98(3): Background Notes on the Noether Theorem

(Lewis H. Ryder, "Quantum Field Theory" (CUP, 1996))

The Noether Theorem is derived from a Lagrangian method described in Ryder's chapter three. The energy-momentum tensor $\theta^{\mu}{}_{\nu}$ is defined by Ryder's equation (3.2

$$\theta^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}{}_{\nu} \mathcal{L} \quad -(1)$$

It is now assumed that the action S is invariant under a group of transformations on x^μ and ϕ which for infinitesimal transformations take the form:

$$\Delta x^{\mu} = X^{\mu}{}_{\nu} \delta\omega^{\nu}, \quad \Delta \phi = \Phi_\mu \delta\omega^{\mu} \quad -(2)$$

characterised by the infinitesimal parameter $\delta\omega^{\nu}$. Here $X^{\mu}{}_{\nu}$ is a matrix and Φ_μ is a set of numbers. Here ϕ is a scalar field, $\delta^{\mu}{}_{\nu}$ is the Kronecker delta, and $\mathcal{L}$ the Lagrangian. If ν is a single index then Φ_μ is a four-parameter group of transformations. However ν can be a double index, as needed to generate Lorentz transformations. It is now assumed that the action is invariant:

$$\delta S = 0, \quad -(3)$$

which implies that:

$$\int_{\delta R} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \theta^{\mu}{}_{\kappa} X^{\kappa}{}_{\mu} \right) \delta\omega^{\nu} d\sigma_\mu = 0$$

$-(4)$

2)

Since $\delta\omega^{\sim}$ is arbitrary:

$$\int_{\partial R} J^{\mu}_{\sim}\, d\sigma_{\mu} = 0 \quad -(5)$$

where

$$J^{\mu}_{\sim} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \Phi_{\sim} - \theta^{\mu}{}_{\kappa} X^{\kappa}_{\sim} \quad -(6)$$

From Gauss' Theorem:

$$\int_{R} \partial_{\mu} J^{\mu}_{\sim}\, d^4x = 0 \quad -(7)$$

and since R is arbitrary:

$$\boxed{\partial_{\mu} J^{\mu}_{\sim} = 0} \quad -(8)$$

The current $J^{\mu}_{\sim}$ is conserved (divergentless), and the ordinary derivative ∂_{μ} is used because the background spacetime is flat with zero connection. The existence of $J^{\mu}_{\sim}$ follows from the invariance of action under the transformations (2). The conserved or time-independent charge is now defined as:

$$Q_{\sim} = \int_{\sigma} J^{\mu}_{\sim}\, d\sigma_{\mu} \quad -(9)$$

where σ_{μ} is a space-like hypersurface in Minkowski spacetime. If this surface is defined by:

3)

$$t = \text{constant} \quad -(10)$$

Then

$$Q_\sim = \int_V J^0_\sim \, d^3x \quad -(11)$$

where the integral is taken over the three volume V. The conservation of $Q_\sim$ follows by integrating (8) over

$$\int_V \partial_0 J^0_\sim \, d^3x + \int_V \partial_i J^i_\sim \, d^3x = 0 \quad -(12)$$

The second term is transformed into a surface integral by the 3-D Gauss Theorem and vanishes, so we obtain <u>Noether's Theorem</u>.

$$\boxed{\frac{d}{dt}\int J^0_\sim \, d^3x = \frac{dQ_\sim}{dt} = 0} \quad -(13)$$

<u>Translational Invariance</u>

This is the translation of the origin of space and time, in which case:

$$\Delta x^\mu = \epsilon^\mu, \quad \Delta\phi = 0 \quad -(14)$$

i.e.

$$X^\mu_\sim = \delta^\mu_\sim, \quad \Phi_\mu = 0 \quad -(15)$$

From eqs. (15) and (6):

4) The corresponding conservation law is

$$\frac{d}{dt}\int \theta^{0}_{\sim}\, d^3x = 0 \qquad -(17)$$

It can be shown that:

$$p_{\sim} = \int \theta^{0}_{\sim}\, d^3x \qquad -(18)$$

is the 4-momentum or energy-momentum of the scalar field ϕ.

$$-(19)$$

<u>Proof</u>

$$\int \theta^{0}_{0}\, d^3x = \int \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L} \right) d^3x$$

$$= \int \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right) d^3x \qquad -(20)$$

In point particle dynamics the relation between the Hamiltonian and Lagrangian is:

$$H = \sum_i p_i \dot{q}_i - L \qquad -(21)$$

and the momentum is defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad -(22)$$

The right hand side of eq. (20) is therefore the energy of the field. Thus $\int \theta^{0}_{i}\, d^3x$ is a momentum because $\partial \phi / \partial x^{\mu}$ is a 4-vector under Lorentz transformations

5) The canonical energy-momentum tensor is defined as:

$$\partial_\mu T^{\mu\nu} = 0 \qquad -(23)$$

in flat or Minkowski spacetime. In curved spacetime

$$D_\mu T^{\mu\nu} = 0 \qquad -(24)$$

and the Einstein field equation was derived by:

$$D_\mu G^{\mu\nu} = k \underbrace{D_\mu T^{\mu\nu}}_{=0} \qquad -(25)$$

where $G^{\mu\nu}$ is the Einstein tensor and k is the Einstein constant. Here:

$$D_\mu G^{\mu\nu} = 0 \qquad -(26)$$

is the second Bianchi identity.

