57(4): Fundamental Commutator, Wave Equation, Expectation Value and Probability.

The basic equations of wave mechanics in ECE theory are all obtained from basic equations of Cartan geometry, the wave function being the tetrad q^a_μ. The basic geometrical equations are:

$$q^a_\mu q^\mu_a = 1 \quad -(1)$$

and

$$D_\mu q^a_\nu = 0. \quad -(2)$$

From the tetrad postulate (2) the ECE Lemma is obtained:

$$\Box q^a_\mu := R q^a_\mu \quad -(3)$$

where:

$$R = q^\lambda_a \partial^\mu \left(\Gamma^\nu_{\mu\lambda} q^a_\nu - \omega^a_{\mu b} q^b_\lambda \right). \quad -(4)$$

Eq (3) is equivalent to:

$$D^\mu \left(D_\mu q^a_\nu \right) := 0. \quad -(5)$$

It is always possible to write eq (1) as the classical equation:

$$q^a_\mu R q^\mu_a = R. \quad -(6)$$

Using eq (3) in eq (6):

2)

$$\boxed{q^{\mu}_{a} \, \square \, q^{a}_{\mu} = R} \quad -(7)$$

which is an operator equation indicating that R is the expectation value of $\square$. These equations define the geometrical basis of generally covariant wave mechanics.

In general q^{μ}_{a} is not the same as q^{a}_{μ}, so:

$$\left.\begin{aligned} \square q^{a}_{\mu} &= R q^{a}_{\mu} \\ \square q^{\mu}_{a} &= R_1 q^{\mu}_{a} \end{aligned}\right\} \quad -(8)$$

where R_1 is to be defined. It follows from eq (8) that:

$$\left.\begin{aligned} q^{\mu}_{a} \square q^{a}_{\mu} &= q^{\mu}_{a} R q^{a}_{\mu} = R \\ q^{a}_{\mu} \square q^{\mu}_{a} &= q^{a}_{\mu} R_1 q^{\mu}_{a} = R_1 \end{aligned}\right\} \quad -(9)$$

In wave mechanics, Hermitian matrices are used, because their eigenvalues are real-valued and physical. In Dirac bracket notation a Hermitian system is defined by:

$$\langle n | \Omega | m \rangle = \langle m | \Omega | n \rangle^{*} \quad -(10)$$

where * denotes complex conjugate. A Hermitian matrix is a square matrix unchanged by taking the transpose of its complex conjugate, e.g. if

$$A = \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}, \quad A^* = \begin{pmatrix} 1 & 1-i \\ 1+i & 3 \end{pmatrix} \quad — (11)$$

then

$$\tilde{A}^* = A = \begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}. \quad — (12)$$

Eq (10) translates into:

$$q^a_\mu \Box q^\mu_a = \left(q^\mu_a \Box q^a_\mu \right)^* \quad — (13)$$

which implies:

$$R_1 = R^*. \quad — (14)$$

This means that the tetrad has been assumed to be a Hermitian matrix, its eigenvalues are real and physical. Now denote:

$$R = R' + iR'' \quad — (15)$$

$$R^* = R' - iR'' \quad — (16)$$

to obtain:

$$R' = \frac{1}{2} \left(q^\mu_a \Box q^a_\mu + q^a_\mu \Box q^\mu_a \right) \quad - (17)$$

$$R'' = -\frac{i}{2} \left(q^\mu_a \Box q^a_\mu - q^a_\mu \Box q^\mu_a \right). \quad - (18)$$

Single particle relativistic equations in the

4) limit of special relativity are regained by assuming:

$$|R'| \rightarrow (mc/\hbar)^2, \quad -(19)$$

$$|R''| \rightarrow 0 \quad -(20)$$

In this limit it is self-consistently apparent that:

$$\frac{1}{2}\left(q^{\mu}_{a}\Box q^{a}_{\mu} + q^{a}_{\mu}\Box q^{\mu}_{a}\right) = \frac{1}{2}(R+R) = R$$

$$= R' = -(mc/\hbar)^2. \quad -(21)$$

<u>The Klein Gordon equation</u> is regained in the limit:

$$\phi = q^{0}_{0} = q^{1}_{1} = q^{2}_{2} = q^{3}_{3} \Rightarrow 1. \quad -(22)$$

In this limit eq. (3) gives:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0. \quad -(23)$$

Therefore ϕ is a <u>scalar field</u> without <u>spin</u>.

Eq. (23) is:

$$\left(\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(24)$$

If we define:

$$\left.\begin{aligned} p^{\mu} &= \left(\frac{E}{c}, \underline{p}\right) \\ p_{\mu} &= \left(\frac{E}{c}, -\underline{p}\right) \end{aligned}\right\} \quad -(25)$$

5) Eq. (24) is the same as the Einstein equation of Special relativity:

$$p^{\mu} p_{\mu} = m^2 c^2 \quad -(26)$$

if:

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \underline{p} \rightarrow -i\hbar \underline{\nabla}. \quad -(27)$$

In the non-relativistic approximation to eq (27):

$$E = \frac{1}{2} \frac{p^2}{m} \quad -(28)$$

and from eq. (27) we obtain the Schrödinger equation for a free particle:

$$\frac{\hbar^2}{2m} \nabla^2 \phi = -i\hbar \frac{\partial \phi}{\partial t}. \quad -(29)$$

Eq (28) defines the kinetic energy in Newtonian mechanics.

