1)

Notes 57(5): Second Quantization

The approach taken to second quantization in ECE field theory is to define the fundamental field as the tetrad, and to define the position and conjugate momentum tetrads from the lagrangian. Then the field tetrad is expanded in creation and annihilation operators. The number operator is defined from the creation and annihilation operators, and a multi-particle interpretation of quantum mechanics is developed in the context of a generally covariant unified field theory.

The fundamental wave equation is the ECE Lemma:

$$\square q^a_\mu := R q^a_\mu \quad -(1)$$

where:

$$R = -kT. \quad -(2)$$

Here q^a_μ is the tetrad, the eigenfunction or wavefunction. The eigenvalues are R, scalar curvature. In eq. (2) k is the Einstein constant and T the index contracted canonical energy-momentum density. It is known that q^a_μ is also the fundamental field, so eq. (1) is an equation of rigorous quantum field theory, giving a multi-particle interpretation of quantum mechanics for all known fields of physics.

To illustrate this property these notes adopt the usual methods of second quantization in quantum field theory (e.g. L. H. Ryder, "Quantum Field Theory"

) (Cambridge Univ. Press, 2nd. ed., 1996). First note that eq. (1) is generally covariant. It is derived from the tetrad postulate and is the same equation as:

$$D^\mu \left(D_\mu \underset{\sim}{q^a} \right) := 0. \quad -(3)$$

Eq. (1) may be derived from an Euler - Lagrange equation (see M.W. Evans, vol. one of "Generally Covariant Unified Field Theory" (Abramis 2005)), and also the final papers of vol. 3 (Abramis, 2006 in press). For example, if:

$$\frac{\partial \mathcal{L}}{\partial \tilde{q}_a} = -\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \tilde{q}_a)} \right) \quad -(4)$$

and

$$\mathcal{L} = -\frac{c^2}{k} \left(\frac{1}{2} \left(\partial_\mu \underset{\sim}{q^a} \right) \left(\partial^\mu \tilde{q}_a \right) + \frac{R}{2} \underset{\sim}{q^a} \tilde{q}_a \right) \quad -(5)$$

then eq. (1) follows. There is a freedom of choice in the Lagrangian, it is chosen to give the ECE Lemma through the Euler - Lagrange equation (4).

Define the position and conjugate momentum tetrads by:

$$x^a_\mu = x^{(0)} q^a_\mu \quad -(6)$$

$$p^a_\mu = p^{(0)} q^a_\mu . \quad -(7)$$

The conjugate momentum is related to the position

3) by:

$$\boxed{p^a_\mu = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\partial_\nu x^a_\mu)}}. \quad —(8)$$

Eq (8) is a canonical equation, in the sense that it generalizes the well known classical result:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}. \quad —(9)$$

Classical dynamics can be developed with the Lagrange equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad —(10)$$

and the Hamilton equations of motion, the canonical equations:

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad -\dot{p}_k = \frac{\partial H}{\partial q_k}. \quad —(11)$$

These are found in every textbook on classical dynamics. Note that eq. (8) is an equation of general relativity, whereas eqs (9) to (11) are classical and non-relativistic.

In the presence of Cartan torsion:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad —(12)$$

and if:

$$\omega^a{}_b = -\frac{1}{2}\kappa\epsilon^a{}_{bc}q^c \quad -(13)$$

then:

$$T^a = d\wedge q^a - \kappa q^b \wedge q^c, \quad -(14)$$

i.e.:

$$\left.\begin{aligned} T^1 &= d\wedge q^1 - \kappa q^2\wedge q^3 \\ T^2 &= d\wedge q^2 - \kappa q^3\wedge q^1 \\ T^3 &= d\wedge q^3 - \kappa q^1\wedge q^2 \\ T^0 &= d\wedge q^0 - \kappa q^2\wedge q^1 \end{aligned}\right\} \quad -(15)$$

