1) Note 145(1): Magnetostatics on a Rotating Platform (In general Moving Platform).

The basic equations are:

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \quad -(1)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(2)$$

$$\partial \underline{B} / \partial t = 0 \quad -(3)$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad -(4)$$

where:

$$\underline{A} = \underline{A}^{(1)} + \underline{A}^{(2)} + \underline{A}^{(3)} \quad -(5)$$

$$\underline{B} = \underline{B}^{(1)} + \underline{B}^{(2)} + \underline{B}^{(3)} \quad -(6)$$

$$\underline{\omega} = \underline{\omega}^{(1)} + \underline{\omega}^{(2)} + \underline{\omega}^{(3)} \quad -(7)$$

$$\underline{J} = \underline{J}^{(1)} + \underline{J}^{(2)} + \underline{J}^{(3)} \quad -(8)$$

Eqs (5) to (8) are examples of the basic theorem that any vector is the sum of (1), (2) and (3) components (Moses / Silver / Reed / Evans).

The spin connection vector $\underline{\omega}$ is a wavenumber, so introduce the topological phase:

$$\phi = \exp\left(\pm i \underline{\omega} \cdot \underline{r}\right) \quad -(9)$$

2) extra

The effect of $\underline{\omega}$ on the potential is therefore:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) e^{-i \underline{\omega} \cdot \underline{r}} \quad -(10)$$

$$\underline{A}^{(2)} = \frac{A^{(0)}}{\sqrt{2}} \left(\underline{i} + i \underline{j} \right) e^{i \underline{\omega} \cdot \underline{r}} \quad -(11)$$

$$\underline{A}^{(3)} = A^{(0)} \underline{k} \quad -(12)$$

therefore:

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{A}^{(1)} &= \omega \underline{A}^{(1)} \quad -(13) \\ \underline{\nabla} \times \underline{A}^{(2)} &= -\omega \underline{A}^{(2)} \quad -(14) \\ \underline{\nabla} \times \underline{A}^{(3)} &= 0 \underline{A}^{(3)} \quad -(15) \end{aligned}}$$

which are Beltrami flow equations.

The spin connection vector is defined in general by eq. (7), with:

$$\underline{\omega}^{(1)} = \frac{\omega}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \exp\left(-i \underline{\omega} \cdot \underline{r} \right) \quad -(16)$$

$$\underline{\omega}^{(2)} = \frac{\omega}{\sqrt{2}} \left(\underline{i} + i \underline{j} \right) \exp\left(i \underline{\omega} \cdot \underline{r} \right) \quad -(17)$$

$$\underline{\omega}^{(3)} = \omega \underline{k} \quad -(18)$$

Therefore:

$$\underline{B}^{(1)*} = \underline{\nabla} \times \underline{A}^{(1)*} + i\,\underline{\omega}^{(2)} \times \underline{A}^{(3)} \qquad (19)$$

$$\underline{B}^{(2)*} = \underline{\nabla} \times \underline{A}^{(2)*} + i\,\underline{\omega}^{(3)} \times \underline{A}^{(1)} \qquad (20)$$

$$\underline{B}^{(3)*} = \underline{\nabla} \times \underline{A}^{(3)*} + i\,\underline{\omega}^{(1)} \times \underline{A}^{(2)} \qquad (21)$$

because the basis vectors are defined by:

$$\left.\begin{aligned} \underline{e}^{(1)} \times \underline{e}^{(2)} &= i\,\underline{e}^{(3)*} \\ \underline{e}^{(3)} \times \underline{e}^{(1)} &= i\,\underline{e}^{(2)*} \\ \underline{e}^{(2)} \times \underline{e}^{(3)} &= i\,\underline{e}^{(1)*} \end{aligned}\right] \qquad (22)$$

The extra effect of the motion of the platform is therefore:

$$\boxed{\begin{aligned} \Delta\underline{B}^{(1)*} &= i\,\underline{\omega}^{(2)} \times \underline{A}^{(3)} \\ \Delta\underline{B}^{(2)*} &= i\,\underline{\omega}^{(3)} \times \underline{A}^{(1)} \\ \Delta\underline{B}^{(3)*} &= i\,\underline{\omega}^{(1)} \times \underline{A}^{(2)} \end{aligned}} \qquad (23)$$

so:

$$\Delta\underline{B}^{*} = \Delta\underline{B}^{(1)*} + \Delta\underline{B}^{(2)*} + \Delta\underline{B}^{(3)*} \qquad (24)$$

In general this can be any motion, which can be built up by Fourier synthesis.

