DERIVATION OF THE BELTRAMI EQUATION FROM CARTAN GEOMETRY.

by

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ABSTRACT

The Beltrami equation is derived from the Cartan identity assuming that the magnetic monopole is zero. It is shown that there are always longitudinal solutions of the Beltrami equation for the free electromagnetic field and some of these solutions are discussed in the contexts of electrodynamics, hydrodynamics, magnetohydrodynamics and cosmology. It is shown that B(3) type solutions always exist in the free electromagnetic field. Therefore photon mass is never identically zero and the standard model of physics is refuted.

Keywords: Beltrami equation, Cartan geometry, photon mass, longitudinal components of the free electromagnetic field.

UFT 257

1. INTRODUCTION

In recent papers of this series {1 - 10} it has been shown that the well known Cartan identity {11} reduces to a simple vector identity if it assumed that there is no magnetic monopole in nature. It has also been shown that magnetic and electric charge current densities can be constructed from Cartan geometry as part of a unified field theory. In previous papers of the series it was shown that plane waves and the B(3) field of free space electromagnetism obeys Beltrami equations. Reed {12} has reviewed the various solutions of the Beltrami equation and in Section 2 it is shown that the solutions of the free electromagnetic field are Beltrami solutions multiplied by a phase factor. This means that all the solutions of the Bealtrami equation known {12} to exist in areas such as hydrodynamics, magnetohydrodynamics, plasma physics and cosmology are also solutions of the free electromagnetic field. Therefore it is essential to derive the Beltrami equation as part of a unified field theory rather than assert its existence a priori, which has been the practice to date {12}. This derivation is given in the first part of section 2, which refers to the extensive details given in the notes accompanying this paper, UFT257 on www.aias.us. As usual the accompanying or background notes are an intrinsic part of the summary given in this paper. An overview of the contents of the notes is given in Section 2. In Section 3, the results are analyzed and graphed by computer, showing the nature of longitudinal flow in free field electrodynamics. The existence of these longitudinal solutions immediately refutes the standard model of physics, which asserts erroneously that the free electromagnetic field has only transverse components.

2. DERIVATION OF THE BELTRAMI EQUATION

It has been shown in recent work that the spatial part of the well known Cartan identity {11} can be expressed as a vector identity:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b = \underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_c - \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b \quad -(1)$$

where $\underline{q}^a$ is the well defined tetrad vector and $\underline{\omega}^a{}_b$ the spin connection vector. In the absence of a magnetic monopole:

$$\underline{\nabla} \cdot \underline{\omega}^a{}_b \times \underline{q}^b = 0 \quad -(2)$$

so:

$$\underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^a{}_b = \underline{\omega}^a{}_b \cdot \underline{\nabla} \times \underline{q}^b. \quad -(3)$$

Assume that the spin connection vector is an axial vector dual in its index space to an antisymmetric tensor:

$$\underline{\omega}^a{}_b = \epsilon^a{}_{bc}\, \underline{\omega}^c \quad -(4)$$

where $\epsilon^a{}_{bc}$ is the totally antisymmetric unit tensor in three dimensions. Then Eq. (3) reduces to:

$$\underline{q}^b \cdot \underline{\nabla} \times \underline{\omega}^c = \underline{\omega}^c \cdot \underline{\nabla} \times \underline{q}^b. \quad -(5)$$

An example of this is:

$$\underline{A}^{(2)} \cdot \underline{\nabla} \times \underline{\omega}^{(1)} = \underline{\omega}^{(1)} \cdot \underline{\nabla} \times \underline{A}^{(2)} \quad -(6)$$

in the complex circular basis ((1), (2), (3)). The vector potential is defined by the ECE hypothesis {1 - 10}:

$$\underline{A}^a = A^{(0)}\, \underline{q}^a \quad -(7)$$

It has also been shown in recent work that the geometrical condition for the absence of a magnetic monopole is:

$$\underline{\omega}^a{}_b \cdot \underline{B}^b = \underline{A}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad -(8)$$

where the spin curvature vector is defined by:

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(9)$$

and where $\underline{B}^a$ is the magnetic flux density vector. Using Eq. (4):

$$\underline{R}^c(\text{spin}) = \underline{\nabla} \times \underline{\omega}^c - \underline{\omega}^b \times \underline{\omega}^a \quad -(10)$$

The complex circular basis {1 - 10} is defined by:

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i\,\underline{e}^{(3)*} \quad -(11)$$

