation (3.205) but the interpretation of I is different. It is the intensity of the laser beam transmitted to regions outside the beam by the spin connection, i.e. by the spinning of space-time itself. Similarly the **B** field of an iron whisker is transmitted to regions outside by the spinning of space-time [1]– [7]. All AB effects are therefore experimental evidence for the spinning of space-time and thus for ECE theory. The AB effects are all effects of the generally covariant electromagnetic field and its commutators used in canonical quantization as already argued. The latter is always defined (indexless shorthand notation [1]–[7]) by:

$$F = D \wedge A \tag{3.206}$$

The EAB experiment therefore has to accurately reproduce the conditions:

$$F = D \wedge A = 0, \quad A \neq 0 \tag{3.207}$$

This is done for the static magnetic field in the Chambers experiment, and in various other designs that reproduce this experiment. The electric and gravitational AB effects have also been observed, but the EAB has not yet been observed. The AB experiments define what is meant by a field in generally covariant electrodynamics. It is a field of force accompanied by kinetic energy. When the field of force is zero the potential energy may still be non-zero, and there is a potential for the creation of a field of force. In the magnetic AB effect for example there is no magnetic force field outside the iron whisker, but there is still a potential for the creation of a magnetic field of force. This potential is generated by the spinning of space-time itself. The gravitational AB effect is due to the potential for force generated by the curving of space-time. In regions of zero gravity, i.e. locally zero Riemann curvature:

$$R = D \wedge \omega = 0, \quad \omega \neq 0 \tag{3.208}$$

and the spin connection still exists. So the usual properties of an electromagnetic beam for example are due to non-zero force fields (in this case oscillating electric and magnetic force fields). These properties include intensity (heat), color (spectrum), transmission of signals form transmitter to receiver, and so on. In ECE theory there exists a property of an electromagnetic field hitherto unmeasured, the EAB effect due to the potential for the creation of a field of force in regions where the field itself is zero. The potential is the tetrad, the properties that create the EAB effect are the spin connection and the commutator needed for canonical quantization.

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