1) Metrical Method Applied to Orbits, Notes 148(1).

In paper 126 the orbits observed in astronomy were developed using constant angular momentum of spacetime. In paper 147 it was shown that the Minkowski metric has an intrinsic angular velocity:

$$\omega = \left(\frac{\gamma^2 - 1}{\gamma^2}\right)^{1/2} \frac{c}{r} = \frac{v}{r} = \frac{d\phi}{dt} \quad -(1)$$

given by the metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(2)$$

under the condition:

$$dr = dZ = 0. \quad -(3)$$

This is the condition for the X Y plane and:

$$r = \left(r_X^2 + r_Y^2\right)^{1/2} = \text{constant} \quad -(4)$$

i.e the circle. Under these conditions:

$$r^2 d\phi^2 = c^2 \left(dt^2 - d\tau^2\right) = v^2 dt^2 \quad -(5)$$

i.e

$$\omega = \frac{d\phi}{dt} = \frac{v}{r}, \quad -(6)$$

$$dt = \gamma \, d\tau. \quad -(7)$$

The Minkowski metric is alone sufficient to produce angular velocity of spacetime under all conditions.

The velocity in eq. (6) is defined by:

$$\underline{v} = \frac{d\underline{r}}{dt}. \quad -(8)$$

The Einstein energy equation is essentially:

$$\underline{p} = \gamma m \underline{v} \quad -(9)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(10)$$

2)

Therefore eqns. (5) and (8) imply eqn. (9). The relativistic angular momentum must therefore be worked out from the $\underline{v}$ and $\underline{p}$ in eqns (8) and (9) respectively.

Therefore use:

$$\underline{v} = \underline{\omega} \times \underline{r} \qquad -(11)$$

$$\underline{J} = \underline{r} \times \underline{p}, \qquad -(12)$$

giving:

$$\underline{J} = \gamma m \, \underline{r} \times (\underline{\omega} \times \underline{r}) \qquad -(13)$$

$$\underline{J} = \gamma m \left(r^2 \underline{\omega} - (\underline{r} \cdot \underline{\omega}) \underline{r} \right) \qquad -(14)$$

i.e.

$$\underline{J} = \gamma m r^2 \omega \underline{k}. \qquad -(15)$$

In the limit

$$v \ll c \qquad -(16)$$

$$\underline{J} \rightarrow m r^2 \omega \underline{k} = m r v \underline{k} \qquad -(17)$$

and

$$J = m r v \qquad -(18)$$

This is the starting point of paper 126, in which

$$J = \text{constant} \qquad -(19)$$

The Minkowski metric is alone sufficient to produce the spacetime angular momentum (18).

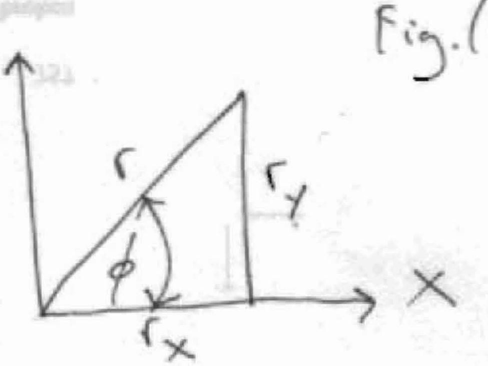

Fig. (1)

For the circular orbit:

$$r = \text{constant} \qquad -(20)$$

$$dr = 0 \qquad -(21)$$

For other types of orbit however:

$$dr \neq 0 \qquad -(22)$$

3)

So:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(23)$$

in the plane defined by

$$dZ = 0 \quad -(24)$$

Elliptical Orbit

$$r(t) = \frac{\alpha}{1 + \epsilon \cos \phi(t)} \quad (25)$$

Relativistic Keplerian Orbit

$$r(t) = \frac{\alpha}{1 + \epsilon \cos\left(\left(1 - \frac{\beta}{\alpha}\right)\phi(t)\right)} \quad -(26)$$

Log Spiral Orbit

$$r(t) = r \exp(b\,\phi(t)) \quad -(27)$$

The metrics for these orbits can be worked out by expressing dr as a function of $d\phi$. The general metric of this planar type is obtained from

$$r = r(\phi, t). \quad -(28)$$

For example, from eq. (27):

$$v^2 dt^2 = dr^2 + r^2 d\phi^2 \quad -(29)$$

$$v = \left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right)^{1/2} \quad -(30)$$

as given in paper 126.

4)

From eq. (27):

$$\boxed{dr = b r \, d\phi} \quad -(31)$$

so in eq. (25)

$$v^2 dt^2 = r^2 (1+b^2) \, d\phi^2 \quad -(32)$$

and

$$\omega = \frac{d\phi}{dt} = \frac{v}{r}(1+b^2)^{-1}. \quad -(33)$$

The metric of a whirlpool galaxy is therefore:

$$\boxed{ds^2 = c^2 d\tau^2 = c^2 dt^2 - (1+b^2) r^2 d\phi^2.} \quad -(34)$$

As

$$b \rightarrow 0 \quad -(35)$$

this becomes the Minkowski metric in an X-Y plane. Both types of metric have an intrinsic angular velocity ω and constant angular momentum:

$$J = mrv \quad -(36)$$

$$= mr^2\omega$$

For a circular orbit:

$$r = \text{constant}, \quad \omega = \text{constant} \quad -(37)$$

and for a log spiral orbit:

$$J = \frac{mvr}{(1+b^2)} = \text{constant} \quad -(38)$$

5) The velocity curve of a spiral galaxy is such that as:

$$r \to \infty \quad \text{— (39)}$$

then

$$v \to \text{constant} \quad \text{— (40)}$$

In the limit (39):

$$\frac{r}{1+b^2} \to \text{constant} \quad \text{— (41)}$$

This equation was further developed in paper 123. In Newtonian dynamics approximated very well by a circular orbit in the solar system, then as:

$$r \to \infty \quad \text{— (42)}$$

we have

$$v \to 0 \quad \text{— (43)}$$

This result is completely different from the observed eq. (40). Note carefully that the Einstein field equation gives the result (26), which reduces to the accurate Newtonian / Keplerian orbit (25) when:

$$\beta << \alpha. \quad \text{— (44)}$$

Neither eq. (25) nor eq. (26) gives the observed result (40), whereas the simple adjustment (34) of the Minkowski metric gives this result.

1) 148(2): The Link between the Relativistic Momentum and the Minkowski Metric.

The following derivation implies that the relativistic momentum

$$p = \gamma m v = \gamma m \frac{dr}{dt} = m \frac{dr}{d\tau} \quad -(1)$$

implies the Minkowski metric and vice versa. This derivation illustrates the importance of the metric to all physics, because the Einstein energy equation and Dirac equation are also derived directly from eq. (1). Therefore by appropriate definition of the metric, different types of classical and quantum dynamics are obtained. This method can also be extended to classical and quantum electrodynamics.

From eq. (1):

$$p^2 = \gamma^2 m^2 v^2 \quad -(2)$$

where

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad -(3)$$

Therefore

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \quad -(4)$$

and eq. (2) is

$$p^2 c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) \quad -(5)$$

i.e.

$$m^2 \left(1 - \frac{1}{\gamma^2}\right) = \frac{p^2}{\gamma^2 c^2} = \frac{m^2}{\gamma^2 c^2} \left(\frac{dr}{d\tau}\right)^2 \quad -(6)$$

which implies

$$\frac{1}{\gamma^2} + \frac{1}{\gamma^2 c^2} \left(\frac{dr}{d\tau}\right)^2 = 1 \quad -(7)$$

and

$$\left(\frac{dt}{d\tau}\right)^2 = \gamma^2 = 1 + \frac{1}{c^2} \left(\frac{dr}{d\tau}\right)^2 \quad -(8)$$

2) i.e.

$$c^2 dt^2 = c^2 d\tau^2 + d\underline{r} \cdot d\underline{r} \quad -(9)$$

and the Minkowski metric:

$$c^2 d\tau^2 = ds^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(10)$$

Q.E.D.