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and in this basis:

$$\begin{aligned}
\underline{R}^{(1)}(\text{spin}) &= \underline{\nabla} \times \underline{\omega}^{(1)} + i\,\underline{\omega}^{(3)} \times \underline{\omega}^{(1)} \\
\underline{R}^{(2)}(\text{spin}) &= \underline{\nabla} \times \underline{\omega}^{(2)} + i\,\underline{\omega}^{(2)} \times \underline{\omega}^{(3)} \\
\underline{R}^{(3)}(\text{spin}) &= \underline{\nabla} \times \underline{\omega}^{(3)} + i\,\underline{\omega}^{(1)} \times \underline{\omega}^{(2)}
\end{aligned} \quad -(12)$$

Similarly, the magnetic flux density vectors are:

$$\begin{aligned}
\underline{B}^{(1)} &= \underline{\nabla} \times \underline{A}^{(1)} + i\,\underline{\omega}^{(3)} \times \underline{A}^{(1)} \\
\underline{B}^{(2)} &= \underline{\nabla} \times \underline{A}^{(2)} + i\,\underline{\omega}^{(2)} \times \underline{A}^{(3)} \\
\underline{B}^{(3)} &= \underline{\nabla} \times \underline{A}^{(3)} + i\,\underline{\omega}^{(1)} \times \underline{A}^{(2)}
\end{aligned} \quad -(13)$$

Eq. (8) may be exemplified by:

$$\underline{\omega}^{(1)} \cdot \underline{B}^{(2)} = \underline{A}^{(1)} \cdot \underline{R}^{(2)}(\text{spin}) \quad -(14)$$

which may be developed as:

$$\underline{\omega}^{(1)} \cdot \left(\underline{\nabla} \times \underline{A}^{(2)} + i\underline{\omega}^{(2)} \times \underline{A}^{(3)} \right) \quad -(15)$$

$$= \underline{A}^{(1)} \cdot \left(\underline{\nabla} \times \underline{\omega}^{(2)} + i\underline{\omega}^{(2)} \times \underline{\omega}^{(3)} \right).$$

Possible solutions are:

$$\underline{\omega}^{(i)} = \pm \frac{\kappa}{A^{(0)}} \underline{A}^{(i)}, \quad i = 1, 2, 3 \quad -(16)$$

In order to be consistent with the original definition of B(3) {1 - 10} the negative sign is chosen so:

$$\underline{B}^{(3)} = \underline{\nabla} \times \underline{A}^{(3)} - \frac{i\kappa}{A^{(0)}} \underline{A}^{(1)} \times \underline{A}^{(2)}$$

$$\underline{B}^{(1)} = \underline{\nabla} \times \underline{A}^{(1)} - \frac{i\kappa}{A^{(0)}} \underline{A}^{(3)} \times \underline{A}^{(1)} \quad -(17)$$

$$\underline{B}^{(2)} = \underline{\nabla} \times \underline{A}^{(2)} - \frac{i\kappa}{A^{(0)}} \underline{A}^{(2)} \times \underline{A}^{(3)}.$$

From Eq. (2):

$$\underline{\nabla} \cdot \underline{\omega}^{(3)} \times \underline{A}^{(1)} = 0 \quad -(18)$$

and the following is an identity of vector analysis:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^{(1)} = 0. \quad -(19)$$

A possible solution of Eqs. (18) and (19) is:

$$\underline{\nabla} \times \underline{A}^{(1)} = i\underline{\omega}^{(3)} \times \underline{A}^{(1)} = -\frac{i\kappa}{A^{(0)}} \underline{A}^{(3)} \times \underline{A}^{(1)}. \quad -(20)$$

Similarly:

$$\underline{\nabla} \times \underline{A}^{(2)} = i\underline{\omega}^{(2)} \times \underline{A}^{(3)} = -\frac{i\kappa}{A^{(0)}} \underline{A}^{(2)} \times \underline{A}^{(3)}. \quad -(21)$$

Now multiply both sides of the basis equations (11a) to (11c) by $A^{(0)2} e^{i\phi} e^{-i\phi}$:

$$A^{(0)2} e^{i\phi} e^{-i\phi} \underline{e}^{(1)} \times \underline{e}^{(2)} = i A^{(0)2} e^{i\phi} e^{-i\phi} \underline{e}^{(3)*} \quad -(22)$$

where the electromagnetic phase is:

$$\phi = \omega t - \kappa Z \quad -(23)$$

to find the cyclic equations:

$$\underline{A}^{(1)} \times \underline{A}^{(2)} = i A^{(0)} \underline{A}^{(3)*} \quad -(24)$$

et cyclicum

where:

$$\begin{aligned} \underline{A}^{(1)} &= A^{(0)} \underline{e}^{(1)} e^{i\phi} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \\ \underline{A}^{(2)} &= A^{(0)} \underline{e}^{(2)} e^{-i\phi} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{-i\phi} \\ \underline{A}^{(3)} &= A^{(0)} \underline{e}^{(3)} = A^{(0)} \underline{k} \end{aligned} \quad -(25)$$

From Eqs. (20), (21) and (24):

$$\underline{\nabla} \times \underline{A}^{(1)} = \kappa \underline{A}^{(1)} = -\frac{i\kappa}{A^{(0)}} \underline{A}^{(3)} \times \underline{A}^{(1)} \quad -(26)$$

$$\underline{\nabla} \times \underline{A}^{(2)} = \kappa \underline{A}^{(2)} = -\frac{i\kappa}{A^{(0)}} \underline{A}^{(2)} \times \underline{A}^{(3)} \quad -(27)$$

which are Beltrami equations, QED. The Beltrami equations have been derived from first principles of geometry.