Eqs (14) and (15) indicate the existence of <u>commutators</u>. These appear at the classical level in general relativity. In regions where the fundamental field is non-zero and where the torsion vanishes:

$$d\wedge q^a = \kappa q^b \wedge q^c \quad -(16)$$

In tensor notation, eqn. (16) is <u>a</u> <u>commutator</u> equation:

$$\boxed{[\partial_\mu, q^a_\nu] = \kappa[q^b_\mu, q^c_\nu]} \quad -(17)$$

For the position tetrad:

$$[\partial_\mu, x^a_\nu] = \kappa[q^b_\mu, x^c_\nu]. \quad -(18)$$

Multiply both sides of eq (18) by $p^{(0)}$ to

5) obtain:

$$p^{(0)}\left[\partial_\mu, \underset{\sim}{x}^{a}\right] = \kappa\left[p^{b}_{\mu}, \underset{\sim}{x}^{c}\right]. \quad -(19)$$

This is an equation of classical general relativity, but it contains the commutator of position and conjugate momentum. The latter is fundamentally important to quantum mechanics, as is well known. Second quantization proceeds from setting up the equal time commutator of position and and momentum. This is because position and conjugate momentum in the Heisenberg equation are defined at the same instant in time.

Eq. (19) shows that the momentum tetrad p^{b}_{μ} and the partial derivative operator ∂_μ play a similar role. This observation leads to the fundamental operator relation of quantum mechanics, but in a generally covariant context.

In the de Broglie limit:

$$p^{(0)} = \hbar\kappa \quad -(20)$$

and

$$\left[p^{b}_{\mu}, \underset{\sim}{x}^{c}\right] = \hbar\left[\partial_\mu, \underset{\sim}{x}^{a}\right]. \quad -(21)$$

Eq. (21) in the complex circular basis is:

$$\left[p^{(b)}_{\mu}, \underset{\sim}{x}^{(c)}\right] = i\hbar\left[\partial_\mu, \underset{\sim}{x}^{(a)*}\right] \quad -(22)$$

6) and is equivalent to the angular momentum commutator relations

$$[J^{(b)}_\mu, J^{(c)}_\nu] = i\hbar J^{(a)*}_{\mu\nu} \quad -(23)$$

where:

$$J^{(a)*}_{\mu\nu} = iJ^{(0)}[\partial_\mu, x^{(a)}_\nu]^* \quad -(24)$$

In vector notation:

$$\underline{J}^{(1)} \times \underline{J}^{(2)} = i\hbar \underline{J}^{(3)*} \quad -(25)$$

et cyclicum.

In general relativity, eq. (19) gives:

$$\boxed{p^{(0)} = \left([p^b_\mu, x^c_\nu] / [\partial_\mu, x^a_\nu]\right)\kappa} \quad -(26)$$

so the Planck constant is the limit:

$$\boxed{[p^b_\mu, x^c_\nu] \rightarrow \hbar [\partial_\mu, x^a_\nu],} \quad -(27)$$

or, in the complex circular basis:

$$\boxed{[p^{(b)}_\mu, x^{(c)}_\nu] \rightarrow i\hbar [\partial_\mu, x^{(a)*}_\nu]} \quad -(27a)$$

The Planck constant is defined by a particular type of Cartan geometry in a well-defined limit.

7) Having recognized that eq. (19) leads to quantization, its fully quantized form is:

$$\boxed{[p^b_\mu, x^c_\nu]\, q^d_\rho = \left(\frac{\hbar^{(0)}}{\kappa}\right)[\partial_\mu, x^a_\nu]\, q^d_\rho} \quad -(28)$$

where it is recognized that the commutators act on the tetrad q^d_ρ. We may write:

$$\delta^a_{\mu\nu} = [\partial_\mu, x^a_\nu] \quad -(29)$$

In shorthand notation, eq. (28) is:

$$\boxed{(p \wedge x)\, q = \left(\frac{\hbar^{(0)}}{\kappa}\right)(d \wedge x)\, q} \quad -(30)$$

where q is both an eigenfunction and a field.

