13(2): Comparison of Line Elements

The line element given by Carroll is:

$$ds^2 = m(t,r)\,dt^2 + n(t,r)\,dr^2 + r^2\,d\Omega^2 \quad — (1)$$

and the line element given by Crothers is:

$$ds^2 = B(R_c)\,dR_c^2 + R_c^2\,d\Omega^2 \quad — (2)$$

These are difficult to compare directly. Experimentally, however, the NASA Cassini result shows that the bending of light by gravity is well described by a form:

$$ds^2 = -\left(1+\frac{\mu}{r}\right)dt^2 + \left(1+\frac{\mu}{r}\right)^{-1}dr^2 + r^2\,d\Omega^2 \quad —(3)$$

The only thing that has to be done to take into account that there is no singularity is to emphasize that in eq (3), r cannot be identically zero.

From experimental data:

$$\mu = \frac{2MG}{rc^2} \quad — (4)$$

So:

$$ds^2 = -\left(1-\frac{2MG}{rc^2}\right)dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2\,d\Omega^2 \quad — (5)$$

More generally this is:

$$ds^2 = -\exp\left(\frac{2\Phi}{c^2}\right)dt^2 + \exp\left(-\frac{2\Phi}{c^2}\right)dr^2 + r^2\,d\Omega^2 \quad — (6)$$

) In eqs. (5) and (6) the experimental data show that r is the radial coordinate and M the mass that appears in Newton's inverse square law. The metric coefficients for this are:

$$g_{00} = -\left(1 - \frac{2GM}{c^2 r}\right), \quad g_{11} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}; \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta \qquad -(7)$$

<u>The Christoffel Symbols</u>

These are defined by:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \qquad -(8)$$

The relevant Christoffel symbol to be checked is: $(\sigma = 1, \mu = 0, \nu = 0)$

$$\Gamma^{1}_{00} = \frac{1}{2} g^{1\rho}\left(\partial_0 g_{0\rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}\right) \qquad -(9)$$

$$= -\frac{1}{2} g^{11} \partial_1 g_{00}$$

$$\Gamma^{1}_{00} = \frac{2GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \qquad -(10)$$

Therefore:

$$\boxed{\Gamma^{1}_{00} = \frac{2GM}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}} \qquad -(11)$$

using the method of paper 91.

3)

If:

$$2GM << c^2 r \quad -(12)$$

then:

$$S^1_{00} \rightarrow \frac{2GM}{r^2} \quad -(13)$$

From eq. (11):

$$S^1_{00} \sim \frac{2GM}{r^2}\left(1 + \frac{2GM}{c^2 r}\right) \quad -(14)$$

The inverse cube term in this expression gives rise to the relativistic correction. For example, if:

$$g := -\frac{1}{2} S^1_{00} = -\frac{GM}{r^2}\left(1 + \frac{2GM}{c^2 r}\right) \quad -(15)$$

these are corrections to the Newtonian:

$$g = -\frac{GM}{r^2}, \quad F = mg \quad -(16)$$

Therefore the force from eq (15) is:

$$\boxed{F = mg = -\frac{GmM}{r^2} - \frac{2G^2 mM^2}{c^2 r^3}} \quad -(17)$$

As shown by Marion and Thornton on page 286, the orbit due to this is a <u>precessing ellipse</u>. This qualitatively accounts for advances in the perihelion.