87(3): Derivation of the SCR Poisson Equation from the Schrodinger Equation of Hydrogen & Vice-Versa

The SCR Poisson equation of paper 63 is:

$$\underline{\nabla} \cdot \left(\left(\underline{\nabla} + \underline{\omega} \right) \phi \right) = -\frac{\rho}{\epsilon_0} \quad -(1)$$

where the spin connection is:

$$\omega_r = -\frac{1}{r} \quad -(2)$$

eq. (1) becomes:

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{1}{r^2} \phi = -\frac{\rho}{\epsilon_0} \quad -(3)$$

where ϕ is the scalar potential in volts and ρ is the charge density in $C m^{-3}$. The units of the vacuum permittivity ϵ_0 are $J^{-1} C^2 m^{-1}$.

In this note it is shown that eq. (3) is a special case of the SE of hydrogen.

We first define the four potential and four current density:

$$A_\mu = \left(\phi, -c \underline{A} \right), \quad j_\mu = \left(c\rho, -\underline{J} \right) \quad -(4)$$

2)

The units of $\underline{A}$ are $J s C^{-1} m^{-1}$ and the units of $\underline{J}$ are $C m^{-2} s^{-1}$. It is found that:

$$\rho = \frac{\epsilon_0}{Ar} \phi \quad -(5)$$

where Ar has the units of an area (m^2). It is well known from classical electrodynamics that A_μ is proportional to j_μ, and this is also shown by ECE theory. So eq. (3) is:

$$\boxed{\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{1}{r^2}\phi = -\frac{\phi}{Ar}} \quad -(6)$$

The Schrödinger equation of H is: $\quad -(7)$

$$\boxed{\frac{d^2P}{dr^2} + \frac{2\alpha\bar{\nu}_c}{r}P - \frac{\ell(\ell+1)}{r^2}P = -\frac{2mE}{\hbar^2}P}$$

where:

$$\left.\begin{array}{l} \alpha = \dfrac{e^2}{4\pi\epsilon_0\hbar c}, \quad \bar{\nu}_c = \dfrac{mc}{\hbar}, \\ P = rR \end{array}\right\} -(8)$$

where R is the radial wave function.

3) By units analysis:

$$A r_H = \frac{2mE}{\hbar^2} \quad — (9)$$

has the units of area. So:

$$\boxed{\frac{d^2 P}{dr^2} + \frac{2\alpha \tilde{r}_c}{r} P - \frac{\ell(\ell+1)}{r^2} P = -\frac{P}{A r_H}} \quad — (10)$$

It is seen that eqs. (6) and (10) are similar in structure. There is therefore an analogy between the scalar potential ϕ and the function $P = rR$ of the Schrödinger equation.

We now write eqs. (10) and (3) in the same form using the results of pages 63 and 85, 86. Eq (10) is:

$$-\frac{\hbar^2}{2m}\frac{d^2 P}{dr^2} = (E + \bar{V}_{eff}) P, \quad — (11)$$

i.e

$$\boxed{\frac{d^2 P}{dr^2} + \frac{2m}{\hbar^2} \bar{V}_{eff} P = -E \cdot \frac{2m}{\hbar^2} P} \quad — (12)$$

where:

$$\bar{V}_{eff} = \frac{e^2}{4\pi\epsilon_0 (r + r(vac))} - \frac{\ell(\ell+1)\hbar^2}{2m(r + r(vac))^2} \quad — (13)$$

4) and eq. (3) was transformed in paper 63 to the same structure as eq. (12) using a change of variable. In paper 63 it was shown that the general structure of eq. (3) is:

$$\frac{\partial^2 \phi}{\partial r^2} + \left(\frac{2}{r} + \omega_r\right)\frac{\partial \phi}{\partial r} + \frac{\phi}{r^2}\left(2r\omega_r + r^2 \frac{\partial \omega_r}{\partial r}\right)$$

$$= -\frac{\rho}{\epsilon_0} \quad -(14)$$

Using eq. (5), eq. (14) is:

$$\frac{\partial^2 \phi}{\partial r^2} + \left(\frac{2}{r} + \omega_r\right)\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\left(2r\omega_r + r^2 \frac{\partial \omega_r}{\partial r}\right)\phi = -\frac{\phi}{Ar}$$

$$-(15)$$

Comparing eqs. (10) and (15) term by term, the two equations are the same if, mathematically:

$$\frac{d^2 P}{dr^2} = \frac{d^2 \phi}{dr^2} \quad -(16)$$

$$\left(\frac{2}{r} + \omega_r\right)\frac{d\phi}{dr} = \frac{2\alpha \tilde{\nu}_c P}{r} \quad -(17)$$

$$\left(2r\omega_r + r^2 \frac{\partial \omega_r}{\partial r}\right)\phi = -\ell(\ell+1)P \quad -(18)$$

$$\frac{P}{Ar_H} = \frac{\phi}{Ar} \quad -(19)$$

 and a constant of proportionality has to be introduced to allow for the fact that the units of $\rho = r R$ and of ϕ are different.

Discussion

By solving eqs. (17) and (18) simultaneously a spin connection may be found that produces the Schrödinger equation structure from eq. (15). However, it may be easier to transform eq. (3) into the same structure as eq. (12) using a change of variable. In paper 13, eq. (3) was considered assuming:

$$\rho = \rho(0) \cos(\kappa_r r) \quad -(20)$$

The change of variable was:

$$\kappa_r r = \exp(i \kappa_r R) \quad -(21)$$

so eq. (3) becomes:

$$\boxed{\frac{d^2 \phi}{dR^2} + \kappa_r^2 \phi = \frac{\rho(0)}{\epsilon_0} \mathrm{Real}\left(e^{2 i \kappa_r R} \cos\left(e^{i \kappa_r R}\right)\right) \quad -(22)}$$

5)

Eq. (22) was shown to produce three resonances in the potential ϕ. We now write eq. (22) as:

$$\boxed{\frac{d^2\phi}{dR^2} + \kappa_r^2\phi = -\frac{\phi}{Ar}} \quad — (23)$$

using eq. (5). This determines the area Ar.

Eq. (12) is:

$$\boxed{\frac{d^2 P}{dr^2} + \kappa_H^2 P = -\frac{P}{Ar_H}} \quad — (24)$$

where:

$$\kappa_H^2 = \frac{2m V_{eff}}{\hbar^2} \quad — (25)$$

and

$$Ar_H = \frac{2mE}{\hbar^2} \quad — (26)$$

So we may relate κ_H to κ_r and Ar_H to Ar. This is because the classical theory of resonance is a precursor to the quantum theory, as described by Marion and Thornton.

Resonances occur directly from eqs. (23) & (24)