Chapter 6

Solutions Of The ECE Field Equations

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Abstract

Solutions of the ECE field equations are given in the dielectric formulation of the theory. The effect of gravitation on electromagnetism is to change the amplitudes of the plane wave solutions of free electromagnetism, to change the phase velocity and to shift the frequency. In the presence of gravitation the in the free electric field strength $\mathbf{E}^a$ and magnetic flux density $\mathbf{B}^a$ become plane waves in the displacement $\mathbf{D}^a$ and magnetic field strength $\mathbf{H}^a$.

Keywords: Solutions of the ECE field equations, unified field theory, interaction of gravitation and electromagnetism

6.1 Introduction

The dielectric formulation of the Einstein Cartan Evans (ECE) unified field theory [1]– [32] has recently been developed in order to provide a framework for relatively straightforward numerical solutions without having to go immediately into the full details of Cartan geometry. In this paper it is shown that in a specific approximation, the effect of gravitation on the plane waves of the free electromagnetic field is to produce plane waves in the displacement ($\mathbf{D}^a$) and magnetic field strength ($\mathbf{H}^a$) instead of the free space electric field strength ($\mathbf{E}^a$) and the magnetic flux density ($\mathbf{B}^a$). In this approximation the ECE field equations have a well defined analytical solution which can be used to test computer code before embarking on the numerical solution of the dielectric formulation of the ECE field equations. In Section 6.2 the stages involved in deriving the dielectric formulation are summarized. Some important mathematical details are given, starting from the form notation [33] of Cartan geometry on which ECE unified field theory is directly based. These details are not easily found elsewhere, but are important for coding purposes. In section 6.3 the dielectric ECE field equations are solved in a well defined approximation to give an analytical solution. In general the ECE field equations must be solved numerically, and so enough mathematical detail is given in this paper to help to achieve this aim.

6.2 Details In The Derivation Of The ECE Field Equations, Form, Tensor, Vector And Dielectric Notation

The most elegant statement of the ECE field equations in mathematics uses the form notation of differential or Cartan geometry [1]– [33]. However in field theory in physics the tensor notation is more often used and in engineering the vector notation is used. In chemistry the dielectric formulation is often used. All these descriptions are interchangeable and equivalent, so it is useful to summarize them in this section and to give sufficient mathematical detail for coding up the equations to give graphs and animations.

The geometrical fundamentals of ECE theory are the well known fundamentals of standard Cartan geometry: the two structure equations, the two Bianchi identities, and the tetrad postulate. Two ansatzen are used to transform the geometry into an objective unified field theory in general relativity [1]– [32]. The two Cartan structure equations are sometimes known in contemporary mathematics as the master equations of differential geometry. The first Cartan structure equation defines the torsion form (T^a) as the covariant exterior derivative of the tetrad (q^a), the fundamental field of ECE theory:

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b. \tag{6.1}$$

The second structure equation of Cartan defines the Riemann or curvature form ($R^a{}_b$) in terms of the spin connection ($\omega^a{}_b$):

$$R^a{}_b = D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{6.2}$$

Here $d\wedge$ is the exterior derivative of Cartan. The covariant exterior derivative [33] is the operator:

$$D\wedge = d \wedge + \omega \wedge . \tag{6.3}$$

It can be seen that the fundamental variables are the tetrad, (the fundamental field), and the spin connection, which defines the way the frame is curved and / or spun in ECE spacetime. The Latin indices are those of the tangent spacetime at P to the base manifold. The latter is indexed with Greek letters, and the convention [33] of standard differential geometry has been followed. In this convention the Greek indices are omitted because they are always the same on each side of an equation. If however the Greek indices are temporarily restored to equations 6.1 and 6.2 for the sake of instruction, they become the differential form equations

$$T^a{}_{\mu\nu} = (d \wedge q^a)_{\mu\nu} + \left(\omega^a{}_b \wedge q^b\right)_{\mu\nu} \tag{6.4}$$

$$R^a{}_{b\mu\nu} = (d \wedge \omega^a{}_b)_{\mu\nu} + (\omega^a{}_c \wedge \omega^c{}_b)_{\mu\nu} \tag{6.5}$$

The tetrad is a vector valued one-form, a rank two mixed index tensor, so has only one Greek subscript, the torsion form is a vector valued two-form which is antisymmetric in its Greek indices:

$$T^a{}_{\mu\nu} = -T^a{}_{\nu\mu}\,. \tag{6.6}$$

The Riemann form is a tensor valued two-form:

$$R^a{}_{b\mu\nu} = -R^a{}_{b\nu\mu}. \tag{6.7}$$

The spin connection is a tensor valued one-form, but is not a tensor because it does not transform as a tensor [33] under coordinate transformation. This property is analogous to that of the Christoffel connection [33], which is not a tensor for the same reason. In order to be able to transform these form equations to tensor equations the following fundamental definitions are needed.

The exterior derivative of the differential form A [33] is defined in general by:

$$(d \wedge A)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\,\partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]}. \tag{6.8}$$

Thus the exterior derivative of a one-form is:

$$\begin{aligned}(d \wedge A)_{\mu_1\mu_2} &= (d \wedge A)_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu\end{aligned} \tag{6.9}$$

and the exterior derivative of a two-form is:

$$(d \wedge A)_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1} A_{\mu_2\mu_3]} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu} \tag{6.10}$$

The wedge product of a form A and a form B is defined in general by [33]:

$$(A \wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}}. \tag{6.11}$$

Therefore the wedge product of two one-forms is defined by:

$$p = 1, q = 1, \mu_1 = \mu, \mu_2 = \nu \tag{6.12}$$

and is:

$$(A \wedge B)_{\mu\nu} = \frac{2!}{1!1!} A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu. \tag{6.13}$$

The wedge product of a one-form and a two-form is given by:

$$p = 1, q = 2, \mu_1 = \mu, \mu_2 = \nu, \mu_3 = \rho \tag{6.14}$$

and is:

$$\begin{aligned}(A \wedge B)_{\mu_1\mu_2\mu_3} = \frac{3!}{2!1!} A_{[\mu_1} B_{\mu_2\mu_3]} &= 3A_{[\mu} B_{\nu\rho]} \\ &= A_\mu B_{\nu\rho} + A_\nu B_{\rho\mu} + A_\rho B_{\mu\nu}.\end{aligned} \tag{6.15}$$

The form notation of Cartan is more elegant than the notation of tensor analysis, but both are equivalent. The advantage of Cartan geometry is that it allows much more insight into the fundamental structure of equations than the more complicated tensor notation. However the latter may be more useful for coding and the standard vector notation used in engineering is almost always derived from the tensor notation. Form notation and tensor notation are rarely if ever used in engineering.

