1) 86(2): Plane Wave Solutions of the Dirac Equation

We first investigate the fundamental origin of the Klein-Gordon and Dirac equations. The origin is in:

$$\underline{p} = \gamma m \underline{v} \,, \quad E = \gamma E_0 \qquad - (1)$$

where:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad E_0 = mc^2 \qquad - (2)$$

This is the fundamental idea of classical special relativity (Einstein 1905), where $\underline{p}$ is the relativistic momentum, E is the relativistic energy, E_0 is the rest energy. If we take the equation:

$$\boxed{E = \gamma E_0} \qquad - (3)$$

the energy is taken as positive, as originally inferred since Newton's time. It cannot be proven mathematically from eq. (3) that E must be negative. On the contrary, from observation, E is positive. Total energy is positive and always conserved.

We now square eq. (3):

$$E^2 = E_0^2 \Big/ \left(1 - \frac{v^2}{c^2}\right) \qquad - (4)$$

So:

$$E^2 = \frac{v^2 E^2}{c^2} + E_0^2 \qquad - (5)$$

$$\boxed{E^2 = c^2 p^2 + E_0^2} \qquad - (6)$$

because

$$E = \frac{pc^2}{v} = \frac{\gamma m v c^2}{v} = \gamma m c^2 = \gamma E_0. \qquad - (7)$$

2) It is obvious that eq. (3) is always the positive root of eq. (6). So the energy in eq. (6) is always positive, <u>Q.E.D.</u> Both the Klein Gordon and Dirac equations come from eq. (3). The Klein Gordon equation is obtained directly from eq. (6) using:

$$p^\mu = i\hbar \partial^\mu \quad -(8)$$

The Dirac equation was used to make sure that probability current-density is positive. The equation (8) is used in both.

The Dirac equation is:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right) \psi = 0 \quad -(9)$$

In which:

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad -(10)$$

are Dirac matrices:

$$\left.\begin{aligned} \gamma^0 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\ \gamma^2 &= \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \end{aligned}\right\} \quad -(11)$$

Here:

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3$$

3) where:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \right) \quad -(13)$$

If we consider propagation in Z, eqn. (9) is:

$$i\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} + ic \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial Z} \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} = \frac{mc^2}{\hbar} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} \quad -(14)$$

which is:

$$i\frac{\partial}{\partial t}\left(\psi_1^L + \psi_2^L + \psi_1^R + \psi_2^R\right) + ic\frac{\partial}{\partial Z}\left(\psi_2^L - \psi_1^L + \psi_1^R - \psi_2^R\right) = \frac{mc^2}{\hbar}\left(\psi_1^R + \psi_2^R + \psi_1^L + \psi_2^L\right) \quad -(15)$$

whose solutions are:

$$\psi = \psi_1^L + \psi_2^L + \psi_1^R + \psi_2^R = \exp\left(-\frac{imc^2}{2\hbar}\left(t - \frac{Z}{c}\right)\right) \quad -(16)$$

$$\psi^* = \psi_2^L - \psi_1^L + \psi_1^R - \psi_2^R = \exp\left(\frac{imc^2}{2\hbar}\left(t - \frac{Z}{c}\right)\right) \quad -(17)$$

$$\psi\psi^* = 1 \quad -(18)$$

<u>It is noted that $E_0 = mc^2$ is always positive,</u> and this is clear from the original equation (3). Therefore eq. (15) is:

$$\boxed{i\hbar\left(\frac{\partial\psi}{\partial t} + c\frac{\partial\psi^*}{\partial Z}\right) = mc^2\psi = E_0\psi} \quad -(19)$$

where:

$$\langle E_0 \rangle = \psi E_0 \psi^* = E_0. \quad -(20)$$

<u>NO INDICATION OF NEGATIVE ENERGY</u>