Rotational Invariance

In this case the action is invariant under spatial rotations:

$$\delta x^i = \epsilon^{ij} x^j, \quad \epsilon^{ij} = -\epsilon^{ji} \ (i, j = 1, 2, 3) \qquad -(27)$$

$$\Delta \phi = 0 \qquad -(28)$$

The rotation group is a sub-group of the Lorentz group, so:

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \qquad -(29)$$

This is written as:

6) $\delta x^\mu = X^\mu{}_{\rho\sigma}\,\epsilon^{\rho\sigma}, \quad X^\mu{}_{\rho\sigma} = \frac{1}{2}\left(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho\right)$ — (30)

This is the same form as eq. (2) but the index ν in eq. (2) becomes the double index $\rho\sigma$ of eq. (30). Now use eq. (6) with:

$$\Phi = 0, \quad \theta^{\mu\nu} \Rightarrow T^{\mu\nu} \quad — (31)$$

to find the three index canonical Noether current:

$$J^{\mu\rho\sigma} = -T^\mu{}_\kappa X^{\kappa\rho\sigma} \quad — (31)$$

$$J^{\mu\rho\sigma} = -\frac{1}{2}\left(T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho\right) \quad — (32)$$

or

$$\boxed{J^\mu{}_{\rho\sigma} = -\frac{1}{2}\left(T^\mu{}_\rho x_\sigma - T^\mu{}_\sigma x_\rho\right)} \quad — (33)$$

In Minkowski spacetime:

$$\partial_\mu J^\mu{}_{\rho\sigma} = 0 \quad — (34)$$

and for non-zero connections:

$$\boxed{D_\mu J^\mu{}_{\rho\sigma} = 0} \quad — (35)$$

In analogy with eq. (24) it is seen that $J^\mu{}_{\rho\sigma}$ is the canonical angular

) momentum density tensor. In analogy with the Einstein Hilbert field equation (25) there exists a geometrical quantity to which $J^{\mu}{}_{\rho\sigma}$ is proportional. This is the three-index Cartan torsion tensor:

$$T^{\mu}{}_{\rho\sigma} = \Gamma^{\mu}{}_{\rho\sigma} - \Gamma^{\mu}{}_{\sigma\rho} \quad -(36)$$

As in notes 98(2):

$$J^{\mu}{}_{\rho\sigma} = \frac{eE^{(0)}}{c} T^{\mu}{}_{\rho\sigma}, \quad -(37)$$

which is the rotational analogue of the EH field equation:

$$G^{\mu\nu} = k T^{\mu\nu}. \quad -(38)$$

In the quantum theory:

$$eA^{(0)} = \hbar\kappa, \quad -(39)$$

$$E^{(0)} = cB^{(0)} = \kappa c A^{(0)} \quad -(40)$$

So

$$eE^{(0)} = \hbar\kappa^2 c \quad -(41)$$

The Cartan torsion therefore plays the role of the Einstein tensor $G^{\mu\nu}$ in spatial rotations as opposed to spatial translations.

8) The most fundamental energy-momentum density is therefore $T_{\mu\nu}$, and the most fundamental angular energy-momentum density is $J^{\kappa}{}_{\mu\nu}$. The four-momentum p_μ is the special case:

$$p_\mu = \int T^0{}_\mu \, d^3x \quad -(42)$$

and the angular momentum is the special case:

$$J^{\mu\nu} = \int \left(T^{0\mu} x^\nu - T^{0\nu} x^\mu \right) d^3x \quad -(43)$$

These special cases are defined by the case:

$$t = \text{constant} \quad -(44)$$

in the spacelike hypersurface σ_μ used to define the time independent:

$$Q_\nu = \int_\sigma J^\mu{}_\nu \, d\sigma_\mu \quad -(45)$$

where:

$$\partial_\mu J^\mu{}_\nu = 0. \quad -(46)$$

The Noether theorem is:

$$\frac{dQ_\nu}{dt} = 0, \quad -(47)$$

which the time independence means that momentum / energy and angular momentum energy are conserved, in the sense that they are time-independent. This is because of eq. (44). More generally however energy and momentum can change with time.

Therefore this gives a fundamental justification for defining the electromagnetic field as a rank ~~three~~ canonical tensor:

$$\boxed{F^{\mu}_{\rho\sigma} = \frac{c}{e\omega} J^{\mu}_{\rho\sigma} = \frac{E^{(0)}}{\omega} \mathbf{T}^{\mu}_{\rho\sigma} = A^{(0)} T^{\mu}_{\rho\sigma}} \quad -(48)$$

where

$$E^{(0)} = kcA^{(0)} = \omega A^{(0)}. \quad -(49)$$

In the notation of differential geometry:

$$F^{a}_{\mu\nu} = A^{(0)} T^{a}_{\mu\nu} \quad -(50)$$

which is the ECE Ansatz, Q.E.D.