The Bianchi identities of Cartan geometry [1]– [33] are the rigorous generalizations of the Ricci cyclic equation and the Bianchi identity of the type of Riemann geometry used in Einsteinian general relativity, i.e. used in the Einstein Hilbert (EH) field theory of gravitation proposed in 1915 [34]. The EH field theory is a famous landmark of twentieth century physics but is restricted by its omission from consideration of the torsion tensor. In Cartan geometry the first Bianchi identity is:

$$D \wedge T^a = d \wedge T^a + \omega^a{}_b \wedge T^b := R^a{}_b \wedge q^b \tag{6.16}$$

and the second Bianchi identity is:

$$\begin{aligned} D \wedge R^a{}_b &= d \wedge R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b \\ &:= 0. \end{aligned} \tag{6.17}$$

In Eqs.6.16 and 6.17 we have reverted to the standard notation in which the Greek subscripts are omitted [33]. The Bianchi identities involve the exterior derivatives of the torsion and Riemann forms, and using Eqs.6.1 and 6.2 can be written as differential equations in the tetrad and the spin connection:

$$\begin{aligned} d \wedge \left(d \wedge q^a + \omega^a{}_b \wedge q^b\right) + \omega^a{}_b \wedge \left(d \wedge q^b + \omega^b{}_c \wedge q^c\right)& \\ := \left(d \wedge \omega^a{}_b + \omega^a{}_b \wedge \omega^c{}_b\right) \wedge q^b& \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} d \wedge \left(d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\right) + \omega^a{}_c \wedge \left(d \wedge \omega^a{}_b + \omega^c{}_d \wedge \omega^d{}_b\right)& \\ - \left(d \wedge \omega^a{}_c + \omega^a{}_d \wedge \omega^d{}_c\right) \wedge \omega^c{}_b := 0.& \end{aligned} \tag{6.19}$$

In references [1]– [32] the equivalents of the structure equations and Bianchi identities have been derived in the most general type of Riemann geometry, i.e. the form notation has been translated into tensor notation.

Translation from differential form to tensor notation requires the well known tetrad postulate, which can be proven in several complementary and instructive ways [1]– [33], each proof giving the same result (the tetrad postulate) and each proof reinforcing the other complementary proofs. The most fundamental meaning of the tetrad postulate is perhaps the fact that the same vector field can be expressed equivalently in different coordinate systems. Here vector field means that a vector is defined by its vector components in base coordinate elements such as unit vectors, spinors, or Pauli matrices [33]. The vector field expressed in cartesian or spherical polar coordinates for example is the same vector field but expressed in different coordinates. In Cartan geometry [1]– [33] it follows that the covariant derivative of the tetrad vanishes:

$$D_\nu q^a{}_\mu = 0 \tag{6.20}$$

and this fundamental property is known conventionally as the tetrad postulate. However, nothing is postulated (i.e. nothing is needed to derive Eq.6.20) other than the fact that a vector field in different coordinates is the same vector field. (If it were not the same vector field then the coordinate system would not be a valid coordinate system.) In order to correctly define the covariant derivative of the tetrad it is necessary to define the covariant derivative of a mixed index rank two tensor, a tensor whose upper index is a and whose lower index is μ. In order to do this the fundamental definition of covariant derivative is needed [33]. Examples are given here for clarity of exposition. In general the covariant derivative of a tensor of any rank is defined by [33]:

$$\begin{aligned} D_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\mu_l} &= \partial T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\mu_l} \\ &+ \Gamma^{\mu_1}{}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\mu_l} + \Gamma^{\mu_2}{}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\mu_l} + \cdots \\ &- \Gamma^{\lambda}{}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\mu_l} - \Gamma^{\lambda}{}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\mu_l} - \cdots . \end{aligned} \tag{6.21}$$

For mixed Greek and Latin indices the gamma connection is replaced by the spin connection. Therefore the covariant derivative of the tetrad is, from Eq.6.21:

$$D_\mu q^a{}_\lambda = \partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^b{}_\lambda - \Gamma^\nu{}_{\mu\lambda} q^a{}_\nu . \tag{6.22}$$

Similarly the covariant derivative of a contravariant four vector is:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda , \tag{6.23}$$

the covariant derivative of a covariant vector is:

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda{}_{\nu\mu} V_\lambda , \tag{6.24}$$

and the covariant derivative of a tensor is:

$$D_\mu T^{\nu\rho} = \partial_\mu T^{\nu\rho} + \Gamma^\nu{}_{\mu\lambda} T^{\lambda\rho} + \Gamma^\rho{}_{\mu\lambda} T^{\nu\lambda} . \tag{6.25}$$

The covariant derivative of a rank three tensor is:

$$D_\mu \left(D^\mu q^a{}_\nu \right) = \partial_\mu \left(D^\mu q^a{}_\nu \right) + \Gamma^\mu{}_{\mu\lambda} D^\lambda q^a{}_\nu + \omega^a{}_{\mu b} D^\mu q^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} D^\mu q^a{}_\lambda . \tag{6.26}$$

The above general formulae allow one to rewrite the first Bianchi identity 6.16 in tensor notation as follows:

$$\begin{aligned} \partial_\mu T^a{}_{\nu\rho} + \partial_\rho T^a{}_{\mu\nu} + \partial_\nu T^a{}_{\rho\mu} + \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \omega^a{}_{\rho b} T^b{}_{\mu\nu} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} \\ = R^a{}_{b\mu\nu} q^b{}_\rho + R^a{}_{b\rho\mu} q^b{}_\nu + R^a{}_{b\nu\rho} q^b{}_\mu \\ = R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} \end{aligned} \tag{6.27}$$

and this is the basic tensorial structure of the field equations of ECE theory [1]–[32]. The geometry 6.27 is transformed into the field equation using the ansatz:

$$F^a{}_{\mu\nu} = A^{(0)} T^a{}_{\mu\nu} \tag{6.28}$$

where $A^{(0)}$ is a scalar valued potential magnitude and where:

$$F^a{}_{\mu\nu} = -F^a{}_{\nu\mu} \tag{6.29}$$

is the anti-symmetric field tensor of electromagnetism influenced by gravitation. The basic field equation of ECE theory is therefore:

$$\begin{aligned}&\partial_\mu F^a{}_{\nu\rho} + \partial_\rho F^a{}_{\mu\nu} + \partial_\nu F^a{}_{\rho\mu} \\ &= A^{(0)} \left(R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} - \omega^a{}_{\mu b} T^b{}_{\nu\rho} - \omega^a{}_{\rho b} T^b{}_{\mu\nu} - \omega^a{}_{\nu b} T^b{}_{\rho\mu} \right)\end{aligned} \tag{6.30}$$

and is rewritten as follows to define the homogeneous current of ECE field theory [1]– [32]:

$$\partial_\mu F^a{}_{\nu\rho} + \partial_\rho F^a{}_{\mu\nu} + \partial_\nu F^a{}_{\rho\mu} = \mu_0 \left(j^a{}_{\mu\nu\rho} + j^a{}_{\rho\mu\nu} + j^a{}_{\nu\rho\mu} \right). \tag{6.31}$$

In differential form notation Eq.6.31 is:

$$d \wedge F^a = \mu_0 j^a \tag{6.32}$$

and is the homogeneous field equation of ECE theory. The way in which gravitation influences electromagnetism is defined by the current j^a. In Section 6.3 we give analytical solutions to the homogeneous field equation and its Hodge dual [1]– [32], the inhomogeneous field equation. Firstly in this section enough mathematical detail is given to develop Eq.6.31 into two vector equations, and to derive the Hodge dual of Eq.6.31. This detail is again needed for coding purposes.

