Notes for Paper 52 Part 6

Construction of the Field Tensor

$$F^a_{01} = -F^a_{10} = \partial_0 A^a_1 - \partial_1 A^a_0 + \omega^a_{0b} A^b_1 - \omega^a_{1b} A^b_0 \quad -(1)$$

$$F^a_{02} = -F^a_{20} = \partial_0 A^a_2 - \partial_2 A^a_0 + \omega^a_{0b} A^b_2 - \omega^a_{2b} A^b_0 \quad -(2)$$

$$F^a_{03} = -F^a_{30} = \partial_0 A^a_3 - \partial_3 A^a_0 + \omega^a_{0b} A^b_3 - \omega^a_{3b} A^b_0 \quad -(3)$$

$$F^a_{12} = \partial_1 A^a_2 - \partial_2 A^a_1 + \omega^a_{1b} A^b_2 - \omega^a_{2b} A^b_1 = -F^a_{21} \quad -(4)$$

$$F^a_{13} = \partial_1 A^a_3 - \partial_3 A^a_1 + \omega^a_{1b} A^b_3 - \omega^a_{3b} A^b_1 = -F^a_{31} \quad -(5)$$

$$F^a_{23} = \partial_2 A^a_3 - \partial_3 A^a_2 + \omega^a_{2b} A^b_3 - \omega^a_{3b} A^b_2 = -F^a_{32} \quad -(6)$$

Denote:

$$A^a_\mu = \left(A^a_0, -\underline{A}^a\right) \quad -(7)$$

$$\omega^a_{\mu b} = \left(A^a_{0b}, -\underline{\omega}^a_b\right) \quad -(8)$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla}\right) \quad -(9)$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{bmatrix} \quad -(10)$$

$$A^{a\mu} = \left(A^{a0}, \underline{A}^a\right) \quad -(11)$$

$$\omega^{a\mu}_b = \left(\omega^{a0}_b, \underline{\omega}^a_b\right) \quad -(12)$$

$$\therefore \quad F^{01a} = -\frac{1}{c}E^{1a} = \partial^0 A^{1a} - \partial^1 A^{0a} + \omega^{0a}_b A^{1b} - \omega^{1a}_b A^{0b} \quad -(13)$$

where

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad -(14)$$

2)

$$\therefore -\frac{1}{c}E^{1a} = -\frac{1}{c}E^a_X = \frac{1}{c}\frac{\partial A^a_X}{\partial t} + \frac{\partial}{\partial X}A^{0a} + \omega^{0a}{}_b A^b_X - \omega^a{}_{Xb}A^{0b} \quad -(15)$$

It follows ~~that~~ that:

$$\boxed{\underline{E}^a = -\frac{\partial \underline{A}^a}{\partial t} - c\underline{\nabla}A^{0a} + c\omega^{0a}{}_b\underline{A}^b - c\underline{\omega}^a{}_b A^{0b}} \quad -(16)$$

Similarly:

$$F^{12a} = \partial^1 A^{2a} - \partial^2 A^{1a} + \omega^{1a}{}_b A^{2b} - \omega^{2a}{}_b A^{1b}$$

$$= -B^{3a} = -B^a_Z \quad -(17)$$

$$-B^a_Z = -\frac{\partial}{\partial X}A^a_Y + \frac{\partial}{\partial Y}A^a_X + \omega^a{}_{Xb}A^b_Y - \omega^a{}_{Yb}A^b_X$$

$$\boxed{\underline{B}^a = \underline{\nabla}\times\underline{A}^a - \underline{\omega}^a{}_b\times\underline{A}^b} \quad -(18)$$

Equations (16) and (18) define the fields $\underline{E}^a$ and $\underline{B}^a$ in terms of the potential and spin connection.

The field equations from paper 50 are:

$$\underline{\nabla}\cdot\underline{B}^a = \mu_0\tilde{j}^{0a} \quad -(19)$$

$$\underline{\nabla}\times\underline{E}^a + \frac{\partial\underline{B}^a}{\partial t} = \mu_0\tilde{\underline{j}}^a \quad -(20)$$

3)

$$\underline{\nabla} \cdot \underline{E}^a = c\mu_0 \tilde{J}^{oa} \quad -(21)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2}\frac{\partial \underline{E}^a}{\partial t} = \frac{\mu_0}{c}\tilde{\underline{J}}^a \quad -(22)$$

The currents are defined by:

$$\tilde{j}^{a\mu} = \left(\frac{1}{c}\tilde{j}^{ao}, \tilde{\underline{j}}^a\right) \quad -(23)$$

$$\tilde{J}^{a\mu} = \left(\frac{1}{c}\tilde{J}^{ao}, \tilde{\underline{J}}^a\right) \quad -(24)$$

