63(14): Euler Reduction with Real Exponential.

Start with:

$$\frac{d^2\phi}{dZ^2} - \frac{1}{Z}\frac{d\phi}{dZ} + \frac{\phi}{Z^2} = -\frac{\rho(0)}{\epsilon_0}\cos(\kappa Z) \quad - (1)$$

Let:

$$Z = Z_0 e^{\kappa x} \quad - (2)$$

$$\kappa x = \log_e \frac{Z}{Z_0} \quad - (3)$$

and

$$\frac{dx}{dZ} = \frac{1}{\kappa Z}. \quad - (4)$$

Use

$$\frac{d\phi}{dZ} = \frac{d\phi}{dx}\frac{dx}{dZ} \quad - (5)$$

then

$$Z\frac{d\phi}{dZ} = \frac{1}{\kappa}\frac{d\phi}{dx}, \quad - (6)$$

$$\frac{d\phi}{dZ} = \frac{1}{\kappa Z}\frac{d\phi}{dx}. \quad - (6a)$$

The second derivative is:

$$\frac{d^2\phi}{dZ^2} = \frac{d}{dZ}\left(\frac{d\phi}{dZ}\right) = \frac{d}{dZ}\left(\frac{1}{\kappa Z}\frac{d\phi}{dx}\right)$$

$$= -\frac{1}{\kappa Z^2}\frac{d\phi}{dx} + \frac{1}{\kappa Z}\frac{d}{dZ}\left(\frac{d\phi}{dx}\right) \quad - (7)$$

In eq. (7):

$$\frac{d}{dZ}\left(\frac{d\phi}{dx}\right) = \frac{d^2\phi}{dZ\,dx} = \frac{d^2\phi}{dx\,dZ} = \frac{d}{dx}\left(\frac{d\phi}{dZ}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{\kappa Z}\frac{d\phi}{dx}\right) = \frac{1}{\kappa Z}\frac{d^2\phi}{dx^2} \quad - (8)$$

Therefore:

2)
$$\frac{d^2\phi}{dz^2} = -\frac{1}{kz^2}\frac{d\phi}{dx} + \frac{1}{k^2z^2}\frac{d^2\phi}{dx^2} \quad -(9)$$

Using eq. (9) in eq. (1):

$$-\frac{1}{k}\frac{d\phi}{dx} + \frac{1}{k^2}\frac{d^2\phi}{dx^2} - \frac{1}{k}\frac{d\phi}{dx} + \phi = -\frac{\rho(0)}{\epsilon_0}\cos(kz) \quad -(10)$$

i.e.
$$\boxed{\frac{d^2\phi}{dx^2} - 2k\frac{d\phi}{dx} + k^2\phi = -\frac{\rho(0)}{\epsilon_0}\cos\left(e^{kx}\right) \quad -(11)}$$

where we have assumed:

$$z_0 = \frac{1}{k} \quad -(12)$$

Eq (11) is a damped oscillator with constant coefficents. Eq (1) is a second order differential equation with z dependent coefficients. The cosine term on ~~the~~ right hand side of eq. (11) is always between +1 and −1, so the equation is well suited for coding.