The general Hodge dual of a tensor is defined [33] by:

$$\widetilde{A}_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{n-p}} A_{\nu_1 \cdots \nu_p} \tag{6.33}$$

where

$$\epsilon_{\mu_1 \mu_2 \cdots \mu_n} = |g|^{1/2} \bar{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} \tag{6.34}$$

is the Levi-Civita tensor. The latter is defined as the square root of the modulus of the determinant of the metric multiplied by the Levi-Civita symbol:

$$\bar{\epsilon}_{\mu_1 \mu_2 \cdots \mu_n} = \left| \begin{array}{ll} 1 & \text{for even subscript permutation} \\ -1 & \text{for odd subscript permutation} \\ 0 & \text{otherwise} \end{array} \right| . \tag{6.35}$$

Using the metric compatibility condition [33]:

$$D_\mu g_{\nu\rho} = 0 \tag{6.36}$$

it is seen that:

$$D_\mu |g|^{1/2} = \partial_\mu |g|^{1/2} = 0 \tag{6.37}$$

because the determinant of the metric is made up of individual elements of the metric tensor. The covariant derivative of each element vanishes by Eq.6.36, so we obtain Eq.6.37. The premultiplier $|g|^{1/2}$ is a scalar, and in deriving Eq.6.37 we have used the definition [33]:

$$D_\rho S = \partial_\rho S \tag{6.38}$$

where S is any scalar. The Hodge dual of Eq.6.31 may now be defined using the general formula 6.33 and used: a) to obtain the vector formulation of the homogeneous field equation and b) to obtain the inhomogeneous ECE field equation from the homogeneous field equation.

The first step in obtaining the vector formulation is to prove that Eq.6.31 can be rewritten as:

$$\partial_\mu \widetilde{F}^{a\mu\nu} = \mu_0 \widetilde{j}^{a\nu} \tag{6.39}$$

in which:

$$\widetilde{F}^{a\mu\nu} = \frac{1}{2}|g|^{1/2}\bar{\epsilon}^{\mu\nu\rho\sigma}F^a{}_{\rho\sigma} \tag{6.40}$$

$$\widetilde{j}^{a\sigma} = \frac{1}{6}|g|^{1/2}\bar{\epsilon}^{\mu\nu\rho\sigma}j^a{}_{\mu\nu\rho} \tag{6.41}$$

are Hodge duals. To prove Eq.6.39 consider individual tensor elements such as those defined by $\nu = 1, \quad \mu = 0, 2, 3$. In this case:

$$\begin{aligned}
&\partial_0 \widetilde{F}^{a01} + \partial_2 \widetilde{F}^{a21} + \partial_3 \widetilde{F}^{a31} \\
&\quad = \frac{1}{2}|g|^{1/2}\bar{\epsilon}^{\mu 1\rho\sigma}\partial_\mu F^a{}_{\rho\sigma} \\
&\quad = \frac{1}{2}|g|^{1/2}\left(\bar{\epsilon}^{01\rho\sigma}\partial_0 F^a{}_{\rho\sigma} + \bar{\epsilon}^{21\rho\sigma}\partial_2 F^a{}_{\rho\sigma} + \bar{\epsilon}^{31\rho\sigma}\partial_3 F^a{}_{\rho\sigma}\right) \\
&\quad = |g|^{1/2}\left(\partial_0 F^a{}_{23} + \partial_2 F^a{}_{30} + \partial_3 F^a{}_{02}\right)
\end{aligned} \tag{6.42}$$

which is a special case of the general result:

$$\partial_\mu \widetilde{F}^{a\mu\nu} \rightarrow |g|^{1/2}\left(\partial_\mu F^a{}_{\nu\rho} + \partial_\rho F^a{}_{\mu\nu} + \partial_\nu F^a{}_{\rho\mu}\right). \tag{6.43}$$

Consider Eq.6.41 for $\sigma = 1$ to obtain:

$$\begin{aligned}
\widetilde{j}^{a1} &= \frac{1}{6}|g|^{1/2}(\bar{\epsilon}^{0231}j^a{}_{023} + \bar{\epsilon}^{0321}j^a{}_{032} \\
&\qquad + \bar{\epsilon}^{2031}j^a{}_{203} + \bar{\epsilon}^{3021}j^a{}_{302} \\
&\qquad + \bar{\epsilon}^{2301}j^a{}_{230} + \bar{\epsilon}^{3201}j^a{}_{320}) \\
&= \frac{1}{3}|g|^{1/2}\left(j^a{}_{023} + j^a{}_{302} + j^a{}_{230}\right)
\end{aligned} \tag{6.44}$$

Similarly, the other two current terms:

$$\widetilde{j}^{a\sigma} = \frac{1}{6}|g|^{1/2}\bar{\epsilon}^{\rho\mu\nu\sigma}j^a{}_{\rho\mu\nu} \tag{6.45}$$

and

$$\widetilde{j}^{a\sigma} = \frac{1}{6}|g|^{1/2}\bar{\epsilon}^{\nu\rho\mu\sigma}j^a{}_{\nu\rho\mu} \tag{6.46}$$

give Eq.6.44 two more times. So the right hand side of Eq.6.31 for $\nu = 1$ is:

$$\widetilde{j}^{a1} = |g|^{1/2}\left(j^a{}_{023} + j^a{}_{302} + j^a{}_{230}\right). \tag{6.47}$$

Finally we use Eq.6.37 to find that:

$$\partial_\mu\left(|g|^{1/2}F^a{}_{\nu\rho}\right) = |g|^{1/2}\partial_\mu F^a{}_{\nu\rho} \tag{6.48}$$

and so derive Eq.6.39 from Eq.6.31, Q.E.D. Note that the $|g|^{1/2}$ premultiplier cancels out either side of Eq.6.39. The vector formulation of Eq.6.39 follows by standard methods [1]– [33] and is:

$$\boldsymbol{\nabla} \cdot \mathbf{B}^a = \mu_0 \widetilde{j}^{a0} \tag{6.49}$$

$$\boldsymbol{\nabla} \times \mathbf{E}^a + \frac{\partial \mathbf{B}^a}{\partial t} = \mu_0 \widetilde{\mathbf{j}}^a \tag{6.50}$$

where the four-current is defined by:

$$\widetilde{j}^{a\nu} = \left(\frac{\widetilde{j}^{a0}}{c}, \widetilde{\mathbf{j}}^a \right). \tag{6.51}$$

The currents terms in Eq.6.31 are defined by:

$$j^a{}_{\mu\nu\rho} = \frac{A^{(0)}}{\mu_0} \left(R^a{}_{\mu\nu\rho} - \omega^a{}_{\mu b} T^b{}_{\nu\rho} \right) \tag{6.52}$$

and so on. Since $R^a{}_{\mu\nu\rho}$ and $T^b{}_{\nu\rho}$ are antisymmetric in their last two Greek indices they have Hodge duals defined by:

$$\widetilde{R}^a{}_\tau{}^{\mu\nu} = \frac{1}{2} |g|^{1/2} \bar{\epsilon}^{\mu\nu\rho\sigma} R^a{}_{\tau\rho\sigma} \tag{6.53}$$

$$\widetilde{T}^{a\mu\nu} = \frac{1}{2} |g|^{1/2} \bar{\epsilon}^{\mu\nu\rho\sigma} T^a{}_{\rho\sigma} \,. \tag{6.54}$$

The four-current of the homogeneous ECE field equation is therefore given in terms of these Hodge duals as follows:

$$\widetilde{j}^{a\nu} = \frac{A^{(0)}}{\mu_0} \left(\widetilde{R}^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} \widetilde{T}^{b\mu\nu} \right), \tag{6.55}$$

and defines the way in which gravitation affects the Gauss law applied to magnetism and the Faraday law of induction.