The Wave Equations

These are given by substituting eqns (16) and (18) into (19) - (22). The simplest wave equation is found by substituting eqn. (18) into eqn (19) and using:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{A}^a = 0 \quad -(25)$$

to give:

$$\boxed{\underline{\nabla} \cdot \left(\underline{\omega}^a{}_b \times \underline{A}^b\right) = -\mu_0 \tilde{j}^{ao}} \quad -(26)$$

In this equation summation over repeated b indices is used, so the equation is:

$$\underline{\nabla} \cdot \left(\underline{\omega}^a{}_0 \times \underline{A}^0 + \dots + \underline{\omega}^a{}_3 \times \underline{A}^3\right) = -\mu_0 \tilde{j}^{ao}$$

$$-(27)$$

4)

Therefore the current is:

$$\tilde{j}^{a0} = -\frac{A^{(0)}}{\mu_0} \underline{\nabla} \cdot \left(\underline{\omega}^a{}_0 \times \underline{q}^0 + \dots + \underline{\omega}^a{}_3 \times \underline{q}^3 \right) \quad -(28)$$

where $\underline{q}^a$ denotes the tetrad.

In the first approximation (Appendix J):

$$\omega^a{}_{\mu b} = -\kappa \, \epsilon^a{}_{bc} \, q^c_\mu, \quad -(29)$$

$$\epsilon^a{}_{bc} = g^{ad} \epsilon_{dbc}. \quad -(30)$$

So:

$$\left.\begin{aligned} \omega^1{}_{\mu 2} &= \kappa q^3_\mu \\ \omega^1{}_{\mu 3} &= -\kappa q^2_\mu \\ \omega^3{}_{\mu 2} &= -\kappa q^1_\mu \\ \omega^2{}_{\mu 3} &= \kappa q^1_\mu \\ &\text{etc.} \end{aligned}\right\} \quad -(31)$$

Therefore for $a = 0$

$$\boxed{\tilde{j}^{00} = -\frac{A^{(0)}}{\mu_0} \underline{\nabla} \cdot \left(\underline{\omega}^0{}_b \times \underline{q}^b \right)} \quad -(32)$$

$$\tilde{j}^{01} = \tilde{j}^{02} = \tilde{j}^{03} = 0 \quad -(33)$$

$$\omega^0{}_{\mu b} = -\kappa \, \epsilon^0{}_{bc} \, q^c_\mu. \quad -(34)$$

5)

Therefore:

$$\underline{\omega}^0{}_b = -\kappa\,\epsilon^0{}_{bc}\,\underline{q}^c \qquad -(35)$$

From the properties of $\epsilon^0{}_{bc}$, summation over b in eqn. (32) would be over space-like indices, 1, 2 and 3. From eqn. (35):

$$\underline{\omega}^0{}_1 = -\kappa\,\epsilon^0{}_{1c}\,\underline{q}^c$$

$$\underline{\omega}^0{}_1 = -\kappa\left(\epsilon^0{}_{12}\,\underline{q}^2 + \epsilon^0{}_{13}\,\underline{q}^3\right) \qquad -(36)$$

$$\underline{\omega}^0{}_2 = -\kappa\left(\epsilon^0{}_{21}\,\underline{q}^1 + \epsilon^0{}_{23}\,\underline{q}^3\right) \qquad -(37)$$

$$\underline{\omega}^0{}_3 = -\kappa\left(\epsilon^0{}_{31}\,\underline{q}^1 + \epsilon^0{}_{32}\,\underline{q}^2\right) \qquad -(38)$$

Therefore, in this approximation:

$$\underline{\omega}^0{}_b \times \underline{q}^b = \underline{\omega}^0{}_1 \times \underline{q}^1 + \underline{\omega}^0{}_2 \times \underline{q}^2 + \underline{\omega}^0{}_3 \times \underline{q}^3$$

$$= -\kappa\left(\epsilon^0{}_{12}\,\underline{q}^2 \times \underline{q}^1 + \epsilon^0{}_{13}\,\underline{q}^3 \times \underline{q}^1 + \epsilon^0{}_{21}\,\underline{q}^1 \times \underline{q}^2 + \epsilon^0{}_{23}\,\underline{q}^3 \times \underline{q}^2 + \epsilon^0{}_{31}\,\underline{q}^1 \times \underline{q}^3 + \epsilon^0{}_{32}\,\underline{q}^2 \times \underline{q}^3\right)$$

$$-(39)$$

6)

Now use:

$$\left.\begin{aligned} \epsilon^{\circ}_{12} &= -\epsilon^{\circ}_{21} = 1 \\ \epsilon^{\circ}_{23} &= -\epsilon^{\circ}_{32} = 1 \\ \epsilon^{\circ}_{31} &= -\epsilon^{\circ}_{13} = 1 \end{aligned}\right\} \quad -(40)$$

and use the complex circular basis to obtain:

$$\tilde{j}^{00} = \frac{2KA^{(0)}}{\mu_0} \underline{\nabla} \cdot \left(\underline{q}^{(2)} \times \underline{q}^{(1)} + \underline{q}^{(1)} \times \underline{q}^{(3)} + \underline{q}^{(3)} \times \underline{q}^{(2)} \right). \quad -(41)$$