Therefore:

$$\boxed{\begin{aligned} \underline{p} &= m\gamma \underline{v} = m \frac{d\underline{r}}{d\tau} \quad -(11) \\ &\Updownarrow \\ c^2 d\tau^2 &= ds^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \end{aligned}}$$

In note 148(1) it was shown that for a whirlpool galaxy, the Minkowski metric is changed to:

$$\boxed{ds^2 = c^2 d\tau^2 = c^2 dt^2 - (1+b)^2 r^2 d\phi^2} \quad -(12)$$

Eq. (12) is enough to give the structure of the whirlpool galaxy. It is therefore possible to evaluate the relativistic momentum and Einstein energy equation for a whirlpool galaxy. For the X - Y plane defined by

$$dZ^2 = 0 \quad -(13)$$

the quantity $d\underline{r} \cdot d\underline{r}$ is:

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\phi^2 \quad -(14)$$

in cylindrical polar coordinates. For a circular orbit:

3)

$$d\underline{r} \cdot d\underline{r} = r^2 d\phi^2 \quad -(15)$$

because:

$$dr = 0. \quad -(16)$$

and for the orbit of a star in a whirlpool galaxy:

$$d\underline{r} \cdot d\underline{r} = (1+b^2) r^2 d\phi^2 \quad -(17)$$

Now work out the relativistic momentum from eqns. (15) and (17). This method illustrates that a circular orbit and galactic orbit cause the effect of a property is purely of geometry and the Minkowski metric or metric (17). This conclusion is true for all orbits and orbits in general.

For the circular orbit:

$$\boxed{p^2 = m^2 \left(\frac{dr}{d\tau}\right)^2} \quad -(18)$$

This is a useful equation transforming the metric into the Einstein energy equation:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(19)$$

$$p^\mu p_\mu = m^2 c^4, \quad -(20)$$

i.e.

or

$$\boxed{E^2 - E_0^2 = c^2 m^2 \left(\frac{dr}{d\tau}\right)^2.} \quad -(21)$$

For the circular orbit:

$$\left(\frac{dr}{d\tau}\right)^2 = r^2 \left(\frac{d\phi}{d\tau}\right)^2 = \gamma^2 r^2 \left(\frac{d\phi}{dt}\right)^2 \quad -(22)$$

4) The angular velocity is:

$$\omega = \frac{d\phi}{dt}. \quad -(23)$$

So

$$p = \gamma m r \omega \quad -(24)$$

i.e.

$$\boxed{v = r\omega} : \quad -(25)$$

From eq. (21):

$$E^2 - E_0^2 = (\gamma c m r \omega)^2. \quad -(26)$$

Finally use:

$$E = \gamma E_0 = \gamma m c^2 \quad -(27)$$

so

$$\left(\frac{\gamma^2 - 1}{\gamma^2}\right) m^2 c^4 = m^2 c^2 v^2 \quad -(28)$$

i.e.

$$\frac{v}{c} = \left(\frac{\gamma^2 - 1}{\gamma^2}\right)^{1/2} \quad -(29)$$

Q.E.D.

The total energy or relativistic kinetic energy is

$$E = T = mc^2(\gamma - 1) \quad -(30)$$

$$= mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - mc^2$$

$$\sim mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} - \dots\right) - mc^2$$

$$\sim \frac{1}{2} m v^2$$

5) if: $v << c$. — (31)

So the circular orbit is being considered as a particle going around a circle, i.e as a purely kinetic problem without yet considering potential energy. Therefore the Minkowski metric has no potential energy and is the simplest solution of the ECE Orbital Theorem. In order to find what is actually keeping the particle in a circular orbit, other solutions of the ECE Orbital Theorem are needed.

For the log spiral orbit of a star in a whirlpool galaxy the metric was shown in note 148(1) to be:

$$v^2 dt^2 = c^2 (dt^2 - d\tau^2) = (1+b^2) r^2 d\phi^2 \quad -(32)$$

$$\text{i.e.} \quad \omega = \frac{d\phi}{dt} = \frac{v}{r}(1+b^2)^{-1} \quad -(33)$$

The metric is:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - b^2 r d\phi^2 - r^2 d\phi^2 - dZ^2$$
$$= c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2$$
$$-(34)$$

$$\text{i.e.} \quad \frac{m^2}{\gamma^2}(1+b^2)\frac{r^2}{c^2}\left(\frac{d\phi}{d\tau}\right)^2 = \left(1 - \frac{1}{\gamma^2}\right) m^2 \quad -(35)$$

and so in eq. (18):

6)

$$\left(\frac{dr}{d\tau}\right)^2 = (1+b^2)\, r^2 \left(\frac{d\phi}{d\tau}\right)^2 \quad -(36)$$

$$= \gamma^2 (1+b^2)\, c^2 \omega^2 r^2$$

Therefore

$$\boxed{\underline{p} = \gamma (1+b^2)\, m\underline{v}} \quad -(37)$$

and the familiar Newtonian relation:

$$\underline{p} = m\underline{v} \quad -(38)$$

no longer holds true in a whirlpool galaxy.

In UFT 123 it was shown that the factor b can be expressed in terms of the constant angular momentum of spacetime as follows:

$$b = \left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{1/2} \quad -(39)$$

where

$$J = \frac{mvr}{1+b^2} = \text{constant} \quad -(40)$$

$$r_0 = \frac{J}{mv} \quad -(41)$$

implying a non-Newtonian force law:

$$F = -\frac{mv^2}{r} \quad -(42)$$

7)

DISCUSSION

These are the observed dynamics of the log spiral trajectory of a star in a whirlpool galaxy. The dynamics are not governed at all by "dark matter". Note carefully that the metric (34) is a solution of the orbital theorem of UFT III, in which

$$dr^2 = b^2 r^2 d\phi^2 \quad - (44)$$

of the log spiral orbit:

$$r = r_0 e^{b\phi} \quad - (45)$$

Note carefully that the concept of gravitation has not been used at all. The stars evolve outwards in a whirlpool pattern due to J, the constant angular momentum of spacetime itself.

Eq. (45) is:

$$\boxed{\frac{dr}{d\phi} = br} \quad - (46)$$

i.e.

$$dr = br\,d\phi \quad - (47)$$

giving eq. (44).

148(3): Metric Base

The elliptical orbit is:

$$r = \frac{d}{1+\epsilon\cos\phi} \quad -(1)$$

where $0 < \epsilon < 1$ is the eccentricity. For the parabola:

$$\epsilon = 1 \quad -(2)$$

and for the hyperbola

$$\epsilon > 1. \quad -(3)$$

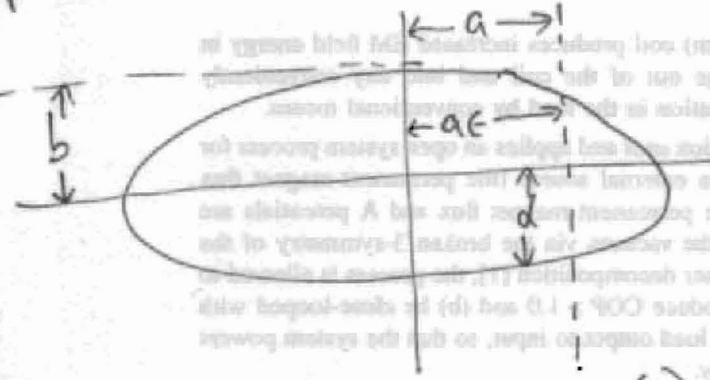

Fig (1)

and for the circle:

$$\epsilon = 0 \quad (4)$$

Consider the metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(5)$$

for the conical section defined by eq. (1) and express dr in terms of $d\phi$. Differentiating eq. (1):

$$\frac{dr}{d\phi} = \frac{d\epsilon\sin\phi}{(1+\epsilon\cos\phi)^2} \quad -(6)$$

So for the circle, ellipse, parabola and hyperbola:

$$dr = \frac{d\epsilon\sin\phi}{(1+\epsilon\cos\phi)^2}\, d\phi \quad -(7)$$

The metric (5) therefore becomes:

$$ds^2 = c^2 dt^2 - \left(\frac{d^2\epsilon^2\sin^2\phi}{(1+\epsilon\cos\phi)^4}\right) d\phi^2 - r^2 d\phi^2 - dZ^2 \quad -(8)$$