The inhomogeneous field equation is derived from the homogeneous field equation by taking the Hodge duals term by term of each two-form in the homogeneous equation:

$$d \wedge F^a = \mu_0 j^a = A^{(0)} \left(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \right). \tag{6.56}$$

The two-form in this equation are: $F^a{}_{\mu\nu}$, $R^a{}_{b\mu\nu}$, and $T^b{}_{\mu\nu}$. Writing out each two-from in tensor notation, the three Hodge duals are:

$$\widetilde{F}^{a\alpha\beta} = \frac{1}{2} |g|^{1/2} \bar{\epsilon}^{\alpha\beta\mu\nu} F^a{}_{\mu\nu} \,, \tag{6.57}$$

$$\widetilde{R}^a{}_b{}^{\alpha\beta} = \frac{1}{2} |g|^{1/2} \bar{\epsilon}^{\alpha\beta\mu\nu} R^a{}_{b\mu\nu}, \tag{6.58}$$

$$\widetilde{T}^{b\alpha\beta} = \frac{1}{2} |g|^{1/2} \bar{\epsilon}^{\alpha\beta\mu\nu} T^b{}_{\mu\nu} \,, \tag{6.59}$$

and each Hodge dual is equivalent to an anti-symmetric rank two tensor. Therefore the inhomogeneous ECE field equation [1]– [32] is:

$$d \wedge \widetilde{F}^a = \mu_0 J^a = A^{(0)} \left(\widetilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \widetilde{T}^b \right) \tag{6.60}$$

and the pre-multiplier $|g|^{1/2}$ cancels out either side of the equation. In tensor notation, Eq.6.60 is:

$$\partial_\mu \widetilde{F}^a{}_{\nu\rho} + \partial_\rho \widetilde{F}^a{}_{\mu\nu} + \partial_\nu \widetilde{F}^a{}_{\rho\mu} = \mu_0 \left(J^a{}_{\mu\nu\rho} + J^a{}_{\rho\mu\nu} + J^a{}_{\nu\rho\mu} \right). \tag{6.61}$$

In summary, the homogeneous and inhomogeneous ECE field equations are:

$$d \wedge F^a = \mu_0 j^a \tag{6.62}$$

$$d \wedge \widetilde{F}^a = \mu_0 J^a. \tag{6.63}$$

In differential form notation, the Maxwell Heaviside (MH) field equations of the standard model are well known to be [33, 34]:

$$d \wedge F = 0, \tag{6.64}$$

$$d \wedge \widetilde{F} = \mu_0 J. \tag{6.65}$$

It is seen that the field form and its Hodge dual appear in the MH equations, but the homogeneous current is missing, indicating that there is no mechanism in MH theory for considering the effect of gravitation on electromagnetism. Also, the inhomogeneous current J of MH theory is introduced empirically (i.e. from experiment), and not from the first theoretical principles of Cartan geometry and generally covariant unified field theory as required in objective physics. The ECE field equations 6.62 and 6.63 identify the source of j^a and J^a in geometry.

The properties of the Hodge dual can be checked with the Schwarzschild metric (SM) [1]– [33]. The SM is a solution of the famous EH field equation of 1915. In EH theory:

$$R^a{}_b \wedge q^b = 0, \tag{6.66}$$

$$T^a = 0. \tag{6.67}$$

In tensor notation Eq.6.66 is the Ricci cyclic equation:

$$R_{\sigma\mu\nu\rho} + R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} = 0 \tag{6.68}$$

and Eq.6.67 is:

$$T^\kappa{}_{\mu\nu} = \Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu} = 0 \tag{6.69}$$

where $\Gamma^\kappa{}_{\mu\nu}$ is the Christoffel connection [33]. In the SM the non-zero elements of the Riemann tensor are:

$$R^0{}_{101}, R^1{}_{212}, R^1{}_{313}, R^2{}_{323}, R^0{}_{202}, R^0{}_{303} \neq 0, \tag{6.70}$$

so Eq.6.68 is true automatically in the SM because the last three subscripts of the Riemann tensors appearing in the Ricci cyclic equation must be all different, i.e. occur in cyclic permutation. However no such elements are non-zero in the SM. The relevant Hodge dual of Eq.6.66 is defined by:

$$R^a{}_b \wedge q^b \longrightarrow \widetilde{R}^a{}_b \wedge q^b \tag{6.71}$$

i.e. by:

$$\widetilde{R}_{\alpha\beta\mu\nu} = \frac{1}{2}|g|^{1/2}\bar{\epsilon}_{\mu\nu}{}^{\rho\sigma} R_{\alpha\beta\rho\sigma}. \tag{6.72}$$

Therefore, upon taking Hodge duals such as:

$$\widetilde{R}_{0123} = |g|^{1/2} R_{0101}, \tag{6.73}$$

$$\widetilde{R}_{0231} = |g|^{1/2} R_{0202}, \tag{6.74}$$

$$\widetilde{R}_{0312} = |g|^{1/2} R_{0303}, \tag{6.75}$$

it is concluded that

$$\widetilde{R}_{0123} + \widetilde{R}_{0231} + \widetilde{R}_{0312} \neq 0 \tag{6.76}$$

i.e.:

$$\widetilde{R}^a{}_b \wedge q^b \neq 0. \tag{6.77}$$

Eq.6.77 means, importantly, that the inhomogeneous current J^a can be very large even if the homogeneous current may be vanishingly small. This result has been illustrated here for the SM, but is true for any metric. This result of ECE field theory explains why the homogeneous current can become zero (as in MH theory), while the inhomogeneous current can become very large. It is a property of geometry.

In deriving Eq.6.63 from Eq.6.62 we have used the Hodge dual of a two-form in four dimensional space-time, a special case of the general Hodge dual formula 6.33. In this case the result is another two-form as argued. The currents in Eqs.6.31 and 6.61 are three-forms, whose Hodge duals in four dimensional space-time are one-forms as we have argued. Denoting the Hodge dual of j^a by $\widetilde{j}^a$, and the Hodge dual of J^a by $\widetilde{J}^a$, then the tensorial homogeneous and inhomogeneous field equations of ECE theory, Eqs.6.31 and 6.61, become [1]–[32]:

$$\partial_\mu \widetilde{F}^{a\mu\nu} = \mu_0 \widetilde{j}^{a\nu}, \tag{6.78}$$

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu}. \tag{6.79}$$

The homogeneous and inhomogeneous currents in tensor notation are:

$$\widetilde{j}^{a\nu} = \frac{A^{(0)}}{\mu_0} \left(\widetilde{R}^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} \widetilde{T}^{b\mu\nu} \right), \tag{6.80}$$

$$\widetilde{J}^{a\nu} = \frac{A^{(0)}}{\mu_0} \left(R^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} T^{b\mu\nu} \right). \tag{6.81}$$