In the preceding paper UFT256 a particular case was used of these solutions:

$$\underline{\omega}^{(3)} = -\underline{k} / A^{(0)} \quad -(28)$$

i. e.

$$\underline{\nabla} \times \underline{v}^{(1)} = -i \underline{k} \times \underline{v}^{(1)} \quad -(29)$$

As described in Note 257(8) the Beltrami equations (26) and (27), together with the ECE field equations for the free field, produce Beltrami equations in $\underline{E}$ and $\underline{B}$ and cyclic equations in $\underline{B}$, the B Cyclic Theorem {1 - 10}.

Free field electromagnetism is therefore described by three Beltrami equations:

$$\begin{aligned} \underline{\nabla} \times \underline{A} &= \kappa \underline{A} \\ \underline{\nabla} \times \underline{B} &= \kappa \underline{B} \\ \underline{\nabla} \times \underline{E} &= \kappa \underline{E} \end{aligned} \quad -(30)$$

and these have a rich variety of solutions as reviewed by Reed {12}. Some of these solutions are described in detail in Notes 257(2) to 257(6). Note 257(3) reduces one of these general solutions to plane wave solutions and solutions of the Proca equation for photon mass. It derives the Helmholtz equation from the Beltrami equation, and defines the phase factor for non zero photon mass, implied by the longitudinal components of the Beltrami solutions of the free electromagnetic field. It shows that the general solutions are generalizations of the B Cyclic Theorem. Note 257(4) describes general solutions in terms of elliptically polarized plane waves, and Note 257(5) describes general solutions of the Beltrami equation in terms of Bessel functions, proving that the longitudinal component is given by the zero order Bessel function and always coexists with the transverse solution given by the first order Bessel function.

The foregoing analysis may be simplified by considering only one component out of the two conjugate components labelled (1) and (2). This procedure however loses information in general. By considering one component, Eq. (1) is simplified to:

$$\underline{\nabla} \cdot \underline{\omega} \times \underline{v} = \underline{v} \cdot \underline{\nabla} \times \underline{\omega} - \underline{\omega} \cdot \underline{\nabla} \times \underline{v} \quad -(31)$$

and the assumption of zero magnetic monopole leads to:

$$\underline{\nabla} \cdot \underline{\omega} \times \underline{v} = 0 \quad -(32)$$

which implies:

$$\underline{\omega} \cdot \underline{\nabla} \times \underline{v} = \underline{v} \cdot \underline{\nabla} \times \underline{\omega} \quad -(33)$$

Proceeding as in Note 257(7) leads to:

$$\underline{\omega} \cdot \underline{B} = \underline{A} \cdot \underline{\nabla} \times \underline{\omega} \quad -(34)$$

where
$$\underline{R}\,(\text{spin}) = \underline{\nabla} \times \underline{\omega} \quad -(35)$$

is the simplified form of the spin curvature. From Eqs. (33) and (34):

$$\underline{\omega} \cdot \underline{B} = \underline{A} \cdot \underline{\nabla} \times \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \times \underline{A} \quad -(36)$$

so:

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(37)$$

However, in ECE theory {1 - 10}:

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \quad -(38)$$

so Eqs. (37) and (38) imply:

$$\underline{\omega} \times \underline{A} = \underline{0}. \quad -(39)$$

In this simplified model the spin conenction vector is parallel to the vector potential. These results are consistent with {1 - 10}:

$$p^\mu = eA^\mu = \hbar \kappa^\mu = \hbar \omega^\mu \quad -(40)$$

from the minimal prescription. So in this simplified model:

$$\omega^{\mu} = (\omega_0, \underline{\omega}) = \frac{e}{\hbar} A^{\mu} = \frac{e}{\hbar} (A_0, \underline{A}). \quad (41)$$