Therefore the Klein-Gordon, Schrödinger and Newton equations have been obtained from Cartan geometry in the limit:

$$R \rightarrow -(mc/\hbar)^2. \quad -(30)$$

The minus sign in eq. (30) is a convention.

6) In the Born interpretation the probability density for the Schrodinger equation is proportional to:

$$\rho = \phi^* \phi \quad — (31)$$

so ϕ is a complex scalar field. The probability current of the Schrodinger equation is defined as:

$$\underline{j} = -\frac{i\hbar}{2m}\left(\phi^* \underline{\nabla} \phi - \phi \underline{\nabla} \phi^*\right) \quad — (32)$$

so that:

$$\partial_\mu j^\mu = 0 \quad — (33)$$

where:

$$j^\mu = (\rho, \underline{j}) \quad — (34)$$

is the relativistic generalization of eqns. (31) and (32). For the Schrödinger equation:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{j} = \frac{\partial}{\partial t}(\phi^* \phi) - \frac{i\hbar}{2m}\left(\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^*\right)$$

$$= \phi^*\left(\frac{\partial \phi}{\partial t} - \frac{i\hbar}{2m} \nabla^2 \phi\right) + \phi\left(\frac{\partial \phi^*}{\partial t} + \frac{i\hbar}{2m} \nabla^2 \phi^*\right)$$

$$= 0. \quad — (35)$$

Eq. (35) is a non-relativistic continuity equation whose special relativistic counterpart

7)

is eq. (33). Thus j^μ is a conserved quantity.

In the Klein-Gordon equation ρ must be the time-like component of j^μ:

$$\rho = \frac{i\hbar}{2m}\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right) \quad -(36)$$

Thus:

$$\partial_\mu j^\mu = \frac{i\hbar}{2m}\left(\phi^* \Box \phi - \phi \Box \phi^*\right) \quad -(37)$$
$$= 0.$$

In ECE theory, eq. the equation (37) must be generalized to involve:

$$\boxed{q_a^{\ \mu} \Box q^a_{\ \mu} - q^a_{\ \mu} \Box q_a^{\ \mu} = R - R_1} \quad -(38)$$

If $R_1 = R^*$ (hermitian $q^a_{\ \mu}$) it is convenient to use eqs (17) and (18). The continuity equation of ECE theory can be defined as:

$$\boxed{\partial_\mu j^\mu = \frac{i\hbar}{2m}\left(q_a^{\ \mu} \Box q^a_{\ \mu} - q^a_{\ \mu} \Box q_a^{\ \mu}\right)} \quad -(39)$$

8) For real and physical $R = R^*$:

$$\partial_\mu j^\mu = R - R^* = 0. \quad -(40)$$

The Klein-Gordon equation was abandoned in favour of the Dirac equation because it gives negative ρ from eq. (36), i.e. a negative probability that makes no sense. In order to make sense of the Klein Gordon equation ϕ must be regarded as a complex field. This procedure is known as second quantization.

In ECE theory:

$$\boxed{\begin{aligned} \rho &= \frac{i\hbar}{2m}\left(q^{\mu}_{a} \frac{\partial}{\partial t} q^{a}_{\mu} - q^{a}_{\mu} \frac{\partial}{\partial t} q^{\mu}_{a} \right) \\ \underline{j} &= -\frac{i\hbar}{2m}\left(q^{\mu}_{a} \underline{\nabla} q^{a}_{\mu} - q^{a}_{\mu} \underline{\nabla} q^{\mu}_{a} \right) \\ j^{\mu} &= (\rho, \underline{j}) \end{aligned}} \quad -(41)$$

and the tetrad is a complex field. These concepts emerge from Cartan geometry.

9) The Born normalization is therefore given directly by the fundamental tetrad normalization (1). The relativistic version of the Born normalization is obtained using eq. (1) in combination with the tetrad postulate (2).