The current four-vectors are defined in S.I. units by:

$$\widetilde{j}^{a\nu} = \left(\frac{1}{c} \widetilde{j}^{a0}, \widetilde{\mathbf{j}}^a \right), \tag{6.82}$$

$$\widetilde{J}^{a\nu} = \left(\frac{1}{c} \widetilde{J}^{a0}, \widetilde{\mathbf{J}}^a \right), \tag{6.83}$$

The two field equations 6.78 and 6.79 in vector notation therefore become the four objective laws of classical electrodynamics in general relativity:

$$\boldsymbol{\nabla} \cdot \mathbf{B}^a = \mu_0 \widetilde{j}^{a0}, \tag{6.84}$$

$$\boldsymbol{\nabla} \times \mathbf{E}^a + \frac{\partial \mathbf{B}^a}{\partial t} = \mu_0 \widetilde{\mathbf{j}}^a, \tag{6.85}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E}^a = \mu_0 c \widetilde{J}^{a0}, \tag{6.86}$$

$$\boldsymbol{\nabla} \times \mathbf{B}^a - \frac{1}{c^2} \frac{\partial \mathbf{E}^a}{\partial t} = \frac{\mu_0}{c} \widetilde{\mathbf{J}}^a. \tag{6.87}$$

Unlike the MH theory, the laws 6.84 to 6.87 can describe the effect of gravitation on electromagnetism. This was a major aim of both Einstein and Cartan.

Eq.6.84 is the Gauss law applied to magnetism; Eq.6.85 is the Faraday law of induction; Eq.6.86 is the Coulomb law; and Eq.6.87 is the Ampère-Maxwell law. It is important to realize that the four laws are now written in the presence of a gravitational field, whereas the familiar MH laws are for electromagnetism without consideration of the gravitational field. This is of course the fundamental aim of a unified field theory.

The relevant S.I. units [35] are as follows:

$$B^a = JsC^{-1}m^{-2} = \text{Tesla}, \tag{6.88}$$

$$E^a = JC^{-1}m^{-1} = Vm^{-1}, \tag{6.89}$$

$$\mu_0 = 4\pi \times 10^{-7} Js^2C^{-2}m^{-1}, \tag{6.90}$$

$$\widetilde{j}^{a0} = Cs^{-1}m^{-2} = Am^{-2}, \tag{6.91}$$

$$\widetilde{j}^{a0}/c = Cm^{-3}, \tag{6.92}$$

$$\frac{1}{c}\widetilde{j}^{a0} = \frac{1}{c}\widetilde{J}^{a0} = \text{charge density}, \tag{6.93}$$

$$\widetilde{\mathbf{j}}^a = \widetilde{\mathbf{J}}^a = \text{current density}. \tag{6.94}$$

Eqs.6.84 to 6.87 use the fundamental S.I. relation [35] in free space:

$$E^{(0)} = cB^{(0)}. \tag{6.95}$$

In the limit of zero gravitation the electromagnetic component of the unified ECE field splits off, and is referred to as the free electromagnetic field. The Cartan geometry of the free electromagnetic field is defined [1]– [32] by the fact that in this limit the homogeneous current j^a vanishes, so it follows that:

$$R^a{}_b \wedge q^b = \omega^a{}_b \wedge T^b. \tag{6.96}$$

A solution of Eq.6.96 is:

$$R^a{}_b = \kappa\epsilon^a{}_{bc}T^c \tag{6.97}$$

$$\omega^a{}_b = \kappa\epsilon^a{}_{bc}q^c, \tag{6.98}$$

where the scalar κ has the units of wave-number (inverse meters). It is important to understand that there may be a Riemann form for a spinning frame. The Riemann form is the curvature form only for the free gravitational field. For rotational motion (i.e. the spinning of the free electromagnetic field) Eqs.6.97 and 6.98 show that for each μ and ν, the Riemann form is the antisymmetric tangent space-time tensor corresponding to the axial vector T^c. Similarly, for each μ, the spin connection is the antisymmetric tensor corresponding to the axial vector q^c. The Hodge dual of Eq.6.96 is:

$$\widetilde{R}^a{}_b \wedge q^b = \omega^a{}_b \wedge \widetilde{T}^b, \tag{6.99}$$

and in consequence, for the free electromagnetic field:

$$\widetilde{j}^{a\nu} = \widetilde{J}^{a\nu} = 0. \tag{6.100}$$

Therefore, for the free electromagnetic field, the four laws 6.84 to 6.87 simplify to:

$$\nabla \cdot \mathbf{B}^a = 0 \tag{6.101}$$

$$\nabla \times \mathbf{E}^a + \frac{\partial \mathbf{B}^a}{\partial t} = \mathbf{0} \tag{6.102}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E}^a = 0, \tag{6.103}$$

$$\boldsymbol{\nabla} \times \mathbf{B}^a - \frac{1}{c^2}\frac{\partial \mathbf{E}^a}{\partial t} = \mathbf{0}. \tag{6.104}$$

These equations have analytical solutions [1]– [32] and describe the electromagnetic field in the hypothetical limit of vanishing mass. These are sometimes known as source-free fields in the standard literature on MH theory, because a source, by definition, must be a radiating electron whose mass is not zero.

The converse limit of zero electromagnetism, (the free gravitational field), is defined [1]– [32] by zero torsion:

$$F^a = A^{(0)}T^a = 0, \tag{6.105}$$

$$R^a{}_b \wedge q^b = 0. \tag{6.106}$$

Polarization and magnetization are defined [35] in ECE theory by a straight forward extension of their MH counterparts to include the polarization index a. Thus:

$$\mathbf{D}^a = \epsilon \mathbf{E}^a = \epsilon_0 \mathbf{E}^a + \mathbf{P}^a, \tag{6.107}$$

$$\mathbf{H}^a = \frac{1}{\mu}\mathbf{B}^a = \frac{1}{\mu_0}\mathbf{B}^a - \frac{1}{\mu_0}\mathbf{M}^a, \tag{6.108}$$

where $\mathbf{D}^a$ is the electric displacement (Cm^{-2}) and $\mathbf{H}^a$ the magnetic field strength (Am^{-1}). Here $\mathbf{P}^a$ is the polarization, $\mathbf{M}^a$ the magnetization, ϵ the dielectric permittivity, μ the magnetic permeability, ϵ_0 the vacuum permittivity and μ_0 the vacuum permeability. The relative permittivity and permeability are therefore [35]:

$$\epsilon_r = \epsilon/\epsilon_0, \tag{6.109}$$

$$\mu_r = \mu/\mu_0, \tag{6.110}$$

and the refractive index is

$$n^2 = \epsilon_r \mu_r. \tag{6.111}$$

In general ϵ_r and μ_r are inhomogeneous functions of spacetime:

$$\epsilon_r = \epsilon_r\,(ct, X, Y, Z) \tag{6.112}$$

$$\mu_r = \mu_r\,(ct, X, Y, Z)\,, \tag{6.113}$$

and in the presence of absorption may become complex valued [35, 36]:

$$\epsilon_r = \epsilon'_r + i\epsilon''_r, \tag{6.114}$$

$$\mu_r = \mu'_r + i\mu''_r. \tag{6.115}$$

The power absorption coefficient (neper m^{-1}) is defined [36] by:

$$\alpha = \frac{\omega \epsilon''_r}{n' c}, \tag{6.116}$$

and by the Beer Lambert law:

$$I = I_0 e^{(-\alpha Z)}. \tag{6.117}$$

Here Z is the sample length, I_0 the intensity of incident and I the intensity of absorbed radiation. The effect of classical gravitation on the classical electromagnetic field is therefore in general to refract, reflect, diffract and absorb electromagnetic radiation, and this is developed in Section 6.3. In other words gravitation acts as a dielectric material which may be a reflector, an absorber, a polarizable medium, a magnetizable medium, a conductor, a superconductor and so forth. Finally in this Section 6.2 the wave equation of ECE field theory is discussed and cross-checked mathematically prior to computation.