The electric field strength E in volts m^{-1} is defined in this simplified model by:

$$\underline{E} = -\underline{\nabla} \phi - \frac{\partial \underline{A}}{\partial t} - c\omega_0 \underline{A} + \phi \underline{\omega} \quad (42)$$

where the scalar potential is:

$$\phi = cA_0 . \quad (43)$$

From Eqs. (42) and (41):

$$\underline{E} = - \underline{\nabla} \phi - \frac{\partial \underline{A}}{\partial t} \quad (44)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad (45)$$

which is the same as the structure given by Heaviside, the structure used in the standard model, but these equations have been derived from general relativity and Cartan geometry, whereas the Heaviside structure is empirical. The equations (31) to (45) are over simplified however because they are derived by consideration of only one out of two possible complex conjugates (1) and (2). Therefore they are derived using real algebra instead of complex algebra. They lose the B(3) field and also spin connection resonance, two of the major results of ECE theory. It is possible to simplify but care has to be taken not to lose information.

In the case of field matter interaction the electric field strength is replaced by the electric displacement $\underline{D}$, and the magnetic flux density $\underline{B}$ by the magnetic field strength $\underline{H}$:

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P} \; ; \; \underline{H} = \frac{1}{\mu_0} (\underline{B} - \underline{M}) \quad (46)$$

where $\underline{P}$ is the polarization, $\underline{M}$ the magnetization, ϵ_0 the vacuum permittivity and μ_0 the vacuum permeability. S. I. Units have been used. The four equations of electrodynamics for each index (1) or (2) are:

$$\underline{\nabla} \cdot \underline{D} = \rho, \quad \underline{\nabla} \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}, \qquad - (47)$$
$$\underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \times \underline{E} + \partial \underline{B} / \partial t = \underline{0}$$

where ρ is the charge density and $\underline{J}$ the current density.

The magnetic Beltrami equation:

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \qquad - (48)$$

is still valid, and is consistent with the absence of a magnetic monopole:

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad - (49)$$

because:

$$\frac{1}{\kappa} \underline{\nabla} \cdot \underline{\nabla} \times \underline{B} = 0. \qquad - (50)$$

So the magnetic Beltrami equation (48) is a consequence of the absence of a magnetic monopole. In other words the Beltrami solution is always a valid solution. From Eqs. (47) and (48):

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \qquad - (51)$$

and for magnetostatics or if the Maxwell displacement current is small:

$$\underline{B} = \frac{\mu_0}{\kappa} \underline{J} \qquad - (52)$$

and the magnetic field is directly proportional to current density. From Eq. (48):

$$\underline{\nabla} \times \underline{B} = \frac{\mu_0}{\kappa} \underline{\nabla} \times \underline{J} = \kappa \underline{B} \qquad - (53)$$

so:

$$\underline{B} = \frac{\mu_0}{\kappa^2} \underline{\nabla} \times \underline{J} \quad - (54)$$

Eqs. (52) and (54) imply that the current density must have the structure:

$$\underline{\nabla} \times \underline{J} = \kappa \underline{J} \quad - (55)$$

in order to produce the Beltrami equation (48) in magnetostatics. Eq. (52) suggests that the jet observed to emanate form the plane of a whirlpool galaxy may be due to the $\underline{B}(3)$ field generated by a $\underline{J}(3)$ current. This suggestion needs further analysis.

However, in field matter interaction the electric Beltrami equation:

$$\underline{\nabla} \times \underline{E} = \kappa \underline{E} \quad - (56)$$

is not valid because it is not consistent with the Coulomb law:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad - (57)$$

From Eqs. (56) and (57):

$$\frac{1}{\kappa} \underline{\nabla} \cdot \underline{\nabla} \times \underline{E} = \rho / \epsilon_0 \quad - (58)$$

which violates the vector identity:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{E} = 0 . \quad - (59)$$

The electric Beltrami equation is valid only for the free electromagnetic field.

Consider finally the four equations of the free electromagnetic field:

$$\underline{\nabla} \cdot \underline{B} = 0 , \quad \underline{\nabla} \cdot \underline{E} = 0 , \quad - (60)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad - (61)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \underline{0} \quad - (62)$$

for each index (1), (2) and (3) of the complex circular basis. It follows from Eqs. (61) and (62) that:

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla \times \mathbf{E} \quad -(63)$$

and:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}\nabla \times \mathbf{B}. \quad -(64)$$

The transverse plane wave solutions are:

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})e^{i\phi} \quad -(65)$$

and:

$$\mathbf{B} = \frac{B^{(0)}}{\sqrt{2}}(i\mathbf{i} + \mathbf{j})e^{i\phi} \quad -(66)$$

where

$$\phi = \omega t - \kappa Z \quad -(67)$$

and where ω is the angular frequency at instant t and κ the magnitude of the wave vector at Z. Complete details of the analysis are given in Note 257(6).

