Meaning of the Electric and Magnetic Field in ECE Field Theory (Notes for Paper 53).

Electric Field

This is a spacetime quantity defined by Cartan geometry as:

$$\boxed{\underline{E}^a = A^{(0)}\left(-\frac{\partial \underline{q}^a}{\partial t} - c\underline{\nabla} q^{0a} - c\omega^{0a}{}_b\underline{q}^b + c\underline{\omega}^a{}_b q^{0b}\right)} \quad -(1)$$

Magnetic Field

This is a spacetime quantity defined by Cartan geometry as:

$$\boxed{\underline{B}^a = A^{(0)}\left(\underline{\nabla}\times\underline{q}^a - \underline{\omega}^a{}_b\times\underline{q}^b\right).} \quad -(2)$$

The tetrad is defined by:

$$q^a_\mu = g_{\mu\nu} q^{\nu a} \quad -(3)$$

and we adopt the convention:

$$q^{\nu a} = (q^{0a}, \underline{q}^a) \quad -(4)$$

$$= (q^{0a}, q^{1a}, q^{2a}, q^{3a})$$

$$= (q^{0a}, q^a_X, q^a_Y, q^a_Z)$$

The spin connection is defined by:

$$\omega^a_{\mu b} = g_{\mu\nu}\,\omega^{\nu a}{}_b \quad -(5)$$

and we adopt the convention:

$$\begin{aligned}\omega^{\nu a}{}_b &= \left(\omega^{0a}, \underline{\omega}^a{}_b\right)\\ &= \left(\omega^{0a}, \omega^{1a}{}_b, \omega^{2a}{}_b, \omega^{3a}{}_b\right) \quad -(6)\\ &= \left(\omega^{0a}, \omega^a_{xb}, \omega^a_{yb}, \omega^a_{zb}\right)\end{aligned}$$

The ECE Ansatz is:

$$A^a_\mu = A^{(0)} q^a_\mu, \quad -(7)$$

$$F^a_{\mu\nu} = A^{(0)} T^a_{\mu\nu}. \quad -(8)$$

The Fundamental Spin Field, $\underline{B}^{(3)}$.

This is defined by adopting the complex circular basis:

$$a = (1), (2), (3) \quad -(9)$$

i.e.:

$$\left.\begin{aligned}\underline{q}^{(1)} \times \underline{q}^{(2)} &= i\,\underline{q}^{(3)*}\\ \underline{q}^{(2)} \times \underline{q}^{(3)} &= i\,\underline{q}^{(1)*}\\ \underline{q}^{(3)} \times \underline{q}^{(1)} &= i\,\underline{q}^{(2)*}\end{aligned}\right\} \quad -(10)$$

3)

The spin field is a special case of:

$$\boxed{\underline{B}^a_{spin} = -A^{(0)}\,\underline{\omega}^a{}_b \times \underline{q}^b} \quad -(11)$$

and therefore exists if and only if the spin connection is non-zero. In other words it is a property of general relativity and vanishes in special relativity.

Now consider:

$$a = 3 \quad -(12)$$

so:

$$\underline{B}^3 = -A^{(0)}\left(\underline{\omega}^3{}_1 \times \underline{q}^1 + \underline{\omega}^3{}_2 \times \underline{q}^2\right) \quad -(13)$$

We have summed over repeated indices and used:

$$\underline{\omega}^3{}_3 = \underline{0}. \quad -(14)$$

Eqn. (14) follows because for circular polarization:

$$\omega^a{}_{\mu b} = -\frac{\kappa}{2}\,\epsilon^a{}_{bc}\, q^c_\mu \quad -(15)$$

i.e. the spin connection is dual to the tetrad in the tangent space. Here:

$$\epsilon^a{}_{bc} = \eta^{ad}\,\epsilon_{dbc} \quad -(16)$$

where η^{ad} is the Minkowski metric and ϵ_{dbc} the 3-D totally antisymmetric unit tensor:

4)

$$\eta^{ad} = \eta_{ad} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{diag}(-1, 1, 1, 1) \quad -(17)$$

$$\epsilon_{abc} = \begin{cases} 1, & \text{even indices} \\ -1 & \text{odd indices} \end{cases} \quad -(18)$$

i.e $\epsilon_{123} = -\epsilon_{132} = 1$ etc. $\quad -(19)$

So: $\epsilon^1{}_{23} = \epsilon_{123}$ etc, $\quad -(20)$

and
$$\omega^1{}_2 = -\frac{\kappa}{2}\epsilon^1{}_{23} q^3$$
$$= -\frac{\kappa}{2} q^3 \quad -(21)$$
$$\omega^3{}_1 = -\frac{\kappa}{2}\epsilon^3{}_{12} q^2 = -\frac{\kappa}{2} q^2 \quad -(22)$$
$$\omega^3{}_2 = -\frac{\kappa}{2}\epsilon^3{}_{21} q^1 = \frac{\kappa}{2} q^1 \quad -(23)$$

Thus:
$$\underline{B}^3 = A^{(0)}\frac{\kappa}{2}\left(\underline{q}^2 \times \underline{q}^1 - \underline{q}^1 \times \underline{q}^2\right)$$
$$\boxed{\underline{B}^3 = -A^{(0)}\kappa\, \underline{q}^1 \times \underline{q}^2} \quad -(24)$$

Finally switch to the complex circular basis

5) and use:

$$A^1 = A^{(0)} q^1, \quad A^2 = A^{(0)} q^2 \quad -(25)$$

to find:

$$\boxed{\begin{aligned} \underline{B}^{(3)} = \underline{B}^{(3)*} &= -i\frac{\kappa}{A^{(0)}} \underline{A}^{(1)} \times \underline{A}^{(2)} \\ &:= -ig\, \underline{A}^{(1)} \times \underline{A}^{(2)} \end{aligned}} \quad -(26)$$

This is the Evans spin field. (Physica B, 182, (1992)) observed in the inverse Faraday effect. It is now known to be a special case of eqn. (2) of Cartan geometry.