2) and because the orbit is in the XY plane:

$$dZ = 0 \qquad -(9)$$

so

$$c^2\left(dt^2 - d\tau^2\right) = v^2 dt^2 = \left(r^2 + \frac{\alpha^2 \epsilon^2 \sin^2\phi}{(1+\epsilon\cos\phi)^4}\right) d\phi^2 \qquad -(10)$$

The angular velocity is:

$$\omega = \frac{d\phi}{dt} = v\left(r^2 + \left(\frac{\alpha\epsilon\sin\phi}{(1+\epsilon\cos\phi)^2}\right)^2\right)^{-1/2} \qquad -(11)$$

$$\boxed{\omega = \frac{v}{r}\left(1 + \left(\frac{\epsilon\sin\phi}{1+\epsilon\cos\phi}\right)^2\right)^{-1/2}} \qquad -(12)$$

For the circle:

$$\omega = \frac{v}{r} \qquad -(13)$$

and for the logarithmic spiral:

$$\omega = \frac{v}{r}\left(1+b^2\right)^{-1/2} \qquad -(14)$$

in which

$$b = \frac{1}{\phi}\log_e\left(\frac{r}{r_0}\right) \qquad -(15)$$

i.e.

$$r = r_0 e^{b\phi} \qquad -(16)$$

By considering the angular velocity the relation between the orbits becomes clear; the angular velocity being easily calculated from the metric of type (5).

In general, the metric for all these orbits is:

3) $$ds^2 = c^2 d\tau^2 = c^2 dt^2 - \left(f^2(r,\phi) + r^2\right) d\phi^2 \quad -(17)$$

i.e. $$v^2 dt^2 = \left(f^2 + r^2\right) d\phi^2 \quad -(18)$$

or $$\boxed{\omega = \frac{d\phi}{dt} = \frac{v}{\left(f^2 + r^2\right)^{1/2}}} \quad -(19)$$

so $$\phi = \int \frac{v}{\left(f^2 + r^2\right)^{1/2}} dt \quad -(20)$$

In eq. (17): $$dr = f(r,\phi)\, d\phi \quad -(21).$$

and $$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(18)$$

which is a simple solution of the orbital theorem of UFT 111 with the additional link (21) between dr and $d\phi$. Using eq. (6) of note 148(2):

$$p = m \frac{dr}{d\tau} \quad -(22)$$

is the magnitude of the relativistic momentum. From eqns (21) and (22)

$$p = m \frac{d}{d\tau}\left(f(r,\phi)\, d\phi\right) \quad -(23)$$

i.e. $$p = \gamma m \frac{d}{dt}\left(f(r,\phi)\, d\phi\right). \quad -(24)$$

4)

Eqn. (24) is:

$$\boxed{p = \gamma m f(r, \phi) \frac{d\phi}{dt}} \qquad -(25)$$

because:

$$p = m \gamma \frac{dr}{dt} = m \gamma \frac{dr}{d\phi} \frac{d\phi}{dt} \qquad -(26)$$

$$= \gamma m f(r, \phi) \frac{d\phi}{dt}$$

Q.E.D.

From eqs. (19) and (25):

$$p = \gamma m f(r, \phi)\, \omega = \frac{\gamma m f(r, \phi)\, v}{\left(f^2(r, \phi) + r^2\right)^{1/2}} . \qquad -(27)$$

and

$$p = \frac{f \gamma m v}{\left(f^2 + r^2\right)^{1/2}} \qquad -(28)$$

SUMMARY

$$p = \gamma m \frac{dr}{dt} = \gamma m \frac{dr}{d\phi} \frac{d\phi}{dt} = \gamma m f \omega \qquad -(29)$$

where $f = \frac{dr}{d\phi}$, $\omega = \frac{d\phi}{dt}$.

and

$$\omega = \frac{v}{\left(f^2 + r^2\right)^{1/2}} . \qquad -(30)$$

5) So

$$p = \left(\frac{f}{(f^2 + r^2)^{1/2}} \right) \gamma m v \quad -(31)$$

where $f = dr / d\phi$

Conical Sections

$$f = \frac{dr}{d\phi} = \frac{d \, \epsilon \sin\phi}{(1 + \epsilon \cos\phi)^2} \quad -(32)$$

Logarithmic Spiral

$$\boxed{f = \frac{dr}{d\phi} = br} \quad (33)$$

The conical section function in eq (32) is

$$f = \frac{\epsilon \sin\phi}{1 + \epsilon \cos\phi} \cdot r \quad -(34)$$

$$\boxed{f = \frac{\epsilon r^2 \sin\phi}{d}} \quad -(35)$$

So

$$\boxed{p = \left(1 + \left(\frac{\epsilon r}{d}\right)^2 \sin^2\phi\right)^{1/2} \gamma m v} \quad -(36)$$

for all conical sections

b) The familiar:

$$\underline{p} = \gamma m \underline{v} \quad -(37)$$

is obtained for the circle:

$$\epsilon = 0. \quad -(38)$$

For the logarithmic spiral:

$$\boxed{\underline{p} = \frac{b}{(1+b^2)^{1/2}} \gamma m \underline{v}} \quad -(39)$$

and this corrects eq. (37) of notes 148(2)

The angular momentum is:

$$\underline{J} = \underline{r} \times \underline{p} \quad -(40)$$

and is a constant of motion.

In the usual lagrangian development of Keplerian orbits, the lagrangian is:

$$\mathcal{L} = \frac{1}{2}\mu v^2 - U(r) \quad -(41)$$

where μ is the reduced mass and:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(42)$$

so the angular momentum is defined as

$$L = mr^2 \frac{d\phi}{dt} = \text{constant} \quad -(43)$$

and it is assumed that:

7)

$$\underline{p} = m\underline{v} \quad -(44)$$

as a matter of definition. The definition (44) gives the elliptical orbit (1) if and only if the potential energy $u(r)$ is defined as non-zero and given by the gravitational potential. The metric used in this method is the Cartesian metric*, not the Minkowski metric or the metrics of this note. In the method of this note there is no potential energy but the orbit is the same, eq. (1)

Finally the angular velocity for the orbits of type (1) can be simplified to:

$$\omega = \frac{d\phi}{dt} = \frac{v}{r}\left(1 + \left(\frac{\epsilon r}{\alpha}\sin\phi\right)^2\right)^{-1/2} \quad -(45)$$

$$\sim \frac{v}{r}\left(1 - \frac{1}{2}\left(\frac{\epsilon r}{\alpha}\sin\phi\right)^2 + \cdots\right) \quad -(46)$$

if

$$\epsilon << 1,$$

as is the case for the Earth.

$$* \quad ds^2 = dX^2 + dY^2 + dZ^2$$

$$= dr^2 + r^2 d\phi^2 + dZ^2$$

not involving dt at all.

148(4): On the Absence of Potential Energy in Orbits

The Newtonian theory of orbits is given by Marion and Thornton pp. 246 ff., third edition, 1988. Using the standard theory itself, it is shown in this note that orbits can be described using a metric, without use of potential energy or gravitation. The standard theory of orbits uses the hamiltonian:

$$H = \text{constant} = T + U \quad -(1)$$

and lagrangian

$$\mathcal{L} = T - U \quad -(2)$$

where T is the kinetic and U the potential energy. It uses the reduced mass μ and constant angular momentum in a plane

$$\underline{L} = \underline{r} \times \underline{p} = \text{constant} \quad -(3).$$

The kinetic energy is

$$T = \frac{1}{2}\mu v^2 = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \quad -(4)$$

First note that T can be derived directly from the metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(5)$$

for the plane

$$dZ^2 = 0. \quad -(6)$$

From eq (5):

$$c^2(dt^2 - d\tau^2) = v^2 dt^2 = dr^2 + r^2 d\phi^2 \quad -(7)$$

so

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(8)$$

giving eq. (4), Q.E.D.

If r is a constant:

$$dr = 0 \quad -(9)$$

2) and
$$v = r\frac{d\phi}{dt} = r\omega \quad -(10)$$

so
$$\omega = v / r. \quad -(11)$$

Note carefully that the kinetic energy and angular velocity are derived directly from the metric, without using the concept of H or $\mathcal{L}$, and without using potential energy U at all.

