85(8): Some Comments on Dirac Algebra

The Dirac equation for the free electron is:

$$(\gamma^\mu p_\mu - mc)\psi = 0 \quad — (1)$$

which becomes the operator equation:

$$(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0 \quad — (2)$$

using

$$p_\mu = i\partial_\mu \quad — (3)$$

Here

$$p_\mu = (p_0, -\underline{p}), \quad p_0 = \frac{E_n}{c}, \quad — (4$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right) \quad — (5)$$

The Dirac matrix is defined as

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad — (6)$$

where:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad — ($$

where the Pauli matrices are:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Dirac matrices obey the equation:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}} \quad — ($$

where

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad — (10)$$

is the Minkowski metric.

Thus:

$$\gamma^0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } (\gamma^0)^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(11)$$

i.e.

$$\{\gamma^0, \gamma^0\} = 2g^{00} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(12)$$

because:

$$g^{00} = 1. \quad -(13)$$

Similarly:

$$(\gamma^1)^2 = \begin{bmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{bmatrix} \quad -(14)$$

$$= \begin{bmatrix} -\sigma^1\sigma^1 & 0 \\ 0 & -\sigma^1\sigma^1 \end{bmatrix} \quad -(15)$$

$$= -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(16)$$

In shorthand notation:

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1. \quad -(17)$$

For any off-diagonal element:

$$\left.\begin{aligned} \gamma^\mu \gamma^\nu &= -\gamma^\nu \gamma^\mu \quad (\nu \neq \mu) \\ \text{and} \quad \{\gamma^\mu, \gamma^\nu\} &= 0 \end{aligned}\right\} \quad -(18)$$

For example:

$$\left.\begin{aligned}\gamma^1\gamma^2 &= \begin{bmatrix} -\sigma^1\sigma^2 & 0 \\ 0 & -\sigma^1\sigma^2 \end{bmatrix} \\ \gamma^2\gamma^1 &= \begin{bmatrix} -\sigma^2\sigma^1 & 0 \\ 0 & -\sigma^2\sigma^1 \end{bmatrix}\end{aligned}\right\} \quad -(19)$$

and
$$\sigma^1\sigma^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\sigma^3$$
$$\sigma^2\sigma^1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = -i\sigma^3.$$

Thus
$$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2}\right] = i\frac{\sigma^3}{2}. \quad -(20)$$

et cyclicum

and
$$\left\{\frac{\sigma^i}{2}, \frac{\sigma^i}{2}\right\} = \frac{\sigma^0}{2}, \quad -(21)$$
$$i = 1, 2, 3$$

where
$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad -(22)$$

Here
$$\left.\begin{aligned}\{\ ,\ \} &= \text{anticommutator} \\ [\ ,\ ] &= \text{commutator}\end{aligned}\right\} -(23).$$

The factor 2 in eqns (20) and (21) means $SU(2)$ symmetry. Similarly, eq. (9) is:
$$\left\{\frac{\gamma^\mu}{2}, \frac{\gamma^\nu}{2}\right\} = \frac{g^{\mu\nu}}{2} \quad -(24)$$

The factor 2 means:
$$\boxed{g = 2} \quad -(25)$$

in the Dirac theory of the electron, eq. (1).

) From paper 18 we know that this should be:

$$g = 2\left(1 + \frac{\alpha}{4\pi}\right)^2 \quad - (26)$$

because the electron is jitterbugged by the vacuum. Here $\alpha / 4\pi$ can be considered as an average influence of the vacuum.