From vector analysis:

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \quad -(68)$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad -(69)$$

and for the free field the divergences vanish so we obtain the Helmholtz wave equations:

$$\nabla^2 \mathbf{B} + \kappa^2 \mathbf{B} = \mathbf{0} \quad -(70)$$

and:

$$\nabla^2 \mathbf{E} + \kappa^2 \mathbf{E} = \mathbf{0} \quad -(71)$$

These are the Trkalian equations {12}:

$$\nabla \times (\nabla \times \mathbf{B}) = \kappa \nabla \times \mathbf{B} = \kappa^2 \mathbf{B} \quad -(72)$$

and:

$$\nabla \times (\nabla \times \mathbf{E}) = \kappa^2 \mathbf{E} \quad -(73)$$

So solutions of the Beltrami equations are also solutions of the Helmholtz wave equations.

From Eqs. (63), (61) and (72):

$$-\nabla^2 \mathbf{B} - \frac{\kappa}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = \mathbf{0} \quad -(74)$$

which is the d'Alembert equation:

$$\Box \mathbf{B} = \mathbf{0} \quad -(75)$$

For finite photon mass, implied by the longitudinal solutions of the free electromagnetic field:

$$\hbar^2 \omega^2 = c^2 \hbar^2 \kappa^2 + m_0^2 c^4 \quad -(76)$$

in which case:

$$\left(\Box + \left(\frac{m_0 c}{\hbar}\right)^2\right)\mathbf{B} = \mathbf{0} \quad -(77)$$

which is the Proca equation. This was first derived in ECE theory from the tetrad postulate of Cartan geometry {1 - 10}.

From Eqs. (63) and (64):

$$\frac{\partial^2}{\partial t^2} \nabla \times \mathbf{B} = -\omega^2 \nabla \times \mathbf{B} \quad -(78)$$

and:
$$\frac{\partial^2}{\partial t^2} \underline{\nabla} \times \underline{E} = -\omega^2 \underline{\nabla} \times \underline{E} \quad -(79)$$

In general:
$$\frac{\partial^2}{\partial t^2} e^{i\phi} = -\omega^2 e^{i\phi} \quad -(80)$$

and:
$$e^{i\phi} = e^{i\omega t} e^{-i\kappa Z} \quad -(81)$$

so the general solution of the Beltrami equation:

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad -(82)$$

will also be a general solution of the equations (60) to (62) if the Beltrami solution is multiplied by the phase factor $\exp(i\omega t)$.

3. NUMERICAL AND GRAPHICAL ANALYSIS OF SOLUTIONS OF THE BELTRAMI EQUATION.

Section by Dr. Horst Eckardt

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Derivation of the Beltrami equation from Cartan geometry

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3 Numerical and graphical analysis of solutions of the Beltrami equation

In this section we inspect some solutions of the Beltrami equation and try to make a connection to physics. Kephart [13] introduces a quite general solution which might give some insight into the nature of Beltrami fields. He defines a vector field $\mathbf{v}$ by

$$\mathbf{v} = \begin{bmatrix} T_3 \cos(T\,Y) + T_2 \sin(T\,Z) \\ T_3 \sin(T\,X) + T_1 \cos(T\,Z) \\ T_2 \cos(T\,X) + T_1 \sin(T\,Y) \end{bmatrix} \tag{83}$$

where X, Y, Z are space coordinates and T, T_1, T_2, T_3 are constants. The curl of this field is

$$\boldsymbol{\nabla} \times \mathbf{v} = T \begin{bmatrix} T_1 (\cos(T\,Y) + \sin(T\,Z)) \\ T_2 (\sin(T\,X) + \cos(T\,Z)) \\ T_3 (\cos(T\,X) + \sin(T\,Y)) \end{bmatrix}, \tag{84}$$

so this is a Beltrami field with

$$\boldsymbol{\nabla} \times \mathbf{v} = T\mathbf{v} \tag{85}$$

if $T_1 = T_2 = T_3$. Projections of this field are plotted in Figs. 1 and 2 for the XY plane and XZ plane, respectively. One can see that the field oscillates vortex-like in all three space dimensions. This becomes also obvious from Fig. 3 where three field planes for different Z are graphed in a 3D plot.