Therefore the circular orbit (11) can be described using pure geometry and no gravitation, which is the standard theory of force. In the standard theory gravitation is introduced through U and the concept of F, specifically the centrally directed force.

The standard theory uses the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \quad -(12)$$

and
$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(13)$$

where
$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) \quad -(14)$$

From eq. (12):
$$\boxed{l = \mu r^2 \omega = \text{constant}} \quad -(15)$$

where
$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega. \quad -(16)$$

This is Kepler's second law. This is a general result for any central orbit and does not use potential energy because it is derived from the kinetic energy as follows:

$$l = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \mu r^2 \omega . \quad -(17)$$

If $$u(r) = f(r) \quad -(18)$$

and if $$u \neq f(\theta) \quad -(19)$$

then Kepler's second law is unaffected, and is a purely kinetic law that can be derived from the metric. Conversely metrics can be constructed from any known orbit as in earlier notes for paper 148.

The standard theory proceeds using:

$$H = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) \quad -(20)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r).$$

so $$\dot{r} = \frac{dr}{dt} = \left(\frac{2}{\mu}(H - U) - \frac{l^2}{\mu^2 r^2}\right)^{1/2} \quad -(21)$$

For a circular orbit:

$$\frac{dr}{dt} = 0 . \quad -(22)$$

The term $$U_c = \frac{l^2}{2\mu r^2} \quad -(23)$$

is defined as the positive valued potential energy of the centrifugal force:

4)

$$\boxed{F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^3} = \mu r \omega^2} \quad -(24)$$

The effective potential energy is then defined as

$$V(r) = U(r) + U_c(r) \quad -(25)$$

From eq. (13):

$$\mu\left(\ddot{r} - r\dot{\phi}^2\right) = -\frac{\partial U}{\partial r} = F(r). \quad -(26)$$

For a circular orbit $\ddot{r} = 0$ $\quad -(27)$

so

$$\boxed{F = -\frac{\partial U}{\partial r} = -\mu r \omega^2} \quad -(28)$$

which is the negative attractive force. Therefore for the circular orbit:

$$\boxed{V(r) = U(r) + U_c(r) = 0} \quad -(29)$$

The standard explanation is that the attractive force is therefore balanced by the centrifugal force. Note carefully that the potential energy is introduced, but then discarded. The angular velocity and kinetic energy of the orbit can be obtained purely by consideration of the metric.

These considerations are true for any orbit, but for simplicity and clarity consider the

5) circular orbit:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(30)$$

with $\epsilon = 0.$ $-(31)$

In this case:

$$F = -F_c = -\frac{\partial U}{\partial r} = \frac{\partial U_c}{\partial r} \quad -(32)$$

$$= -\mu r \omega^2$$

and

$$T = \frac{1}{2}\mu r^2 \omega^2 \quad -(33)$$

so

$$\boxed{F = -F_c = -\frac{2T}{r}} \quad . \quad -(34)$$

Therefore the apparently derived forces are defined purely by the kinetic energy, which is defined purely by the metric. The potential energies are also defined purely by kinetic energy.

$$\frac{\partial U}{\partial r} = \frac{2T}{r} = -\frac{\partial U_c}{\partial r}. \quad -(35)$$

The metric contains all the information needed to define the concepts of U, F, U_c and F_c. These are all defined by T and r. The orbit is also defined by the metric, and is:

6)

$$r = \alpha = \frac{v}{\omega} = \text{constant.} \quad -(36)$$

For a circular orbit, r, v and ω are all constant.

The Concept of Gravitational Force

This is the familiar idea of an attractive force between the masses m and M directed along the line joining the two masses. It was introduced by Newton in about 1665. Earlier, Kepler had shown that the orbit of Mars was an ellipse, eqn (30) with

$$0 < \epsilon < 1 \quad -(37)$$

Mathematically, the ellipse (30) is obtained from the hamiltonian (20) using:

$$U(r) = -\frac{k}{r} \quad -(38)$$

so

$$F(r) = -\frac{k}{r^2} \quad -(39)$$

This procedure gives eqn. (30) with:

$$\alpha = \frac{l^2}{\mu k}, \quad \epsilon = \left(1 + \frac{2Hl^2}{\mu k^2}\right)^{1/2} \quad -(40)$$

For the circular orbit:

$$\alpha = r, \quad \epsilon = 0, \quad -(41)$$

and

$$k = \mu r^3 \omega^2 = GmM \quad -(42)$$

where

$$G = (6.6726 \pm 0.0005) \times 10^{-11} \, m^3 s^{-2} kg^{-1} \quad -(43)$$

7) is known as Newton's constant. For a circular orbit it is:

$$G = \frac{\mu}{mM} r^3 \omega^2 \quad - (44)$$

where

$$\mu = \frac{mM}{m+M} \quad - (45)$$

is the reduced mass. Therefore

$$G = \frac{r^3 \omega^2}{m+M} \sim \frac{r^3 \omega^2}{M} \quad - (46)$$

if

$$M \gg m \quad - (47)$$

Therefore:

$$\boxed{r^3 \omega^2 = GM = \text{constant}} \quad - (47)$$

for a circular orbit wrt $M \gg m$. The same orbit is obtained by using the hamiltonian:

$$H = T + V \quad - (48)$$

where

$$V = -\frac{k}{r} + \frac{k}{r} \quad - (49)$$

$$= -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}.$$

In fact,

$$\left.\begin{aligned} V &= U + U_c \\ &= U - U \end{aligned}\right\} \quad - (50)$$

8) so mathematically, U can be any function of r, but not a function of ϕ.

The circular orbit is in fact the metric:

$$ds^2 = c^2 dt^2 - r^2 d\phi^2 \quad -(51)$$

from which:

$$\omega = \frac{d\phi}{dt} = \frac{v}{r} \quad -(52)$$

and

$$T = \frac{1}{2}\mu r^2 \omega^2 = \frac{1}{2}\mu v^2 \quad -(53)$$

All that is observed experimentally is the orbit:

$$r = \alpha = \frac{v}{\omega} = \text{constant} \quad -(54)$$

The elliptical orbit is in fact the metric:

$$ds^2 = c^2 dt^2 - r^2\left(1 + \left(\frac{\epsilon r \sin\phi}{\alpha}\right)^2\right) d\phi^2 \quad -(55)$$

i.e.

$$\boxed{v^2 dt^2 = r^2 d\phi^2 \left(1 + \left(\frac{\epsilon r \sin\phi}{\alpha}\right)^2\right)} \quad -(56)$$

$$-(57)$$

$$\omega = \frac{d\phi}{dt}$$

giving:

$$T = \frac{1}{2}\mu\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right) \quad -(58)$$

and

$$r = \frac{\alpha}{1 + \epsilon\cos\phi} \quad -(59)$$

$$\frac{dr}{d\phi} = \frac{\epsilon}{\alpha} r^2 \sin\phi \quad -(60)$$

9) Therefore:

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{\epsilon}{\alpha} r^2 \sin\phi \frac{d\phi}{dt} \quad -(61)$$

so:

$$\boxed{\begin{aligned} T &= \frac{1}{2}\mu r^2\left(1 + \left(\frac{\epsilon \sin\phi}{1+\epsilon\cos\phi}\right)^2\right)\left(\frac{d\phi}{dt}\right)^2 \quad -(62) \\ \omega &= \frac{d\phi}{dt} \end{aligned}}$$

All that are ever observed are ω and T. The relativistic Keplerian orbit, the precessing ellipse, is merely a variation on this theme, as are the orbits of stars in a whirlpool galaxy.

CONCLUSION

In cosmology all that is observed is the metric relevant to an orbit. These orbits actually do not prove an inverse square law, as can be seen from equations such as (49) and (50). The hamiltonian of all orbits is purely kinetic:

$$H_1 = T \quad -(63)$$

and H_1 is given directly from the metric which is given by direct observation.