The basic wave equation of ECE field theory [1]– [32] is derived straightforwardly form the tetrad postulate (6.20) through a lemma, or subsidiary geometric proposition, the ECE Lemma. The fundamental structure of the latter is:

$$D^{\mu}\left(D_{\mu}q^{a}{}_{\nu}\right) := 0 \tag{6.118}$$

and is seen from Eq.6.20 to be an identity of Cartan geometry. From Eq.6.38 the lemma is seen to be:

$$\partial^{\nu}\left(D_{\mu}q^{a}{}_{\nu}\right) := 0 \tag{6.119}$$

i.e.

$$\partial^{\mu}\left(\partial_{\mu}q^{a}{}_{\lambda} + \omega^{a}{}_{\mu b}q^{b}{}_{\lambda} - \Gamma^{\nu}{}_{\mu\lambda}q^{a}{}_{\nu}\right) := 0. \tag{6.120}$$

The dAlembertian operator is defined [33, 34] as:

$$\Box := \partial^{\mu}\partial_{\mu} \tag{6.121}$$

so Eq.6.120 is:

$$\Box q^{a}{}_{\lambda} = \partial^{\mu}\left(\Gamma^{\nu}{}_{\mu\lambda}q^{a}{}_{\nu} - \omega^{a}{}_{\mu b}q^{b}{}_{\lambda}\right). \tag{6.122}$$

Now define the scalar curvature:

$$R := q^{\lambda}{}_{a}\partial^{\nu}\left(\Gamma^{\nu}{}_{\mu\lambda}q^{a}{}_{\nu} - \omega^{a}{}_{\mu b}q^{b}{}_{\lambda}\right) \tag{6.123}$$

and use the fundamental [33] inverse identity of tetrads:

$$q^{a}{}_{\lambda}q^{\lambda}{}_{a} = 1 \tag{6.124}$$

to deduce the ECE Lemma [1]– [32]:

$$\Box q^{a}{}_{\lambda} := Rq^{a}{}_{\lambda}. \tag{6.125}$$

To check this derivation use Eq.6.20 in the form:

$$\Gamma^{\nu}{}_{\mu\lambda}q^{a}{}_{\nu} - \omega^{a}{}_{\mu b}q^{b}{}_{\lambda} = \partial_{\mu}q^{a}{}_{\lambda} \tag{6.126}$$

to find:

$$R = q^{\lambda}{}_{a}\partial^{\mu}\partial_{\mu}q^{a}{}_{\lambda} = q^{\lambda}{}_{a}\Box q^{a}{}_{\lambda} \tag{6.127}$$

Q.E.D.

The ECE wave equation [1]– [32] is obtained from the ECE lemma by using the Einstein Ansatz:

$$R = -kT \tag{6.128}$$

where k is the Einstein constant and T the index contracted energy-momentum tensor. Einstein [37] asserted that the ansatz 6.128 must be applied to all the radiated and matter fields of physics, not only the gravitational field. However until the emergence of ECE theory in 2003 [1]– [32] the ansatz necessarily had to be restricted to gravitation. In gravitational theory Eq.6.128 can be deduced directly from the EH field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = kT_{\mu\nu}. \tag{6.129}$$

using the inverse metric definition [1]– [32, 37]:

$$g^{\mu\nu} g_{\mu\nu} := 4. \tag{6.130}$$

Multiply both sides of Eq.6.129 by $g^{\mu\nu}$ to obtain Eq.6.128 [37], as first shown by Einstein. Here $R_{\mu\nu}$ is the symmetric Ricci tensor of EH field theory, R the scalar curvature of EH field theory, $g_{\mu\nu}$ the symmetric metric of EH field theory, and $T_{\mu\nu}$ the symmetric canonical energy-momentum tensor of EH field theory. In the more general unified ECE theory the Einstein Ansatz 6.128 has been proven in several ways. The key point is that the Einstein Ansatz in ECE field theory applies to all radiated and matter fields, i.e., in logic, to the unified field. There is only ONE unified field by definition, and so Einstein's fundamental link of physics and geometry must apply to that unified field. All other fields in nature are components of the unified field: the gravitational, electromagnetic, weak, strong and matter fields are variations of the tetrad field [1]– [33] in the Palatini formulation of general relativity, and thus of causal and objective physics. This is a major philosophical advance of ECE field theory from the standard model.

Therefore from Eqs.6.125 and 6.128 the ECE wave equation is:

$$(\square + kT)\, q^a{}_\mu = 0 \tag{6.131}$$

and is the archetypical wave equation of objective physics, i.e. of general relativity applied to the unified field.

The Dirac equation for the fermionic matter field, for example, is the linear limit of Eq.6.131 defined by:

$$kT \rightarrow \left(\frac{m_e c}{\hbar}\right)^2 \tag{6.132}$$

where m_e is the mass of the fermion, $\hbar$ is the reduced Planck constant and c the vacuum speed of light, a universal constant of all relativity theory. In the linear limit, as the name suggests, the ECE field equation linearizes, because its eigenvalues are no longer intrinsically functions of the tetrad. More generally the ECE wave equation is non-linear because R depends on the tetrad as in Eq.6.123. Numerical methods are needed therefore to solve the ECE wave equation in general. The Dirac equation is therefore:

$$\left(\square + \left(\frac{m_e c}{\hbar}\right)^2\right) q^a{}_\mu = 0 \tag{6.133}$$

where the tetrad defines the Dirac four-spinor [1]– [32]. Using the ECE Ansatz in the form:

$$A^a{}_\mu = A^{(0)} q^a{}_\mu \tag{6.134}$$

we define the electromagnetic potential field. The governing wave equation of the electromagnetic part of the unified field (the electromagnetic field for short) is therefore:

$$(\square + kT)\, A^a{}_\mu = 0. \tag{6.135}$$

In the linear limit 6.132 we obtain the Proca equation from Eq.6.135:

$$kT \to \left(\frac{m_p c}{\hbar}\right)^2 \tag{6.136}$$

where m_p is the mass of the photon, a boson. It is important to understand that a different representation space [1]– [32] is used for the tetrads defining the fermion and boson. Similarly a different representation space of the tetrad is used for gluons and quarks [1]– [32], but the fundamental field is always the tetrad field. Thus quantum electrodynamics in ECE theory proceeds by solving Eqs.6.131 and 6.135 simultaneously with exchange of photons between two electrons [1]– [32]. Similarly quantum chromodynamics proceeds by setting up simultaneous ECE field equations with exchange of gluons between two quarks. These procedures must be carried out numerically to avoid singularities and renormalization. The Feynman calculus and the unobjective path integral formalism [34]are by-passed completely by the numerical methods of ECE field theory. Singularities do not occur in nature, and do not occur in the theory of relativity and in objective and causal physics. In Feynman's path integral formalism the electron "can do anything it likes", "go backwards in time", and so on [34]. These hypothetical trajectories are essentially summed to give what APPEARS superficially to be an accurate result for the anomalous magnetic moment of the electron and so forth. These ideas of quantum electrodynamics are obviously and diametrically at odds with a causal and objective relativity theory such as ECE field theory, wherein each event must be preceded by a cause, as in Newtonian natural philosophy. The claimed accuracy of quantum electrodynamics and quantum chromo-dynamics has more to do with the selective use of several parameters than with a first principles theory of physics such as ECE field theory or EH field theory.