For plane waves:

$$\underline{q}^{(1)} = \underline{q}^{(2)*} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(42)$$

and:

$$\underline{\nabla} \cdot \underline{q}^{(2)} \times \underline{q}^{(1)} = 0. \quad -(43)$$

Also:

$$\underline{q}^{(1)} \times \underline{q}^{(3)} = \frac{1}{\sqrt{2}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{vmatrix} e^{i\phi}$$

$$= -i\,\underline{q}^{(2)*} = -i\,\underline{q}^{(1)}$$

and

$$\underline{q}^{(3)} \times \underline{q}^{(2)} = -i\,\underline{q}^{(1)*} = -i\,\underline{q}^{(2)} \quad -(43)$$

so:

$$\boxed{\tilde{j}^{00} = -\frac{2iKA^{(0)}}{\mu_0} \underline{\nabla} \cdot \left(\underline{q}^{(1)} + \underline{q}^{(2)} \right)}$$

7)

Conclusion

In this pure rotational limit it is found that:

$$\tilde{j}^{00} = \tilde{j}^{01} = \tilde{j}^{02} = \tilde{j}^{03} = 0 \quad -(44)$$

This is self-consistent with the fact that in this limit:

$$\tilde{j}^{a} = 0 \quad -(45)$$

whose solution is eqn. (29), Q.E.D.

Therefore the wave equation (26) is self-consistent.

Linear Inhomogeneous Equation with Resonance

Having checked the self-consistency of our method we may now proceed to the simplest structure that gives resonance, using eqns (16) and (21). These give:

$$\underline{\nabla} \cdot \underline{E}^{a} = c\mu_0 \tilde{J}^{0a} \quad -(46)$$

$$\underline{E}^{a} = -\frac{\partial \underline{A}^{a}}{\partial t} - c\,\underline{\nabla} A^{a0} + c\,\omega^{0a}{}_{b}\,\underline{A}^{b} - c\,\underline{\omega}^{a}{}_{b}\,A^{0b} \quad -(47)$$

These two equations give a linear, inhomogeneous structure:

8)

$$-\underline{\nabla}\cdot\left(\frac{\partial \underline{A}^{a}}{\partial t}\right)-c\,\underline{\nabla}\cdot\left(\underline{\nabla}A^{a0}\right)$$

$$+\,c\,\underline{\nabla}\cdot\left(\omega^{0a}{}_{b}\underline{A}^{b}\right)-c\,\underline{\nabla}\cdot\left(A^{0b}\underline{\omega}^{a}{}_{b}\right)$$

$$=c\mu_0\tilde{J}^{0a} \qquad -(48)$$

i.e.

$$-\frac{\partial}{\partial t}\left(\underline{\nabla}\cdot\underline{A}^{a}\right)-c\,\underline{\nabla}\cdot\underline{\nabla}A^{a0}+c\,\underline{\nabla}\cdot\left(\omega^{0a}{}_{b}\underline{A}^{b}\right)$$

$$-\,c\,\underline{\nabla}\cdot\left(\underline{\omega}^{a}{}_{b}A^{0b}\right)=\mu_0 c\,\tilde{J}^{0a} \qquad -(49)$$

i.e.

$$\boxed{\begin{aligned}&\underline{\nabla}\cdot\underline{\nabla}A^{a0}+\frac{1}{c}\frac{\partial}{\partial t}\left(\underline{\nabla}\cdot\underline{A}^{a}\right)\\ &-\underline{\nabla}\cdot\left(\omega^{0a}{}_{b}\underline{A}^{b}\right)+\underline{\nabla}\cdot\left(\underline{\omega}^{a}{}_{b}A^{0b}\right)\\ &=-\mu_0\tilde{J}^{0a}\end{aligned}}$$

In the standard theory the spin connection vanishes, so we obtain:

$$\frac{1}{c}\,\underline{\nabla}\cdot\frac{\partial \underline{A}^{a}}{\partial t}+\underline{\nabla}\cdot\underline{\nabla}A^{a}_{0}=0$$

where A_0 is the electric scalar potential ϕ. With the Lorentz condition:

$$\frac{1}{c}\frac{\partial\phi}{\partial t}+\underline{\nabla}\cdot\underline{A}=0$$

9) and the Laplace operator:

$$\nabla^2 = \underline{\nabla} \cdot \underline{\nabla}$$

We obtain the wave equation for the scalar potential:

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0$$

i.e. $\square \phi = 0$.

In order to obtain resonances the complete equation (49) must be used, without the Lorentz condition.