1) Note 148(5): Metric for Precessing Ellipse and Free Fall

From experimental observation in the solar system, and from the orbital theorem of UFT 111, the relativistic Kepler problem is described by a precessing ellipse, whose metric is:

$$ds^2 = c^2 d\tau^2 = x^2 c^2 dt^2 - \frac{dr^2}{x^2} - r^2 d\phi^2 - dZ^2 \quad -(1)$$

in the plane XY:

$$dZ = 0. \quad -(2)$$

Here

$$x = \left(1 - \frac{2GM}{R}\right)^{1/2} \quad -(3)$$

where M is the mass of a "gravitating object", G is Newton's constant and R the distance between M and an attracted object of mass m. The proper time interval is:

$$d\tau^2 = \left(1 - \frac{v^2}{c^2}\right) dt^2 \quad -(4)$$

where v is the magnitude of the linear velocity of m and c is the vacuum speed of light.

Eq. (1) is:

$$c^2\left(x^2 dt^2 - d\tau^2\right) = \frac{dr^2}{x^2} + r^2 d\phi^2 \quad -(5)$$

i.e.

$$\boxed{v^2 = \frac{2MG}{R} + \left(1 - \frac{2MG}{R}\right)^{-1}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2} \quad -(6)$$

From previous notes the elliptical orbit is:

$$v^2 = \left(\frac{\epsilon r^2 \sin\phi}{\alpha}\right)^2 \left(\frac{d\phi}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(7)$$

For both eqs. (6) and eq. (7) the kinetic energy

2) in the limit:

$$m << M \quad -(8)$$

is

$$T = \frac{1}{2} m v^2 \quad -(9)$$

and the Hamiltonian is

$$H = T \quad -(10)$$

i.e. is purely kinetic.

For an approximately circular orbit:

$$dr \sim 0 \quad -(11)$$

so

$$v^2 \sim \frac{2MG}{R} + r^2 \left(\frac{d\phi}{dt}\right)^2 \quad (12)$$

$$\boxed{v^2 = \frac{2MG}{R} + r^2 \omega^2} \quad (13)$$

where

$$\omega = \frac{d\phi}{dt} \quad -(14)$$

by definition.

The pure circular orbit is

$$v = \omega r \quad -(15)$$

i.e.

$$r^2 \omega^2 >> \frac{2MG}{R} \quad -(16)$$

If for some reason

$$\frac{2MG}{R} >> r^2 \omega^2 \quad -(17)$$

3) then:

$$v^2 \sim \frac{2MG}{R} \quad - (18)$$

and

$$\boxed{T = \frac{1}{2} m v^2 = \frac{mMG}{R}} \quad - (19)$$

This is a free fall out of orbit. Conventionally it is described by

$$T = -U = \int F dR = m \int g \, dR = \frac{mMG}{R} \quad - (20)$$

where the acceleration due to gravity is

$$g = \frac{MG}{R^2} \quad - (21)$$

The conventional description uses the concept of force F and potential energy U:

$$F = -\frac{\partial U}{\partial R} \quad - (22)$$

However, the metrical description given by eq. (19) is the above [illegible] limits the concepts of force and potential energy are replaced by a purely geometrical metric. The values of m, M and G are derived from observation. This is how an orbit is transformed into direct interaction between m and M.

This development suggests that it is possible to derive a metric for the interaction of two charges e_1 and e_2.

4) This metric for the attraction of two charges is eq. (1) with:

$$x = \left(1 - \frac{2 e_1}{4 \pi \epsilon_0 R} \right)^{1/2} \quad - (23)$$

and for the repulsion of two charges is

$$x = \left(1 + \frac{2 e_1}{4 \pi \epsilon_0 R} \right)^{1/2} \quad - (24)$$

where ϵ_0 is the vacuum permittivity.

In the free fall limit:

$$v^2 \sim \frac{2 e_1}{4 \pi \epsilon_0 R} \quad - (25)$$

so

$$\boxed{T = \mp U = \pm \frac{e_1 e_2}{4 \pi \epsilon_0 R}} \quad - (26)$$

where $+$ denotes attraction and $-$ repulsion. But conventionally, U is a potential energy, but again in the metrical description, the problem is purely kinetic. It is well known that eqn. (26) is the Coulomb law, which in a more complete ECE description involves the spin connection. Here it has been derived from a suggested electrodynamic metric.

148(6): Some Concepts of General Relativity, and Background to Metrical Analysis.

The proper time for the general metric is defined by:

$$d\tau^2 = \frac{1}{c^2} g_{\mu\nu} \, dx^\mu dx^\nu \quad — (1)$$

and the geodesic equation by:

$$\frac{d^2 x^\mu}{dq^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{dq} \frac{dx^\lambda}{dq} = 0 \quad — (2)$$

The Hamilton Jacobi equation is

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = m^2 c^2 \quad — (3)$$

where the action is defined by

$$S = -Et + L\phi + S_r(r) \quad — (4)$$

The kinetic energy is defined by

$$T = \frac{1}{2} m c^2 = \frac{1}{2} m \left(\frac{ds}{d\tau} \right)^2 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} m \quad — (5)$$

and Hamilton's Principle by

$$\delta \int T \, d\tau = 0 \quad (6)$$

The Lagrange equations are

$$\frac{d}{d\tau} \left(\frac{\partial T}{\partial \dot{x}^\mu} \right) = \frac{\partial T}{\partial x^\mu} \quad — (7)$$

In ECE theory the gravitational metric can be derived as a solution of the orbital theorem of UFT 111 and is:

2)

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad -(8)$$

where

$$r_s = \frac{2GM}{c^2} \quad -(9)$$

Now define the constants of motion:

$$\frac{E}{mc^2} = \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau}, \quad \frac{L}{m} = r^2 \frac{d\phi}{d\tau} \quad -(10)$$

These come from the Lagrange equations w.r.t. metric (8), which gives the kinetic energy:

$$\frac{T}{m} = \left(1 - \frac{r_s}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{1}{1 - \frac{r_s}{r}}\right)\left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \quad -(11)$$

Using eq. (7):

$$\frac{d}{d\tau}\left(r^2 \frac{d\phi}{d\tau}\right) = 0 \quad -(12)$$

$$\frac{d}{d\tau}\left(\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau}\right) = 0 \quad -(13)$$

$$\frac{d}{d\tau}\left(\left(1 - \frac{r_s}{r}\right)^{-1}\frac{dr}{d\tau}\right) = 0 \quad -(14)$$

Therefore

$$L = mr^2 \frac{d\phi}{d\tau} = \text{constant} \quad -(15)$$

$$E = mc^2 \left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau} = \text{constant} \quad -(16)$$

and

$$v = \left(1 - \frac{r_s}{r}\right)^{-1}\frac{dr}{d\tau} = \text{constant} \quad -(17)$$

It follows that:

3)

$$\frac{r^2}{bc}\frac{d\phi}{dt} = 1 - \frac{r_s}{r} \quad -(18)$$

i.e.

$$\frac{E}{L}\left(\frac{r}{c}\right)^2 \omega = 1 - \frac{r_s}{r} \quad -(19)$$

where

$$\omega = \frac{d\phi}{dt} \quad -(20)$$

The angular velocity is therefore:

$$\boxed{\omega = \frac{L}{E}\left(\frac{c}{r}\right)^2\left(1 - \frac{r_s}{r}\right)} \quad -(21)$$

It is found that

$$\omega = \frac{d\phi}{d\tau}\frac{d\tau}{dt} \quad -(22)$$

and

$$r^2\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau}\,\omega = \text{constant.} \quad -(23)$$

Using these definitions it is found that:

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}m\left(\left(\frac{E}{mc}\right)^2 - \left(1 - \frac{r_s}{r}\right)\left(c^2 + \frac{L^2}{m^2r^2}\right)\right)$$

$$= \left(\frac{E^2}{2mc^2} - \frac{1}{2}mc^2\right) + \left(\frac{mM}{r}G - \frac{L^2}{2mr^2} + \frac{GML^2}{c^2mr^3}\right) \quad -(24)$$

$$= \frac{E^2}{2mc^2} - T - V \quad -(25)$$

where

$$T = \frac{1}{2}mc^2 \quad -(26)$$

$$V = -\frac{mMG}{r} + \frac{L^2}{2mr^2} - \frac{GML^2}{c^2mr^3} \quad -(27)$$

4) Here T is the kinetic energy defined in eq. (5), E is the total energy, defined in eq. (16), and V is the effective potential energy, made up of Newtonian, centrifugal and relativistic (essentially), the attraction force from eq (27) is

$$F_a = -\frac{\partial V_a}{\partial t} = -\frac{mMG}{r^2} - \frac{3GML^2}{c^2 m r^4} \quad -(28)$$

and the centrifugal force is

$$F_c = -\frac{\partial V_c}{\partial t} = +\frac{L^2}{m r^3} \quad -(29)$$

The attraction and repulsion forces are the same as for Newtonian dynamics except for the second term in eq. (28). The approach taken in a textbook such as Marion and Thornton is to consider this term as a small perturbation to Newtonian dynamics, whose lagrangian is considered to be:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \quad -(30)$$