According to Kephart [13] a general solution can be found by defining derivatives of functions

$$v_X = \frac{\partial \phi(X, Y, t)}{\partial Y}, \tag{86}$$

$$v_Y = -\frac{\partial \phi(X, Y, t)}{\partial X}, \tag{87}$$

$$v_Z = w(X, Y, Z, t). \tag{88}$$

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Equating the components with the curl of $\mathbf{v}$ gives the differential equations

$$\frac{\partial \phi}{\partial Y} = \frac{\partial w}{\partial Y}, \tag{89}$$

$$\frac{\partial \phi}{\partial X} = \frac{\partial w}{\partial X}, \tag{90}$$

$$w = -\left(\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2}\right). \tag{91}$$

There are some typos in the paper of Kephart, this result has been obtained by computer algebra. Using the example

$$\phi = A \ \cos(k_1 X) \sin(k_2 Y) \tag{92}$$

gives

$$w = A(k_1^2 + k_2^2) \cos(k_1 X) \sin(k_2 Y) \tag{93}$$

and the eigenvalues of curl v are

$$k_1^2 + k_2^2, \tag{94}$$

$$k_1^2 + k_2^2, \tag{95}$$

$$1. \tag{96}$$

Since all three factors (or functions in general) have to be equal for the Beltrami condition, we have an additional constraint

$$k_1^2 + k_2^2 = 1 \tag{97}$$

for (92) to be the generator function of a Beltrami field. Reed [12] has shown that the most general Beltrami field can be described by

$$\mathbf{v} = \kappa \ \boldsymbol{\nabla} \times (\psi \mathbf{a}) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\psi \mathbf{a}) \tag{98}$$

where ψ is an arbitrary function, κ is a constant and $\mathbf{a}$ is a constant vector. We present two examples. First we define

$$\psi = \frac{1}{L^3} XYZ \tag{99}$$

with

$$\mathbf{a} = [0, 0, 1]. \tag{100}$$

The resulting field (98) is graphed in Fig. 4. The field has only XY components and describes a hyperbolic vortex. Nevertheless the divergence is zero. A second example is

$$\psi = \sin(\kappa X) \ \sin(\kappa Y) \ \cos(\kappa Z) \tag{101}$$

which has a more complicated structure (Fig. 5). The projection of several Z levels on the XY plane is shown in Fig. 6. One can see that the vectors rotate in Z direction.

Another prominent example, with relation to physics, is defined by Bessel functions J_0 and J_1 in cylindrical coordinates:

$$\mathbf{v} = \begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix} = \begin{bmatrix} 0 \\ J_1(kr) \\ J_0(kr) \end{bmatrix}. \tag{102}$$

From Fig. 7 it can be seen that the vector field changes from transversal to longitudinal when approaching the Z axis. However, there are always longitudinal components in certain distances as well. This is obvious from Fig. 8 where the decomposition into both components is plotted. According to the oscillating nature of the Bessel functions, the field changes periodically from transversal to longitudinal with increasing radius. The flow of test particles in the field (if it is assumed to be hydrodynamic) can best be seen from a streamline picture (Fig. 9). At the center the flow is fast and longitudinal while at the periphery the flow is circling and has only a small Z component. Looking at this picture from the top (Fig. 10) shows up some similarity with galaxies with a jet of masses in the center. This looks like a realization of a cosmic B(3) field.

There is another remarkable similarity with technical developments of Tesla. Fig. 9 looks like a Tesla transformer and the transversal parts in Fig. 8 resemble the current distribution of a Tesla flat coil. The longitudinal part in the middle corresponds to the current goint to the sphere in a Tesla transmitter, see for example a patent of Tesla [14]. As worked out in section 2 of this paper, we have indeed a method for generating electromagnetic Beltrami fields with longitudinal field components. According to Eq.(55) the current density for producing a Beltrami field is a Beltrami field itself:

$$\boldsymbol{\nabla} \times \mathbf{J} = \kappa \mathbf{J}. \tag{103}$$

Therefore it should be possible to transmit free Beltrami fields by constructing a transmitter having a current distribution as for example shown in Fig. 9. A flat coil is only a rough approximation since therein the current density is constant and not depenent from the radius. An improvement could be to use concentric conducting rings with differing currents. At the center a dipole-like structure perpendicular to the rings could be imagined. The spatial dimensions are determined by the wave number which enters Eq.(103) by the relation $\kappa = 2\pi/\lambda$. The wavelenght λ is defined by the frequency which should be in the radio frequency range, otherwise the Maxwell displacement current of Eq.(51) has to be taken into account.

References

[1] [13] T. W. Kephart, Generalized Helicity Conservation and Beltrami Fields, http://www.phys.sinica.edu.tw/~heptheory/2013_RS_talk_files/2013.0724.BeltramiAcademiaSinica24July2013f.pdf and arXiv:1305.4927

[2] [14] http://intalek.com/Index/Projects/SmartLINK/00649621.PDF

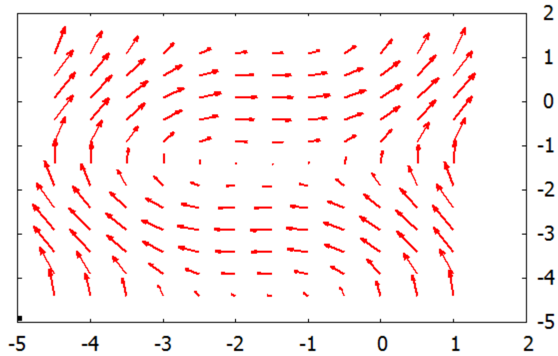

Figure 1: Beltrami field of Eq.(83), XY plane.

