81(3): General Calculation of the Inverse Faraday Effect's Angular Momentum

In general this is calculated from the relativistic momentum:

$$\underline{p} = \gamma m \underline{v} \quad -(1)$$

where:

$$\underline{v} = \frac{d\underline{r}}{d\tau} = m \frac{d\underline{r}}{dt}\frac{dt}{d\tau} \quad -(2)$$

where

$$\frac{dt}{d\tau} = \gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}. \quad -(3)$$

Here one frame moves with respect to another at velocity u, and τ is the proper time.

Thus:

$$\underline{r} = \frac{1}{\gamma m}\int \underline{p}\, dt. \quad -(4)$$

The minimal prescription in this case is:

$$\underline{p} \rightarrow \underline{p} + e\underline{A} = \gamma m \underline{v} + e\underline{A}. \quad -(5)$$

For a plane wave of the type:

$$\underline{A} = A^{(0)}\left(\underline{i}\cos\phi + \underline{j}\sin\phi\right) \quad -(6)$$

the interaction part of $\underline{p}$ is:

$$\left.\begin{aligned} p_x &= eA^{(0)}\cos\phi, & p_y &= eA^{(0)}\sin\phi, \\ r_x &= \frac{eA^{(0)}}{\gamma\omega}\sin\phi, & r_y &= -\frac{eA^{(0)}}{\gamma\omega}\cos\phi. \end{aligned}\right\} \quad -(7)$$

2) The radius vector from eqs. (4) and (5) is:

$$\underline{r} = \frac{1}{m}\left(\int m\underline{v}\,dt + \frac{1}{\gamma}\int e\underline{A}\,dt\right). \quad -(8)$$

Therefore the relativistic angular momentum is:

$$\underline{J} = \underline{r} \times \underline{p} \quad -(9)$$

$$= \frac{1}{m}\left(\int m\underline{v}\,dt + \frac{1}{\gamma}\int e\underline{A}\,dt\right) \times \left(\gamma m\underline{v} + e\underline{A}\right). \quad -(10)$$

Using:

$$\underline{r}_0 = \int \underline{v}\,dt \quad -(11)$$

then:

$$\boxed{\underline{J} = \gamma\,\underline{r}_0 \times \underline{p} + e\underline{r}_0 \times \underline{A} + e\int \underline{A}\,dt \times \underline{v} + \frac{e^2}{m\gamma}\int \underline{A}\,dt \times \underline{A}.} \quad -(12)$$

The $\underline{B}^{(3)}$ term enters through the second order term, the last one on the RHS of eq. (12). The induced dipole moment of the second order term is:

$$\underline{m} = \frac{e}{2m}\,\underline{J} = \frac{e^3 A^{(0)2}}{2m^2\gamma\omega}\,\underline{k}. \quad -(13)$$

In the non-relativistic limit the energy is:

$$E = \frac{1}{2}\omega J = \frac{e^2 A^{(0)2}}{2m} = \frac{e^2 c^2 B^{(0)2}}{2m\omega^2}$$

3)

There are two first order terms in eq. (12). In the ultra-relativistic limit the velocity of the electron becomes:

$$\underline{v} = v^{(0)} \left(\underline{i} \cos\phi + \underline{j} \sin\phi \right) \quad -(15)$$

and the first order angular momentum becomes:

$$J = \frac{eA^{(0)} v^{(0)}}{\omega} \rightarrow \frac{eA^{(0)} c}{\omega} = \frac{ec^2}{\omega^2} B^{(0)} \quad -(16)$$

as in previous notes. In this limit:

$$eA^{(0)} = \hbar \kappa, \quad -(17)$$

and

$$J = \hbar. \quad -(18)$$

So we are now ready to proceed to describe the Faraday effect.