1) Solution of the Resonance equation : Euler Equation

An analytical solution is needed of the equation:

$$\nabla^2 \phi - \frac{A}{z}\frac{\partial \phi}{\partial z} + \frac{A}{z^2}\phi = \frac{\rho_0}{\epsilon_0}\cos(\kappa z) \quad -(1)$$

This is an example of Euler's equation:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = f(x). \quad -(2)$$

For $n = 2$:

$$a_0 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = f(x) \quad -(3)$$

Eq. (3) is:

$$a_0 \frac{d^2 y}{dx^2} + \frac{a_1}{x}\frac{dy}{dx} + \frac{a_2}{x^2} y = \frac{f(x)}{x^2}. \quad -(4)$$

Eqs. (1) and (4) have the same structure - that of a linear differential equation with non-constant coefficients. Using well known mathematical method, eq. (4) can be reduced to a differential equation with constant coefficients using:

$$x = e^t. \quad -(5)$$

Then:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x} \quad -(6)$$

so:

$$x\frac{dy}{dx} = \frac{dy}{dt}, \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}. \quad -(7)$$

So eq. (3) becomes:

$$a_0\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\frac{dy}{dt} + a_2 y = f(e^t)$$

2)

i.e.

$$a_0 \frac{d^2y}{dt^2} + (a_1 - a_0)\frac{dy}{dt} + a_2 y = f(e^t) \quad -(9)$$

Eq (9) may be solved analytically, it is a standard resonance equation. The complementary function of eq. (9) is the solution of:

$$a_0 \frac{d^2y}{dt^2} + (a_1 - a_0)\frac{dy}{dt} + a_2 y = 0. \quad -(10)$$

i.e. of

$$\frac{d^2y}{dt^2} + \frac{(a_1 - a_0)}{a_0}\frac{dy}{dt} + \frac{a_2}{a_0} y = 0. \quad -(11)$$

If:

$$\left(\frac{a_1 - a_0}{a_0}\right) = 2\beta, \quad \frac{a_2}{a_0} = \omega_0^2 \quad -(12)$$

Then the complementary function is:

$$\boxed{\begin{aligned} y_c(t) &= e^{-\beta t}\left(A_1 \exp\left((\beta^2 - \omega_0^2)^{1/2} t\right) + A_2 \exp\left(-(\beta^2 - \omega_0^2)^{1/2} t\right)\right) \quad -(13) \\ \text{where} \quad t &= \log_e x. \quad -(14) \end{aligned}}$$

Eq (1) is:

$$Z^2 \frac{d^2\phi}{dZ^2} - AZ\frac{d\phi}{dZ} + A\phi = \frac{f_0}{\epsilon_0} Z^2 \cos(\kappa Z) \quad -(15)$$

3)

Comparing eqs. (3) and (14):

$$\left.\begin{aligned} a_0 &= 1, \quad a_1 = -A, \quad a_2 = A, \\ f(Z) &= \frac{\rho_0}{\epsilon_0} Z^2 \cos(kZ) \end{aligned}\right\} \qquad -(16)$$

Now assume:

$$Z = Z_0 e^{i\omega_0 t} \qquad -(17)$$

so:

$$t = \frac{-i}{\omega_0} \log_e\left(\frac{Z}{Z_0}\right)$$

$$= -\frac{i}{\omega_0} \log_e Z - \log_e Z_0 \qquad -(18)$$

and:

$$\frac{dt}{dZ} = -\frac{i}{\omega_0 Z} \qquad -(19)$$

Thus:

$$\frac{d\phi}{dZ} = \frac{d\phi}{dt}\frac{dt}{dZ} = -\frac{i}{\omega_0 Z}\frac{d\phi}{dt} \qquad -(20)$$

and

$$Z\frac{d\phi}{dZ} = -\frac{i}{\omega_0}\frac{d\phi}{dt}. \qquad -(21)$$

Therefore:

$$\frac{d^2\phi}{dZ^2} = -\frac{i}{\omega_0}\frac{d}{dZ}\left(\frac{1}{Z}\frac{d\phi}{dt}\right)$$

$$= \frac{i}{\omega_0 Z^2}\frac{d\phi}{dt} - \frac{i}{\omega_0 Z}\frac{d}{dZ}\left(\frac{d\phi}{dt}\right)$$

Thus

$$Z^2\frac{d^2\phi}{dZ^2} = \frac{i}{\omega_0}\frac{d\phi}{dt} - \frac{iZ}{\omega_0}\frac{d}{dZ}\frac{d\phi}{dt} \qquad -(22)$$

4) Using eqn (20) in eqn. (22):

$$z^2 \frac{d^2\phi}{dz^2} = \frac{i}{\omega_0}\frac{d\phi}{dt} - \frac{1}{\omega_0^2}\frac{d^2\phi}{dt^2}. \quad -(23)$$

Using eqns. (21) and (23) in eq. (15):

$$\frac{i}{\omega_0}\frac{d\phi}{dt} - \frac{1}{\omega_0^2}\frac{d^2\phi}{dt^2} + \frac{iA}{\omega_0}\frac{d\phi}{dt} + A\phi = \frac{\rho_0}{\epsilon_0} z^2 \cos(kz) \quad -(24)$$

Taking the real parts of both sides of eq. (24), and using $A=-1$:

$$\frac{d^2\phi}{dt^2} + \omega_0^2 \phi = -\frac{\rho_0}{\epsilon_0} \operatorname{Re}\left(z^2 \cos(kz)\right). \quad -(25)$$

Using eq. (17):

$$\frac{d^2\phi}{dt^2} + \omega_0^2\phi = -\frac{\rho_0}{\epsilon_0} z_0^2 \cos(2\omega_0 t)\cos(kz_0\cos(\omega_0 t))$$

$$:= \alpha \cos \Omega t$$

where: $\alpha := -\frac{\rho_0}{\epsilon_0} z_0^2$

$$\cos \Omega t := \cos(2\omega_0 t)\cos(kz_0 \cos(\omega_0 t))$$

The final equation is:

5)

$$\boxed{\frac{d^2\phi}{dt^2} + \omega_0^2 \phi = \alpha \cos(\Omega t)} \quad -(26)$$

Units Check

$\phi = JC^{-1} = \text{volts}, \; \rho = Cm^{-3},$
$\epsilon_0 = J^{-1} C^2 m^{-1}$

and units balance both sides.

Eqn. (26) is an undamped driven oscillator whose particular solution is:

$$\phi_p = \frac{\alpha}{(\omega_0^2 - \Omega^2)} \cos(\Omega t - \delta) \quad -(27)$$

where:

$$\delta = \tan^{-1} 0 = 0$$

Thus:

$$\boxed{\phi_p = \frac{\alpha}{(\omega_0^2 - \Omega^2)}} \quad -(28)$$

Amplitude and kinetic energy resonance occur at

$$\omega_R = \omega_E = \omega_0 \quad -(29)$$