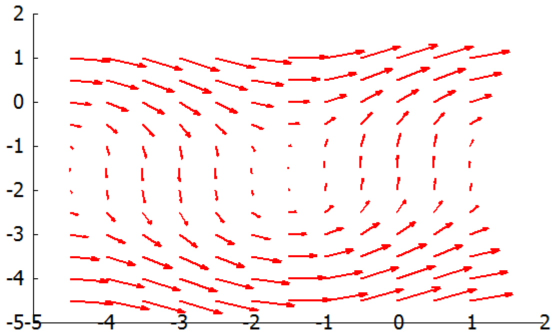

Figure 2: Beltrami field of Eq.(83), XZ plane.

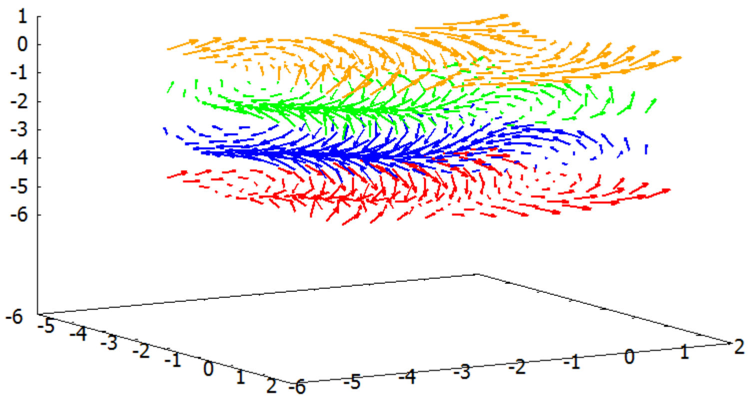

Figure 3: Beltrami field of Eq.(83) in three Z planes.

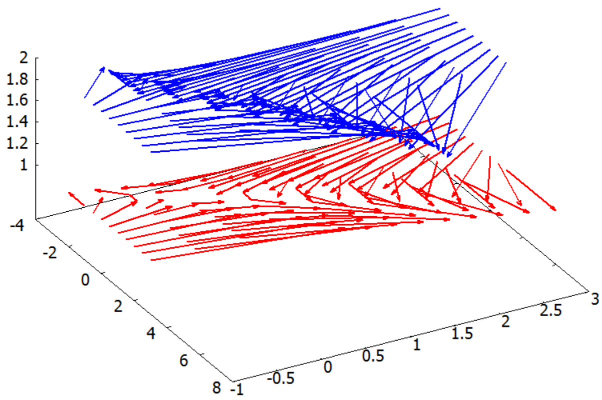

Figure 4: General Beltrami field of Eqs.(98,99,100).

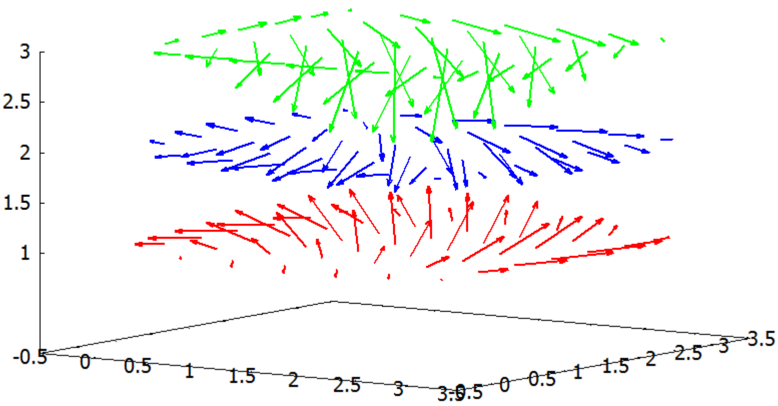

Figure 5: General Beltrami field of Eqs.(98,100,101).

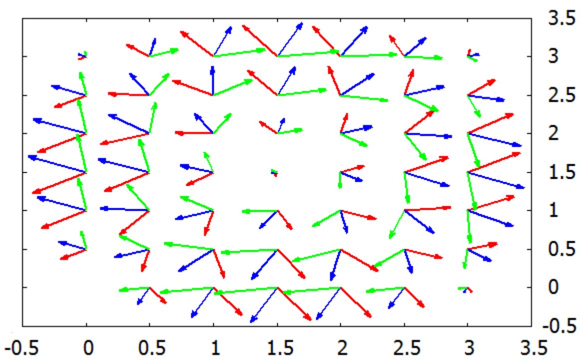

Figure 6: General Beltrami field of Eqs.(98,100,101), projection in XY plane.