4) Eqs. (16) to (18) represent the simplest type of decomposition of $\underline{\omega}$ into (1), (2) and (3) components. More generally, ω can be expressed in terms of Fourier components. Eqs. (5) to (8) are extensions of the Helmholtz Theorem. The latter is used extensively in classical electrodynamics. The phase (9) is related to the Berry phase, Pancharatnam phase, and topological phase. — (25)

Eqns (19) and (20) are:

$$\underline{B}^{(2)} = \underline{\nabla} \times \underline{A}^{(2)} + i\,\underline{\omega}^{(2)} \times \underline{A}^{(3)}$$

$$\underline{B}^{(1)} = \underline{\nabla} \times \underline{A}^{(1)} + i\,\underline{\omega}^{(3)} \times \underline{A}^{(1)} \quad -(26)$$

and eq. (21) is:

$$\underline{B}^{(3)} = \omega A^{(0)}\,\underline{k}\,\Phi \quad -(27)$$

The $\underline{B}^{(3)}$ field in eq. (27) is induced by the spin connection ω, the magnitude of $\underline{\omega}$.

5) This means that an extra (magnetic) flux density is induced by any type of matter in general. The phase Φ in eq. (27) is:

$$\Phi = \exp\left(-i\left(\omega t - \kappa Z\right) + i\,\underline{\omega}\cdot\underline{r}\right) \quad -(28)$$

assuming that the original phase of $\underline{A}^{(1)}$ is a plane wave along Z. More generally, if the original $\underline{A}^{(1)}$ is:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right)e^{-i\underline{\kappa}\cdot\underline{r}} \quad -(29)$$

then

$$\Phi = \exp\left(i\left(\underline{\omega} - \underline{\kappa}\right)\cdot\underline{r}\right) \quad -(30)$$

In eq. (26):

$$\underline{\omega}^{(2)} \times \underline{A}^{(3)} = i A^{(0)}\,\underline{\omega}^{(1)*}_{(2)} \quad -(31)$$

$$= i A^{(0)}\,\underline{\omega}^{(2)}$$

so:

$$\underline{B}^{(2)} = \underline{\nabla} \times \underline{A}^{(2)} - A^{(0)}\,\underline{\omega}^{(2)}$$

$$\underline{B}^{(2)} = -\kappa\,\underline{A}^{(2)} - A^{(0)}\,\underline{\omega}^{(2)}$$

$$\boxed{\Delta\underline{B}^{(2)} = -A^{(0)}\,\underline{\omega}^{(2)}} \quad -(32)$$

145(2): Phase and Spin Connection

The spin connection is written as

$$\omega^\mu = \left(\frac{\omega_0}{c}, \underline{\kappa} \right) \quad -(1)$$

with

$$x_\mu = \left(ct, -\underline{r} \right) \quad -(2)$$

so the ECE phase is

$$\boxed{\begin{aligned} \phi &= \exp\left(i\,\omega^\mu x_\mu \right) \\ &= \exp\left(i\left(\omega_0 t - \underline{\kappa}\cdot\underline{r}\right)\right) \end{aligned}} \quad -(3)$$

All the well observed phase effects of physics derive from the spin connection. Therefore as required, all of physics is general relativity.

Sagnac Effect

This is explained by using a tetrad rotating left:

$$\underline{q}^{(1)}_L = \frac{1}{\sqrt{2}} \left(\underline{i} - i\,\underline{j} \right) \exp\left(i\omega t \right) \quad -(4)$$

and one rotating right:

$$\underline{q}^{(1)}_R = \frac{1}{\sqrt{2}} \left(\underline{i} + i\,\underline{j} \right) \exp\left(i\omega t \right) \quad -(5)$$

so only the time-like component of the spin connection is needed.

2) The time taken to rotate 360° (2π radians) is:

$$\Delta t = \frac{2\pi}{\omega} \quad -(6)$$

left or right.

Now spin the platform to the left and

$$\omega \rightarrow \omega + \Omega, \quad -(7)$$

so:

$$\underline{V}^{(1)}_{L} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \exp\left(i(\omega+\Omega)t\right). \quad -(8)$$

Similarly:

$$\underline{V}^{(1)}_{R} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \exp\left(i(\omega-\Omega)t\right) \quad -(9)$$

Therefore there is a difference in time taken for the light to go left or right:

$$\boxed{\Delta t = 2\pi\left(\frac{1}{\omega-\Omega} - \frac{1}{\omega+\Omega}\right)} \quad -(10)$$

which is the Sagnac effect.