Eq (1) is expanded using:

$$\gamma^\mu p_\mu = \gamma^0 p_0 + \gamma^i \gamma_i \quad - (27)$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} p_0 + \begin{bmatrix} 0 & -\underline{\sigma} \\ \underline{\sigma} & 0 \end{bmatrix} \cdot \underline{p} \quad - (28)$$

The Dirac spinor ϕ is a column four-vector made up of two Pauli spinors, ϕ_L and ϕ_R, so eq. (1) is:

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} p_0 + \begin{bmatrix} 0 & -\underline{\sigma} \\ \underline{\sigma} & 0 \end{bmatrix} \cdot \underline{p} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} mc \right) \begin{bmatrix} \phi_L \\ \phi_R \end{bmatrix} = 0$$

$$- (29)$$

which represents the simultaneous equations:

$$\left(\begin{bmatrix} 0 & p_0 \\ p_0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\underline{\sigma} \cdot \underline{p} \\ \underline{\sigma} \cdot \underline{p} & 0 \end{bmatrix} - \begin{bmatrix} mc & 0 \\ 0 & mc \end{bmatrix} \right) \begin{bmatrix} \phi_L \\ \phi_R \end{bmatrix} = 0$$

$$- (30)$$

i.e.

$$\begin{bmatrix} 0 & p_0 \\ p_0 & 0 \end{bmatrix} \begin{bmatrix} \phi_L \\ \phi_R \end{bmatrix} + \begin{bmatrix} 0 & -\underline{\sigma} \cdot \underline{p} \\ \underline{\sigma} \cdot \underline{p} & 0 \end{bmatrix} \begin{bmatrix} \phi_L \\ \phi_R \end{bmatrix} = \begin{bmatrix} mc & 0 \\ 0 & mc \end{bmatrix} \begin{bmatrix} \phi_L \\ \phi_R \end{bmatrix}$$

i.e.

$$\boxed{\begin{aligned} p_0 \phi_R - \underline{\sigma} \cdot \underline{p}\, \phi_R &= mc\, \phi_L \\ p_0 \phi_L + \underline{\sigma} \cdot \underline{p}\, \phi_L &= mc\, \phi_R \end{aligned}} \quad - (31)$$

These are equations linking left and right spins:

$$\left.\begin{aligned} \phi_L &= \frac{1}{mc}\left(p_0 - \underline{\sigma} \cdot \underline{p}\right) \phi_R \\ \phi_R &= \frac{1}{mc}\left(p_0 + \underline{\sigma} \cdot \underline{p}\right) \phi_L \end{aligned}\right\} \quad -(32)$$

These in turn can be represented by:

$$\left.\begin{aligned} \phi_R(\underline{p}) &= \exp\left(\frac{1}{2}\underline{\sigma} \cdot \underline{\phi}\right) \phi_R(0) \\ \phi_L(\underline{p}) &= \exp\left(-\frac{1}{2}\underline{\sigma} \cdot \underline{\phi}\right) \phi_L(0) \end{aligned}\right\} \quad -(33)$$

These mean essentially that the spinor rotates through half the angle the vector rotates through. This is the origin of $g = 2$.

In paper 18, the influence of the vacuum is accounted for by using:

$$\gamma^\mu \rightarrow \gamma^\mu\left(1 + \frac{\alpha}{4\pi}\right) \quad -(34)$$

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\ &= 2\left(1 + \frac{\alpha}{4\pi}\right)^2 g^{\mu\nu} \end{aligned} \quad -(35)$$

So:

$$\boxed{g = 2\left(1 + \frac{\alpha}{4\pi}\right)^2} \quad -(36)$$

and equivalently:

$$p_\mu \rightarrow p_\mu\left(1 + \frac{\alpha}{4\pi}\right) = p_\mu + eA_\mu^{(vac)}$$

6) So the Dirac equation of the jitterbugging electron is:

$$\left(\gamma^{\mu} p_{\mu}\left(1+\frac{\alpha}{4\pi}\right)-mc\right)\psi = 0 \quad -(38)$$

and in H:

$$\left.\begin{aligned}&\left(\gamma^{\mu} p_{\mu}\left(1+\frac{\alpha}{4\pi}\right)-mc-\frac{\hbar\alpha}{r}\right)\psi = 0\\ \text{or}\quad &\left(i\gamma^{\mu}\left(1+\frac{\alpha}{4\pi}\right)\partial_{\mu}-\frac{mc}{\hbar}-\frac{\alpha}{r}\right)\psi = 0\end{aligned}\right\} -(39)$$