So in this sense, rigorous quantum field theory emerges automatically from Cartan geometry. It is seen that the commutators of eq. (28) are commutators between tetrads, which is a rigorously quantized field. This is true for all the fields of physics, including the electromagnetic field:

$$\boxed{[A^a_\mu, A^b_\nu] = \frac{A^{(0)2}}{\kappa}\, \delta'^a_{\mu\nu} = \frac{A^{(0)2}}{\kappa}[\partial_\mu, \kappa^a_\nu]} \quad -\cancel{(27)}\ (31)$$

The Creation and Annihilation Operators

Eq. (25) is a relation between tetrads:

$$\underline{q}^{(1)} \times \underline{q}^{(2)} = i \underline{q}^{(3)} \quad -(32)$$

et cyclicum

Following the usual development of quantum field theory eqs (32) can be expressed as:

$$\left.\begin{aligned} [q_X, q_Y] &= i q_Z \\ [q_Y, q_Z] &= i q_X \\ [q_Z, q_X] &= i q_Y \end{aligned}\right\} \quad -(33)$$

The raising and lowering operators are (Atkins, p. 107)

$$q^+ = q_X + i q_Y \quad -(34)$$

$$q^- = q_X - i q_Y. \quad -(35)$$

These are similar to the complex circular tetrads:

$$q^{(1)} = \frac{1}{\sqrt{2}}(q_X + i q_Y) \quad -(36)$$

$$q^{(2)} = \frac{1}{\sqrt{2}}(q_X - i q_Y). \quad -(37)$$

The commutator properties of (34) and (35) are:

$$[q^+, q_Z] = -q^+ \quad -(38)$$

$$[q^-, q_Z] = q^- \quad -(39)$$

$$[q^+, q^-] = 2 q_Z \quad -(40)$$

The <u>creation tetrad operator</u> is then

a) defined by:

$$a^+|N\rangle = c_N^+|N+1\rangle \quad -(41)$$

and the annihilation ~~ketrad~~ operator by:

$$a^-|N\rangle = c_N^-|N-1\rangle \quad -(42)$$

In quantum electrodynamics a^+ increases the state $|N\rangle$ to $|N+1\rangle$ for example. The electromagnetic field is described by an infinite number of harmonic oscillators, one for every point in space. In this case:

$$a^-|N\rangle = \sqrt{N}\,|N-1\rangle \quad -(43)$$

$$a^+|N\rangle = (N+1)^{1/2}|N+1\rangle \quad -(44)$$

The number operator is:

$$n = a^+a^- \quad -(45)$$

so that:

$$n|N\rangle = N|N\rangle \quad -(46)$$

The Hamiltonian is:

$$\left.\begin{aligned} H &= \hbar\omega\,(n+1/2) \\ &= \hbar\omega\,(a^+a^- + 1/2) \\ &= \hbar\omega\,(N+1/2) \end{aligned}\right\} \quad -(47)$$

giving the zero point energy:

$$H_0 = \frac{1}{2}\hbar\omega \quad -(48)$$

o) The fundamental field is expanded in a Fourier series:

$$\phi = \sum_{\kappa} \left(a_{\kappa}^{+} e^{-i\underline{\kappa}\cdot\underline{r}} + a_{\kappa}^{-} e^{i\underline{\kappa}\cdot\underline{r}} \right) \quad (49)$$

which may be developed into an integral in a volume $\bar{V}$:

$$\phi = \bar{V} \int \left(a_{\kappa}^{+} e^{-i\underline{\kappa}\cdot\underline{r}} + a_{\kappa}^{-} e^{i\underline{\kappa}\cdot\underline{r}} \right) d^3\kappa \quad - (50)$$

<u>Summary</u>

In ECE ~~theory~~ the fundamental field and wave-function are both tetrads, so second quantization and canonical quantization follow rigorously from Cartan geometry. The latter also indicates the need for ~~the~~ basic operator equivalence of quantum mechanics through the Cartan structure equation. This provides a <u>generally covariant quantum field theory</u>.