This is not a satisfactory approach because it just "forces" the general relativity to become Newtonian dynamics. However, the Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(31).$$

5)

and is rewritten as:

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{L^2}\frac{1}{u^2}F(u) \quad -(32)$$

If we use $\mu \sim m$ $\quad -(33)$

this equation becomes:

$$\frac{d^2u}{d\phi^2} + u = \frac{Gm^2M}{L^2} + \frac{3GM}{c^2}u^2 \quad -(34)$$

where $u = \frac{1}{r}$ $\quad (35)$

(Marion and Thornton, eq. (7.74)). Eq. (34) is the equation of a precessing ellipse:

$$\frac{\alpha}{r} = 1 + \epsilon\cos\left(\phi - \frac{\delta}{\alpha}\phi\right) \quad -(36)$$

where $\alpha = \frac{L^2}{Gm^2M}$, $\delta = \frac{3GM}{c^2}$ $\quad -(37)$

So

$$\boxed{\frac{\alpha}{r} = 1 + \epsilon\cos\left(\phi\left(1 - 3\left(\frac{mMG}{cL}\right)^2\right)\right)} \quad -(38)$$

This is a simple equation:

$$\frac{\alpha}{r} = 1 + \epsilon\cos(x\phi) \quad -(39)$$

where

$$x = 1 - 3\left(\frac{mMG}{cL}\right)^2 \quad -(40)$$

6)

In Newtonian dynamics:

$$x = 1 \quad -(41)$$

Note carefully that the analysis has been reduced to a small perturbation of the Minkowski metric:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(42)$$

with

$$\boxed{\frac{\alpha}{r} = 1 + \epsilon \cos(x\phi)} \quad -(43)$$

The perihelion of the orbit is displaced by an amount calculated from:

$$x\phi = 2\pi, \quad -(44)$$

i.e

$$\phi = \frac{2\pi}{x} \sim 6\pi \left(\frac{GmM}{cL}\right)^2 \quad -(45)$$

The perihelion advance is

$$\phi = 6\pi \left(\frac{GmM}{cL}\right)^2 = \frac{6\pi GM}{ac^2(1-\epsilon^2)} \quad -(46)$$

where

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(47)$$

Sommerfeld's Model of the H Atom

In the old quantum theory, this model was based on:

$$H = mc^2(\gamma - 1) - \frac{kZe^2}{r} \quad -(48)$$

$$= T + U$$

7) This model is one of special relativity using the Minkowski metric (42). It used the special relativistic kinetic energy

$$T = mc^2(\gamma - 1) \quad -(49)$$

and the Coulombic attraction:

$$U = V = -\frac{kZe^2}{r} \quad -(50)$$

but did not use the centrifugal energy that appears in the Schrodinger equation and Dirac equation. However, the Sommerfeld model also produces a precessing ellipse. It can be adapted for the Newtonian force of attraction by replacing:

$$kZe^2 \rightarrow mMG \quad -(51)$$

so:

$$\boxed{W = mc^2(\gamma - 1) - \frac{mMG}{r}} \quad -(52)$$

In the non-relativistic limit it becomes:

$$W = \frac{1}{2}mv^2 - \frac{mMG}{r} \quad -(53)$$

giving the ellipse:

$$\frac{\alpha}{r} = 1 + \epsilon \cos\phi \quad -(54)$$

The Sommerfeld equation (52) can be written as

$$E - V = c(p^2 + m^2c^2)^{1/2} \quad -(55)$$

which for

$$V \rightarrow 0 \quad -(56)$$

8) is the Einstein energy equation. It therefore corresponds to a Minkowski metric modified by the presence of V. It can be written as:

$$\left(\frac{E-V}{c}\right)^2 - p^2 = m^2c^2 \quad -(57)$$

$$= g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}$$

with $S = -(E-V)t + L\phi + S_r(r)$. $-(58)$

This is the Hamilton-Jacobi equation of the system. In this equation:

$$p^2 = p_r^2 + \frac{L^2}{r^2} \quad -(58)$$

$$= m(\dot{r}^2 + r^2\dot{\phi}^2)$$

Thus:

$$m^2c^2 - \frac{1}{c^2}(E-V)^2 + \left(p_r^2 + \frac{L^2}{r^2}\right) = 0 \quad -(59)$$

where $p_r = \frac{\partial S}{\partial r}$, $\phi = \frac{\partial S}{\partial L}$. $-(60)$

The HJ equation is:

$$m^2c^2 - \frac{1}{c^2}(E-V)^2 + \left(\frac{\partial S}{\partial r}\right)^2 + \frac{L^2}{r^2} = 0 \quad -(61)$$

and

$$S = S_0 + \int\left(\frac{1}{c^2}(E-V)^2 - \frac{L^2}{r^2} - m^2c^2\right)^{1/2} dr \quad -(62)$$

9) Finally the dependence of ϕ on r may be found by

$$\phi = \frac{\partial S}{\partial L} \qquad -(63)$$

Sometimes the definition:

$$\phi = \frac{\partial S}{\partial L} \qquad -(64)$$

is used in the literature, giving:

$$\phi = \phi_0 + L \int \frac{1}{r^2} \left(\frac{1}{c^2} \left(E - V \right)^2 - \frac{L^2}{r^2} - m^2 c^2 \right)^{-1/2} dr \qquad -(65)$$

i.e. the precessing ellipse

$$\boxed{r = \frac{\alpha}{1 + \epsilon \cos(\gamma \phi)}} \qquad -(66)$$

where:

$$\alpha = \frac{L \gamma^2}{\beta E}, \quad \epsilon = \frac{1}{\beta} \left(1 - \gamma^2 \frac{m^2 c^4}{E^2} \right)^{1/2}$$

$$\gamma = \left(1 - \beta^2 \right)^{1/2} \qquad -(67)$$

Therefore the Sommerfeld and Einstein gravitational metric methods both give precessing ellipses.

1) 148(7): General Principle of Orbits

Principle

Stable orbits are always described by the Minkowski metric provided the dependence of r on ϕ is known.

Examples

1) Orbits in a Plane, Kepler Problem

If a particle rotates about a fixed force centre the real Newtonian force on it is inwards towards the force centre. In a frame fixed on the particle the observer measures this force and holds at the same time that the particle does not fall inward to the force centre. The force is given by:

$$F(r) = -\frac{\partial U}{\partial r} = m\left(\ddot{r} - r\dot{\phi}^2\right) \quad -(1)$$

which can be re-expressed as:

$$\frac{d^2 u}{d\phi^2} + u = -\frac{mr^2}{L^2} F(r) \quad -(2)$$

where

$$u = 1/r. \quad -(3)$$

If

$$F(r) = -mMG/r^2 \quad -(4)$$

then the orbit is

$$\frac{1}{r} = u = \frac{1}{d}\left(1 + \epsilon\cos\phi\right) \quad -(5)$$

which allows $dr/d\phi$ to be calculated.