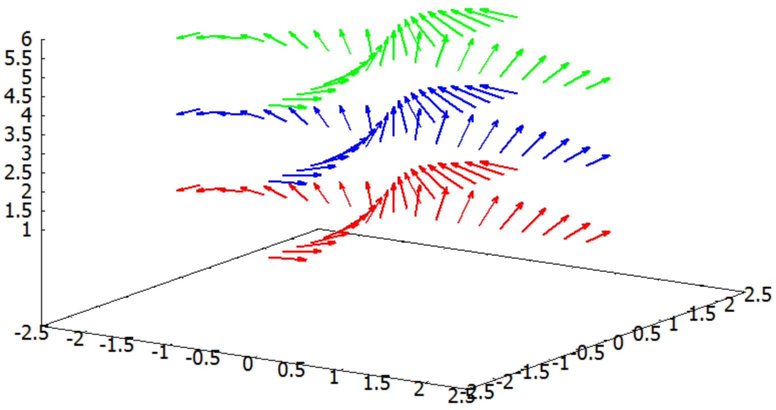

Figure 7: Beltrami field of Bessel functions.

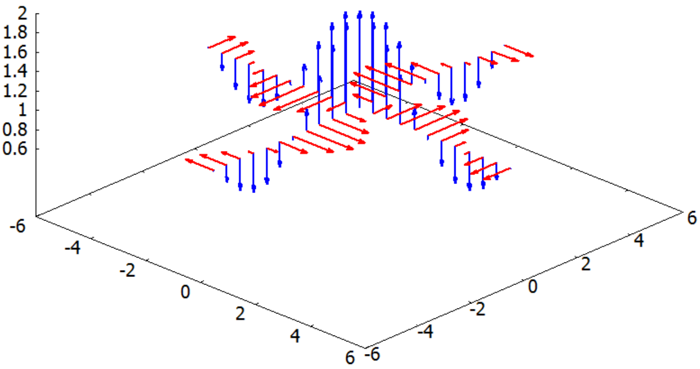

Figure 8: Beltrami field of Bessel functions, decomposition into transversal and longitudinal vectors.

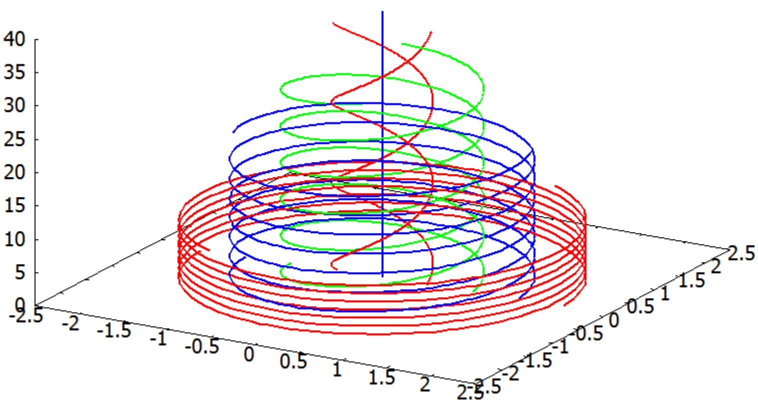

Figure 9: Beltrami field of Bessel functions, streamlines.

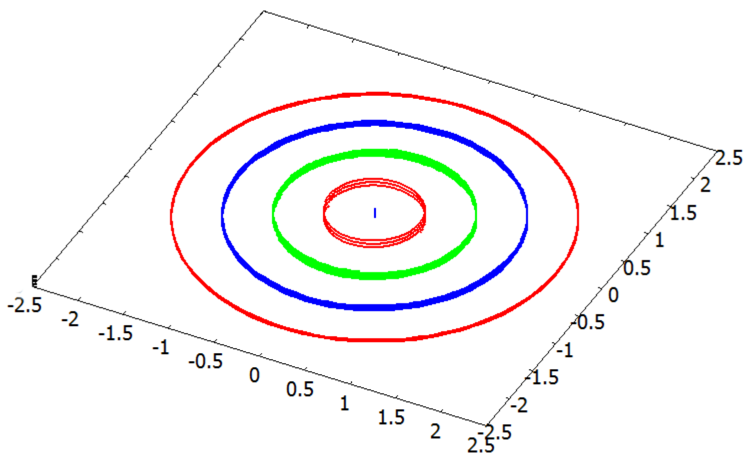

Figure 10: Beltrami field of Bessel functions, streamlines, top view.