The derivation of the ECE Lemma can be cross checked in at least two ways. Apply the Leibnitz Theorem to Eq.6.118:

$$D^\mu \left(D_\mu q^a{}_\nu\right) = \left(D^\mu D_\mu\right) q^a{}_\nu = 0 \tag{6.137}$$

and to Eq.6.119:

$$\partial^\mu \left(D_\mu q^a{}_\nu\right) = \left(\partial^\mu D_\mu\right) q^a{}_\nu + D_\mu \left(\partial^\mu q^a{}_\nu\right) = 0. \tag{6.138}$$

Therefore Eq.6.118 is:

$$\begin{aligned} &D^\mu \left(\partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^b{}_\lambda - \Gamma^\nu{}_{\mu\lambda} q^a{}_\nu\right) \\ &\quad \left(D^\mu \partial_\mu\right) q^a{}_\lambda + \left(D^\mu \omega^a{}_{\mu b}\right) q^b{}_\lambda - \left(D^\mu \Gamma^\nu{}_{\mu\lambda}\right) q^a{}_\nu = 0, \end{aligned} \tag{6.139}$$

where we have used Eq.6.20 again. Now use the results:

$$D_\mu \partial^\mu = \Box + \Gamma^\mu{}_{\mu\lambda} \partial^\lambda, \tag{6.140}$$

$$D^\mu = g^{\mu\nu} D_\nu, \tag{6.141}$$

$$\partial_\mu = g_{\mu\nu} \partial^\nu, \tag{6.142}$$

to find:

$$D^\mu \partial_\mu = g^{\mu\nu} D_\nu g_{\mu\nu} \partial^\nu = 4 D_\mu \partial^\mu. \tag{6.143}$$

From Eq.6.143 in Eq.6.139:

$$4\left(D_{\mu}\partial^{\mu}\right)q^{a}{}_{\lambda}+\left(D^{\mu}\omega^{a}{}_{\mu b}\right)q^{b}{}_{\lambda}-\left(D^{\mu}\Gamma^{\nu}{}_{\mu\lambda}\right)q^{a}{}_{\nu}=0 \tag{6.144}$$

Now use the Leibnitz Theorem again:

$$\left(D_{\mu}\partial^{\mu}\right)q^{a}{}_{\lambda}=D_{\mu}\left(\partial^{\mu}q^{a}{}_{\lambda}\right)+\partial^{\mu}\left(D_{\mu}q^{a}{}_{\lambda}\right) \tag{6.145}$$

to find:

$$4\left(D_{\mu}\left(\partial^{\mu}q^{a}{}_{\lambda}\right)+\partial^{\mu}\left(D_{\mu}q^{a}{}_{\lambda}\right)+\left(D^{\mu}\omega^{a}{}_{\mu b}\right)q^{b}{}_{\lambda}-\left(D^{\mu}\Gamma^{\nu}{}_{\mu\lambda}\right)q^{a}{}_{\nu}\right)=0. \tag{6.146}$$

Comparing Eqs.6.146 and 6.119:

$$4D_{\mu}\left(\partial^{\mu}q^{a}{}_{\lambda}\right)+D^{\mu}\omega^{a}{}_{\mu b}-\left(D^{\nu}\Gamma^{\nu}{}_{\mu\lambda}\right)q^{a}{}_{\nu}=0 \tag{6.147}$$

i.e.

$$D^{\nu}\left(\partial_{\mu}q^{a}{}_{\lambda}+\omega^{a}{}_{\mu b}q^{b}{}_{\lambda}-\Gamma^{\nu}{}_{\mu\lambda}q^{a}{}_{\nu}\right)=0 \tag{6.148}$$

which is:

$$D^{\mu}\left(D_{\mu}q^{a}{}_{\lambda}\right)=0 \tag{6.149}$$

implying self-consistently the tetrad postulate 6.20, Q.E.D.

Secondly the ECE Lemma may be cross-checked using the general formula 6.20 for the covariant derivatve of any tensor. Regarding $D^{\mu}q^{a}{}_{\nu}$ as a rank three mixed index tensor with two upper indices, μ and a, and one lower index, ν, Eq.6.21 gives:

$$\begin{aligned}D_{\mu}\left(D^{\mu}q^{a}{}_{\nu}\right)=\partial_{\nu}\left(D^{\mu}q^{a}{}_{\nu}\right)+\Gamma^{\mu}{}_{\mu\lambda}D^{\lambda}q^{a}{}_{\nu}\\+\omega^{a}{}_{\mu b}D^{\mu}a^{b}{}_{\nu}-\Gamma^{\lambda}{}_{\mu\nu}D^{\mu}q^{a}{}_{\lambda}=\partial_{\mu}\left(D^{\mu}q^{a}{}_{\nu}\right)\end{aligned} \tag{6.150}$$

where we have used Eq.6.20 again, Q.E.D.

6.3 Dielectric ECE Theory, Analytical And Numerical Solutions

In this Section an analytical solution is given in a well defined approximation of the simultaneous equations 6.85 and 6.87; in general these must be solved numerically along with the other two equations 6.84 and 6.86. First develop the free fields $\mathbf{E}^a$ and $\mathbf{B}^a$ as follows using Eqs.6.107 and 6.108:

$$\mathbf{E}^{a}=\frac{1}{\epsilon_{0}}\left(\mathbf{D}^{a}-\mathbf{P}^{a}\right) \tag{6.151}$$

$$\mathbf{B}^{a}=\mu_{0}\left(\mathbf{H}^{a}+\mathbf{M}^{a}\right) \tag{6.152}$$

From Eqs.6.151 and 6.152 in Eq.6.85 we obtain:

$$\frac{1}{\epsilon_{0}}\boldsymbol{\nabla}\times\mathbf{D}^{a}+\mu_{0}\frac{\partial\mathbf{H}^{a}}{\partial t}=\mu_{0}\widetilde{\mathbf{j}}^{a}+\frac{1}{\epsilon_{0}}\boldsymbol{\nabla}\times\mathbf{P}^{a}-\mu_{0}\frac{\partial\mathbf{M}^{a}}{\partial t}. \tag{6.153}$$

Therefore if the homogeneous current is defined as:

$$\widetilde{\mathbf{j}}^{a}:=\frac{\partial\mathbf{M}^{a}}{\partial t}-c^{2}\boldsymbol{\nabla}\times\mathbf{P}^{a}, \tag{6.154}$$

then we obtain:

$$\boldsymbol{\nabla} \times (\epsilon_r \mathbf{E}^a) + \frac{\partial}{\partial t}\left(\frac{1}{\mu_r}\mathbf{B}^a\right) = 0, \tag{6.155}$$

which can be reexpressed as:

$$\boldsymbol{\nabla} \times \mathbf{D}^a + \frac{1}{c^2}\frac{\partial \mathbf{H}^a}{\partial t} = \mathbf{0}. \tag{6.156}$$

Eqs.6.155 and 6.156 are true if and only if Eq.6.154 is true. However, the homogeneous current can always be expressed as a combination of polarization and magnetization as in Eq.6.154. The latter can therefore serve as a general definition of the homogeneous current. In other words there is no loss of generality in the derivation of Eqs.6.155 and 6.156 from Eq.6.85.