The well known problem with this description is that the Newton equation is being applied in a non-inertial frame, so although there is a net force

2)

given by eq. (1), the particle is not attracted. It remains in the stable orbit (5). Newton's law:

$$F = mg = -\frac{mMG}{r^2} \quad — (6)$$

applies only to an inertial frame, in which:

$$F = m\ddot{r}. \quad — (7)$$

In order to force Newton's law to apply to orbits, an outward force is introduced, called "the centrifugal force". The net force on the particle is zero. So:

$$F = m\ddot{r} - mr\dot{\phi}^2 + mr\dot{\phi}^2 \quad — (8)$$

consisting of the Newtonian $m\ddot{r}$, the "centripetal force" $-mr\dot{\phi}^2$ and the centrifugal force $mr\dot{\phi}^2$. The only real forces in Newtonian dynamics are the first two. The net force is:

$$F = m\ddot{r} \quad — (9)$$

and on average in any stable orbit:

$$\langle F \rangle = m \langle \ddot{r} \rangle = 0. \quad — (10)$$

In a circular orbit, r does not change with time, so

$$F = 0. \quad — (11)$$

In an elliptical orbit, $\langle F \rangle$ is zero on average, because r changes with time:

$$\frac{dr}{dt} = \left(\frac{2}{m}(E-U) - \frac{L^2}{m^2 r^2} \right)^{1/2} \quad — (11)$$

3) The elliptical orbit comes from the hamiltonian:

$$H = T + U = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{mr^2} + U \quad -(12)$$

in which

$$U = -\frac{k}{r}. \quad -(13)$$

The force of attraction (1) comes from the lagrangian

$$\mathcal{L} = T - U, \quad -(14)$$

and the Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}}. \quad -(15)$$

Use

$$\frac{d\phi}{dr} = \frac{d\phi}{dt} \frac{dt}{dr} \quad -(16)$$

and

$$\omega = \frac{d\phi}{dt} = \frac{L}{mr^2} \quad -(17)$$

so

$$\phi(r) = \int \left(\frac{1}{r^2} \left(2m \left(H - U - \frac{L^2}{2mr^2} \right) \right)^{-1/2} \right) dr \quad -(18)$$

i.e

$$\cos \phi = \left(\frac{L^2}{mkr} - 1 \right) \left(1 + \frac{2HL^2}{mk^2} \right)^{-1/2} \quad -(19)$$

i.e.

$$\frac{\alpha}{r} = 1 + \epsilon \cos \phi \quad -(20)$$

where

4)

$$\alpha = \frac{L^2}{mk}, \quad \epsilon = \left(1 + \frac{2HL^2}{mk^2}\right)^{1/2} \quad -(21)$$

It is seen that the orbit (20) is the direct result of H and L. However, the orbit is not stable in Newtonian dynamics, because the centrifugal force is missing from the analysis. Furthermore, the only thing that is observed experimentally is eq. (20). This equation gives:

$$\boxed{\frac{dr}{d\phi} = \frac{\epsilon}{\alpha} r^2 \sin\phi} \quad -(22)$$

in a stable orbit. The purely Newtonian analysis gives eq. (22), but the orbit is unstable.

The above is a non-relativistic analysis in two dimensional space, the plane XY.

The new general principle of orbits comes from a relativistic analysis based on the Minkowski metric in the XY plane:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \quad -(23)$$

i.e. $$c^2 (dt^2 - d\tau^2) = dr^2 + r^2 d\phi^2$$

$$= v^2 dt^2$$

$$\boxed{v^2 dt^2 = dr^2 + r^2 d\phi^2} \quad -(24)$$

in which we have used:

$$d\tau = \gamma dt, \quad -(25)$$

The proper time. Although eq. (24) looks non-relativistic, it is in fact a Minkowski metric in XY, in which

$$dZ = 0 \quad -(26)$$

Eq. (24) gives the kinetic energy directly:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \right)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right), \quad -(27)$$

and the Hamiltonian and Lagrangian:

$$H = \mathcal{L} = T. \quad -(28)$$

The concepts of potential energy, centripetal and centrifugal force, and the problem of trying to use a Newtonian analysis in a non-inertial frame have disappeared. They have been replaced by eqs. (22) and (24), which can be combined to give:

$$v^2 dt^2 = \left(\frac{\epsilon}{\alpha} r^2 \sin\phi + r^2 \right) d\phi^2 \quad -(29)$$

giving the angular velocity:

$$\boxed{\omega = \frac{d\phi}{dt} = \frac{v}{r} \left(1 + \frac{\epsilon}{\alpha} \sin\phi \right)^{-1/2}} \quad -(30)$$

6)

and kinetic energy:

$$T = \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \right) \quad -(31)$$

$$= \frac{1}{2} m r^2 \left(1 + \frac{\epsilon}{\alpha} \sin\phi \right) \left(\frac{d\phi}{dt} \right)^2$$

$$\boxed{\begin{aligned} T &= \frac{1}{2} m r^2 \omega^2 \left(1 + \frac{\epsilon}{\alpha} \sin\phi \right) \\ \omega &= \frac{v}{r} \left(1 + \frac{\epsilon}{\alpha} \sin\phi \right)^{-1/2} \end{aligned}} \quad -(32)$$

In a circular orbit:

$$T = \frac{1}{2} m v^2, \quad \omega = \frac{v}{r}. \quad -(33)$$

The conventional description uses the same T, but also uses u and the effective potential V, so:

$$H = T + V \quad -(34)$$

$$= T + u + u_c$$

ADDED

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) \quad \boxed{- \frac{k}{r} + \frac{1}{2} r^2 \dot{\phi}^2 m}$$

in which

$$\left. \begin{aligned} u &= - \frac{k}{r} \\ u_c &= \frac{1}{2} m r^2 \dot{\phi}^2 \\ k &= mG \end{aligned} \right\} \quad -(35).$$

7)

Therefore in a stable orbit:

$$-\frac{mG}{r} = \frac{1}{2} m r^2 \dot{\phi}^2$$

or

$$F = mg = -\frac{mMG}{r^2} = m r \dot{\phi}^2 \quad -(36)$$

which means that the attractive inverse square force inwards is balanced by the centrifugal force outward. The metrical description is simply:

$$H = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) \quad -(37)$$

and is simpler and also fully relativistic, so can be extended self consistently to relativistic orbits. The centrifugal force is:

$$F_c = -\frac{\partial U_c}{\partial r} = m r \dot{\phi}^2 = \frac{L^2}{m r^3} \quad -(38)$$

so in eq. (36)

$$-\frac{mMG}{r^2} = \frac{L^2}{m r^3} \quad -(39)$$

i.e

$$\frac{L^2}{r} \left(- m^2 MG = 0 \right. \quad -(40)$$

giving the characteristic radius:

$$\boxed{r_0 = \frac{L^2}{m^2 MG} = \text{constant of motion}} \quad -(41)$$

2) Relativistic Orbits in a Plane

a) Relativistic Kepler Problem

The metrical description is:

$$H = T = \mathcal{L} \qquad (42)$$

where

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \cos(\phi + x\phi)\right) \qquad (43)$$

$$= \frac{1}{\alpha}\left(1 + \cos(y\phi)\right)$$

where

$$y = 1 + x. \qquad (44)$$

Here y and x are determined observationally.

Therefore

$$\frac{dr}{d\phi} = \frac{y\epsilon}{\alpha} r^2 \sin(y\phi) \qquad (45)$$

The kinetic energy is:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m r^2 \omega^2 \left(1 + \left(\frac{\epsilon y r}{\alpha} \sin(y\phi)\right)^2\right) \qquad (46)$$

and the angular velocity is:

$$\omega = \frac{d\phi}{dt} = \frac{v}{r}\left(1 + \left(\frac{\epsilon y r}{\alpha} \sin(y\phi)\right)^2\right)^{-1/2}. \qquad (47)$$

The metric is:

$$\boxed{\begin{aligned} ds^2 &= c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 && (48)\\ dr^2 &= \left(\frac{\epsilon y r}{\alpha} \sin(y\phi)\right)^2 r^2 d\phi^2 && (49)\end{aligned}}$$

, and is fully equivalent to the gravitational metric from the orbital theorem:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad -(50)$$

As is well known, this gravitational metric produces the centrifugal force from geometry, i.e. from the ECE orbital theorem.