The Dirac equation is a limit of the ECE Lemma using eq. (30), but the Dirac field is a spin half field defined by the Dirac spinor. As shown in previous work the Dirac spinor is a tetrad in SU(2) representation space. The Dirac equation is therefore:

$$\left(\Box + \left(\frac{mc}{\hbar} \right)^2 \right) q^a_\mu = 0 \quad - (42)$$

where the a and μ labels represent half-integral spin with components q^R_1, q^R_2, q^L_1 and q^L_2. The Pauli spinors are:

$$\phi^R = \begin{bmatrix} q^R_1 \\ q^R_2 \end{bmatrix}, \quad \phi^L = \begin{bmatrix} \phi^L_1 \\ \phi^L_2 \end{bmatrix} \quad - (43)$$

and the Dirac spinor is:

$$q^a_\mu = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} \quad - (44)$$

10) In quantum field theory the Dirac spinor is denoted by ψ and the convention that $c = \hbar = 1$ is used, so the Dirac equation is:

$$(\Box + m^2)\psi = 0. \quad -(45)$$

Eq. (45) can be written as:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad -(46)$$

where γ^μ is a Dirac matrix:

$$\gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^i \partial_i \quad -(47)$$

Written out in full, eqn. (46) is:

$$\left(i\gamma^\mu \partial_\mu - \left(\frac{mc}{\hbar}\right)^2\right) V^a_\mu = 0. \quad -(48)$$

Using the shorthand notation (46) it is possible to write the Hermitian conjugate equation:

$$\psi^+ \left(-i\gamma^0 \overleftarrow{\partial_0} + i\gamma^i \overleftarrow{\partial_i} - m\right) = 0. \quad -(49)$$

Now define: $$\overline{\psi} = \psi^+ \gamma^0 \quad -(50)$$

where: $$\gamma^i \gamma^0 = -\gamma^0 \gamma^i \quad -(51)$$

to give: $$\overline{\psi}\left(i\gamma^\mu \overleftarrow{\partial_\mu} + m\right) = 0. \quad -(52)$$

11) Here $\overline{\psi}$ is the adjoint spinor: The current

$$j^\mu = \overline{\psi}\gamma^\mu\psi \quad -(53)$$

is conserved:

$$\partial_\mu j^\mu = 0, \quad -(54)$$

i.e.

$$\begin{aligned}\partial_\mu j^\mu &= (\partial_\mu\overline{\psi})\gamma^\mu\psi + \overline{\psi}\gamma^\mu(\partial_\mu\psi) \\ &= (im\overline{\psi})\psi + \overline{\psi}(-im\psi) \\ &= 0. \quad -(55)\end{aligned}$$

The all important point is that:

$$\begin{aligned}j^0 &= \overline{\psi}\gamma^0\psi = \psi^+\psi \\ &= |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \\ &> 0 \quad -(56)\end{aligned}$$

and is the Born probability for the free Dirac particle.

In ECE theory: $\quad -(57)$

$$\psi^+ \rightarrow q_a^{\mu+}; \quad \overline{\psi} \rightarrow q_a^{\mu+}\gamma^0 = \overline{q}_a^\mu;$$

$$\boxed{j^\mu = \overline{q}_a^{\nu}\gamma^\mu q^a_{\nu}} \quad -(58)$$

12)

The negative energy states of the Dirac equation came from the fact that the spin 1/2 particles obey the Pauli exclusion principle. Negative energy states are completely filled, so the exclusion principle prevents any more electrons entering the Dirac sea, made up of negative energy states. The Dirac sea is the vacuum, made up of negative energy electrons, protons, neutrinos, neutrons and all fermions. If there is a vacancy, or "hole" in the fermion sea, with energy $-|E|$, an electron with energy E fills it, emitting energy $2E$:

$$e^- + \text{hole} \rightarrow \text{energy}. \quad -(59)$$

The hole has charge e^+ and positive energy, and is called a positron. The energy difference is:

$$\Delta E = E - (-|E|) = 2E. \quad -(60)$$

Eq. (59) in particle theory is:

$$e^- + e^+ = 2\gamma, \quad -(61)$$

and this means that an electron and positron mutually annihilate to give two photons.

13)

In ECE theory the Dirac sea is defined by tetrads obeying the Pauli exclusion principle. Positive energy states are also given by tetrads obeying the Pauli exclusion principle. The latter is understood by regarding the tetrad as a field (second quantization).

Quantities such as $p^\mu p_\mu$ are understood in ECE theory through the momentum tetrad:

$$p^a_{\ \mu} = p^{(0)} q^a_{\ \mu} \quad -(62)$$

Planck / Einstein / de Broglie quantization is given by:

$$p^a_{\ \mu} = \hbar \kappa^a_{\ \mu} = eA^a_{\ \mu} = eA^{(0)} q^a_{\ \mu} \quad -(63).$$

The Heisenberg commutator equation is understood using the structure invariants of Cartan geometry (e.g. p. 410 of vol 1):

$$x^a = \int_S T^a \quad -(64)$$

$$\theta^a_{\ b} = \int_S R^a_{\ b} \quad -(65)$$

 and the Heisenberg equation is:

$$[x^a, p_b] = i J^a{}_b = \hbar \theta^a{}_b \quad -(66)$$

where:

$$\theta^a{}_b = \frac{i}{\hbar} J^a{}_b . \quad -(67)$$

Thus:

$$[x^a, p_b] = \hbar \int_S R^a{}_b \quad -(68)$$

where:

$$R^a{}_b = -\frac{\kappa}{2} \epsilon^a{}_{bc} T^c \quad -(69)$$

appropriate to spinning spacetime. Here T^c is the Cartan torsion.

The next stage in development is the second quantization of the tetrad field.