Similarly, using Eqs.6.151 and 6.152 in Eq.6.87, we obtain:

$$\boldsymbol{\nabla} \times \mathbf{H}_1^a - \frac{\partial \mathbf{D}_1^a}{\partial t} = \frac{1}{c}\widetilde{\mathbf{J}}^a - \boldsymbol{\nabla} \times \mathbf{M}_1^a - \frac{\partial \mathbf{P}_1^a}{\partial t}. \tag{6.157}$$

Therefore if we define the inhomogeneous current as:

$$\widetilde{\mathbf{J}}^a := c\left(\boldsymbol{\nabla} \times \mathbf{M}_1^a + \frac{\partial \mathbf{P}_1^a}{\partial t}\right) \tag{6.158}$$

we obtain the equation:

$$\boldsymbol{\nabla} \times \mathbf{H}_1^a - \frac{\partial \mathbf{D}_1^a}{\partial t} = \mathbf{0}. \tag{6.159}$$

This equation can be expressed in terms of the relative permittivity ϵ_{r1} and permeability μ_{r1} as:

$$\boldsymbol{\nabla} \times \left(\frac{\mathbf{B}^a}{\mu_{r1}}\right) - \frac{1}{c^2}\frac{\partial}{\partial t}(\epsilon_{r1}\mathbf{E}^a) = \mathbf{0}. \tag{6.160}$$

Therefore the analytical and computational problem has been reduced to solving the simultaneous equations 6.155 and 6.160. It is important to note [38] that ϵ_{r1} is in general different from ϵ_r, and that μ_{r1} is in general different from μ_r. The reason is that the current $\widetilde{\mathbf{J}}^a$ is in general different from the current $\widetilde{\mathbf{j}}^a$. Therefore the input parameters for the numerical solution of the simultaneous equations 6.155 and 6.160 are ϵ_r, ϵ_{r1}, μ_r and μ_{r1}.

In the special case:

$$\epsilon_r = \epsilon_{r1}, \quad \mu_r = \mu_{r1} \tag{6.161}$$

analytical solutions can be obtained of the simultaneous equations 6.155 and 6.160, because in this special case:

$$\mathbf{D}_1^a = \mathbf{D}^a, \tag{6.162}$$

$$\mathbf{H}_1^a = \mathbf{H}^a, \tag{6.163}$$

giving the simultaneous equations:

$$\boldsymbol{\nabla} \times \mathbf{D}^a + \frac{1}{c^2}\frac{\partial \mathbf{H}^a}{\partial t} = \mathbf{0}, \tag{6.164}$$

$$\boldsymbol{\nabla} \times \mathbf{H}^a - \frac{\partial \mathbf{D}^a}{\partial t} = \mathbf{0}. \tag{6.165}$$

These can be written as:

$$\nabla \times (c\mathbf{D}^a) + \frac{\partial}{\partial t}\left(\frac{\mathbf{H}^a}{c}\right) = \mathbf{0}, \tag{6.166}$$

$$\nabla \times (\frac{\mathbf{H}^a}{c}) - \frac{1}{c^2}\frac{\partial}{\partial t}(c\mathbf{D}^a) = \mathbf{0}, \tag{6.167}$$

and so have the same structure as:

$$\nabla \times \mathbf{E}^a + \frac{\partial \mathbf{B}^a}{\partial t} = \mathbf{0}, \tag{6.168}$$

$$\nabla \times \mathbf{B}^a - \frac{1}{c^2}\frac{\partial \mathbf{E}^a}{\partial t} = \mathbf{0}. \tag{6.169}$$

The plane wave solutions of Eqs.6.168 and 6.169 are well known. For example, for $a = (1)$ in the complex circular basis [1]– [32], the plane wave solutions are:

$$\mathbf{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}}\,(\mathbf{i} - i\mathbf{j})\,e^{(\omega t - \kappa Z)}, \tag{6.170}$$

$$\mathbf{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}\,(i\mathbf{i} + \mathbf{j})\,e^{i(\omega t - \kappa Z)}. \tag{6.171}$$

It follows that Eqs.6.164 and 6.165 have solutions such as:

$$\mathbf{D}^{(1)} = \frac{D^{(0)}}{\sqrt{2}}\,(\mathbf{i} - i\mathbf{j})\,e^{(\omega t - \kappa Z)}, \tag{6.172}$$

$$\mathbf{H}^{(1)} = \frac{H^{(0)}}{\sqrt{2}}\,(i\mathbf{i} + \mathbf{j})\,e^{i(\omega t - \kappa Z)}, \tag{6.173}$$

where:

$$H^{(0)} = cD^{(0)}. \tag{6.174}$$

Now use:

$$D^{(0)} = \epsilon E^{(0)}, \tag{6.175}$$

$$H^{(0)} = \frac{1}{\mu}B^{(0)}, \tag{6.176}$$

to find:

$$E^{(0)} = vB^{(0)} = \frac{c}{n^2}B^{(0)} \tag{6.177}$$

where the refractive index is:

$$n^2 = \epsilon_r \mu_r \tag{6.178}$$

and the phase velocity is:

$$v = \frac{c}{n^2}. \tag{6.179}$$

In the special case 6.161 there are also the simultaneous equations:

$$\widetilde{\mathbf{j}}^a = \frac{\partial \mathbf{M}^a}{\partial t} - c^2 \nabla \times \mathbf{P}^a, \tag{6.180}$$

$$\widetilde{\mathbf{J}}^a = c\left(\nabla \times \mathbf{M}^a + \frac{\partial \mathbf{P}^a}{\partial t}\right), \tag{6.181}$$

where $\widetilde{\mathbf{J}}^a$ and $\widetilde{\mathbf{j}}^a$ are linked by Hodge duality (Section 6.2). Therefore in the special case 6.161 the effect of gravitation on electromagnetism can be deduced analytically from ECE theory as follows.

1. Gravitation changes the amplitudes of the plane waves:

$$E^{(0)} \longrightarrow D^{(0)}, \tag{6.182}$$

$$B^{(0)} \longrightarrow H^{(0)}. \tag{6.183}$$

2. Gravitation changes the phase velocity of the free space plane waves from c to v, causing diffraction as in the Eddington effect [1]– [32].

3. Gravitation causes a red shift in angular frequency for a given κ because the phase velocity is defined by

$$v = \frac{\omega}{\kappa} \tag{6.184}$$

and has been decreased from c to v if the refractive index is greater than unity.

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