Note carefully that both the metrics (48) and (50) produce the orbit (43), a precessing ellipse in which the perihelion advances. Since both metrics produce the same orbit, they are equivalent.

b) Sommerfeld's Model of the H Atom

Essentially this is a solution of the hamiltonian

$$H = T + U \quad -(51)$$

where

$$T = mc^2(\gamma - 1) \quad -(52)$$

$$U = -k/r \quad -(53)$$

$$\gamma = \left(1 - v^2/c^2\right)^{-1/2} \quad -(54)$$

to produce a precessing elliptical orbit of the electron around the proton in a H atom in the old quantum theory (Bohr / Bury / Sommerfeld atom). Here T is the relativistic kinetic energy:

$$T^2 = p^\mu p_\mu = m^2 c^2 p^2 \quad -(55)$$

10)

where the relativistic momentum is:

$$\underline{p} = \gamma m \underline{v} = m \frac{d\underline{r}}{d\tau} = \gamma m \frac{d\underline{r}}{dt} \quad -(56)$$

It was shown in paper 1147 that eq. (56) is simply a re-expression of the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - (d\underline{r} \cdot d\underline{r})^2 \quad -(57)$$

Eq. (56) is:

$$p^2 = \left(\frac{T}{c}\right)^2 = \gamma^2 m^2 v^2 \quad -(58)$$

From eq. (51)

$$T = H - U \quad -(59)$$

so

$$p^2 = \left(\frac{H - U}{c m}\right)^2 = \gamma^2 m^2 v^2 \quad -(59)$$

so the Sommerfeld description of the H atom in eq. (59), written in a Minkowski metric. It is known that the functional dependence of the Sommerfeld H atom must be given in r or ϕ in the Sommerfeld atom has the by eq. (45), so the Sommerfeld atom has the metric (48) and (49). This is again a Minkowski metric with a functional dependence of r on ϕ:

From eq. (59) the hamiltonian and

11) lagrangian are:

$$H = \gamma mvc - \frac{k}{r} \quad -(60)$$

$$\mathcal{L} = \gamma mvc + \frac{k}{r} \quad -(61)$$

and in the non-relativistic limit:

$$v << c \quad -(62)$$

reduce to

$$H = \frac{1}{2}mv^2 - \frac{k}{r} \quad -(63)$$

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{k}{r} \quad -(64)$$

Eqs. (60) and (61) give a precessing ellipse, and eqs. (63) and (64) give an ellipse, for the orbit of the electron around the proton in the old quantum theory. Both orbits are again illustrations of the new general principle introduced in this note.

1) 148(8): Expression of the Gravitational Metric.

The ECE metrical description of gravitation is based on observation and the Minkowski metric. For a precessing elliptical orbit:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \cos(y\phi)\right) \quad -(1)$$

in the Minkowski metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(2)$$

in the plane XY:

$$dZ^2 = 0. \quad -(3)$$

From eq. (1):

$$\boxed{\frac{dr}{d\phi} = ar} \quad -(4)$$

where

$$a = \frac{y\epsilon}{\alpha} r \sin(y\phi) \quad -(5).$$

so:

$$\boxed{d\tau^2 = dt^2 - (1+a^2)\left(\frac{r}{c}\right)^2 d\phi^2} \quad -(6)$$

Therefore

$$\left(\frac{d\tau}{dt}\right)^2 = (1+a^2)\left(\frac{r}{c}\right)^2 d\phi^2 \quad -(7)$$

and

$$c^2(dt^2 - d\tau^2) = (1+a^2)\left(\frac{r}{c}\right)^2 d\phi^2 \quad -(8)$$

However, in the Minkowski metric (2), by definition:

$$\underline{v} = \frac{d\underline{r}}{dt} \quad -(9)$$

so

$$d\tau^2 = dt^2 - \frac{1}{c^2} d\underline{r} \cdot d\underline{r}$$

$$= dt^2\left(1 - \frac{v^2}{c^2}\right) \quad -(10)$$

2)

Comparing eqs. (6) and (10):

$$\omega = \frac{d\phi}{dt} = \left(\frac{v^2}{(1+a^2)r^2}\right)^{1/2} = \frac{v}{r}\left(1+a^2\right)^{-1/2}$$

$$\boxed{\omega = \frac{v}{r} \cdot \frac{1}{(1+a^2)^{1/2}}} \quad -(11)$$

Therefore $\quad dt = \frac{d\phi}{\omega} = \frac{r}{v}\left(1+a^2\right)^{1/2} \quad -(12)$

In the limit:

$$\epsilon \to 0 \quad -(13)$$

then

$$\omega = \frac{v}{r} \quad -(14)$$

for a <u>circular</u> <u>orbit</u>.

The gravitational metric is described completely by eqs. (2), (4) and (12).

Thus:

$$\boxed{\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \left(\frac{a}{(1+a^2)^{1/2}}\right)v} \quad -(15)$$

Here v is defined by:

$$v = |\underline{v}| = \left|\frac{d\underline{r}}{dt}\right| \quad -(16)$$

and

$$v^2 = \frac{|d\underline{r} \cdot d\underline{r}|}{dt^2} \quad -(17)$$

3) with

$$\left.\begin{aligned} |d\underline{r} \cdot d\underline{r}| &= dr^2 + r^2 d\phi^2 + dZ^2 \\ &= dX^2 + dY^2 + dZ^2 \end{aligned}\right\} \quad -(18)$$

In the limit (13) for a circle, it is seen from eq (15) that:

$$\frac{dr}{dt} \to 0 \quad -(19)$$

i.e

$$dr = 0, \quad -(20)$$

the circle has constant r.

Therefore the gravitational metric is:

$$\boxed{ds^2 = c^2 d\tau^2 = \frac{c^2}{\omega^2} d\phi^2 - |d\underline{r} \cdot d\underline{r}|^2} \quad -(21)$$

using eq. (12).

From eqs. (11) and (15):

$$\boxed{\frac{dr}{dt} = a\omega r} \quad -(22)$$

so from eq. (22) the gravitational metric is:

$$\boxed{ds^2 = c^2 d\tau^2 = \frac{c^2 dr^2}{a^2\omega^2 r^2} - |d\underline{r} \cdot d\underline{r}|} \quad -(23)$$

4) <u>Comparison with the Eddington Isotropic Metric</u>

The conventional format of the gravitational metric is :

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\phi^2 - dZ^2 \quad -(24)$$

where

$$r_s = \frac{2MG}{c^2} \quad -(25)$$

The isotropic spherical coordinates of Eddington are defined by :

$$r = r_1 \left(1 + \frac{GM}{2c^2 r_1}\right)^2, \quad -(25)$$

$$dr_1^2 = dX_1^2 + dY_1^2 + dZ_1^2 \quad -(26)$$

So :

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{GM}{2c^2 r_1}\right)^2 \left(1 + \frac{GM}{2c^2 r_1}\right)^{-2} - \left(1 + \frac{GM}{2c^2 r_1}\right)^4 dr_1^2 \quad -(27)$$

$$\boxed{c^2 d\tau^2 = c^2 dt^2 b^2 - d\underline{r} \cdot d\underline{r}} \quad -(28)$$

here

$$b^2 = \left(1 - \frac{GM}{2c^2 r_1}\right)\left(1 + \frac{GM}{2c^2 r_1}\right)^{-2} \quad -(29)$$

the metric (28) can be expressed as :

$$c^2 d\tau^2 = c^2 dt'^2 - d\underline{r} \cdot d\underline{r} \quad -(30)$$

where

$$\boxed{dt'^2 = b^2 dt^2} \quad -(31)$$

5) Comparing eqns. (21), (23) and (30):

$$\boxed{dt'^2 = b^2 dt^2 = \frac{d\phi^2}{\omega^2} = \frac{dr^2}{a\omega r}} \quad -(32)$$

where

$$\omega = \frac{d\phi}{dt} = \frac{v}{r}\left(\frac{1}{(1+a^2)^{1/2}}\right) \quad -(33)$$

so

$$\frac{d\phi}{dt'} = \frac{\omega}{b} \quad -(34)$$

$$\frac{dr}{dt'} = \frac{a\omega r}{a\omega r b} \quad -(35)$$

$$dt' = b\,dt \quad -(36)$$

In the limit:

$$r_1 \to \infty, \quad -(37)$$

then

$$b \to 1, \quad -(38)$$

$$\gamma \to 1. \quad -(39)$$

Conclusion

In the conventional approach (24) dt is changed to $b\,dt$. In the equivalent ECE metric theory dt is changed to $\frac{d\phi}{\omega} = \frac{dr}{a\omega r}$

so

$$-(40)$$

the same as

$$\boxed{\begin{array}{l} dt \longrightarrow b\,dt \\ dt \longrightarrow \dfrac{d\phi}{\omega} = \dfrac{dr}{a\omega r} \end{array}}$$