Now write:

$$\omega = \frac{c}{r} \quad -(11)$$

and

$$t_1 = \frac{2\pi}{\omega - \Omega}, \quad -(12)$$

$$t_2 = \frac{2\pi}{\omega + \Omega} \quad -(13).$$

3) Thus:

$$t_1 = 2\pi \left(\frac{1}{\frac{c}{r} - \Omega} \right) = \frac{2\pi r}{c} \left(\frac{1}{1 - \frac{r\Omega}{c}} \right) \quad -(14)$$

$$t_2 = 2\pi \left(\frac{1}{\frac{c}{r} + \Omega} \right) = \frac{2\pi r}{c} \left(\frac{1}{1 + \frac{r\Omega}{c}} \right) \quad -(15)$$

i.e.

$$t_1 = \frac{1}{c} \left(2\pi r + \Delta l_1 \right) \quad -(16)$$

$$t_2 = \frac{1}{c} \left(2\pi r - \Delta l_2 \right) \quad -(17)$$

where:

$$\Delta l_1 = r\Omega t_1, \quad -(18)$$

$$\Delta l_2 = r\Omega t_2, \quad -(19)$$

$$v = r\Omega. \quad -(20)$$

Now use

$$v \ll c \quad -(21)$$

so:

$$t_1 \sim \frac{2\pi r}{c} \left(1 + \frac{r}{c}\Omega \right) \quad -(22)$$

$$t_2 \sim \frac{2\pi r}{c} \left(1 - \frac{r}{c}\Omega \right) \quad -(23)$$

and

$$\Delta t = t_1 - t_2 = \frac{4\pi r^2 \Omega}{c^2}. \quad -(24)$$

$$\boxed{\Delta t \sim \frac{4 A r \Omega}{c^2}} \quad -(25)$$

4)

The Sagnac effect is an effect of general relativity, which is why it cannot be explained by the MH equation of special relativity.

In the MH equation the angular frequency ω is that of light, and there is no connection between light and dynamics.

In ECE theory the angular frequency ω is the time like part of ω^μ, within a factor c, and light is the frame of reference itself. Therefore the extra mechanical angular frequency Ω can be added to ω or subtracted from ω:

$$\boxed{\omega \rightarrow \omega \pm \Omega} \quad -(26)$$

This simple law gives the standard expression (25) of the Sagnac effect, verified experimentally to many orders of magnitude precision in the ring laser gyro. Electrodynamics and dynamics are unified in eq. (26).

1) 145(3): Diagram of the Sagnac Effect

Consider a beam of light rotating in a circle in the X - Y plane. In ECE theory (general relativity) the beam of light is a rotating frame, the tetrad:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \exp(i\omega t) \quad -(1)$$

$$= \underline{e}^{(1)} \exp(i\omega t)$$

where $\underline{e}^{(1)}$ is a unit vector of the complex circular basis:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}} \left(\underline{i} + i \underline{j} \right) \quad -(2)$$

$$\underline{e}^{(3)} = \underline{k}$$

with

$$\underline{e}^{(1)} \times \underline{e}^{(2)} = i \, \underline{e}^{(3)*}$$

$$\underline{e}^{(2)} \times \underline{e}^{(3)} = i \, \underline{e}^{(1)*} \quad -(3)$$

$$\underline{e}^{(3)} \times \underline{e}^{(1)} = i \, \underline{e}^{(2)*}$$

In eq. (1), ω is the angular frequency of rotation (radians per second).

The fundamental ECE hypothesis is:

$$A_\mu^a = A^{(0)} q_\mu^a \quad -(4)$$

so

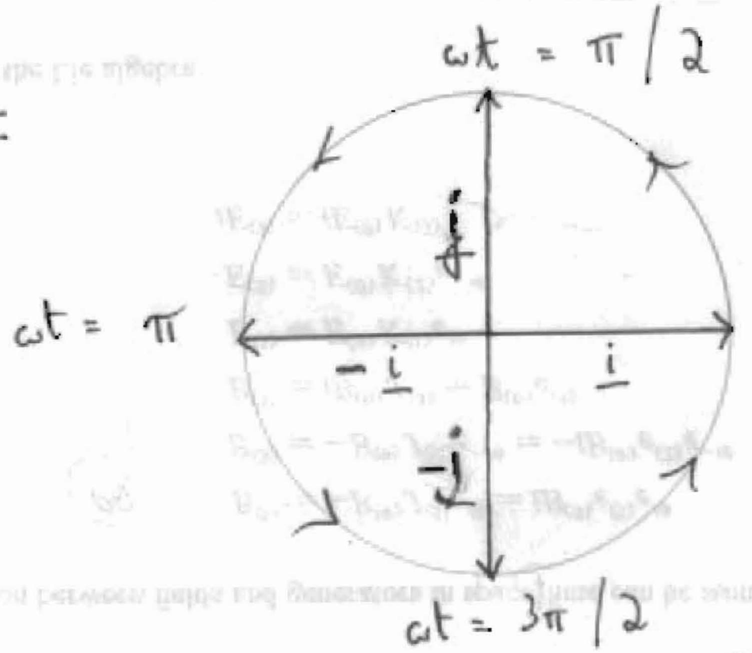
$$\boxed{\underline{A}^{(1)} = A^{(0)} \underline{q}^{(1)}} \quad -(5)$$

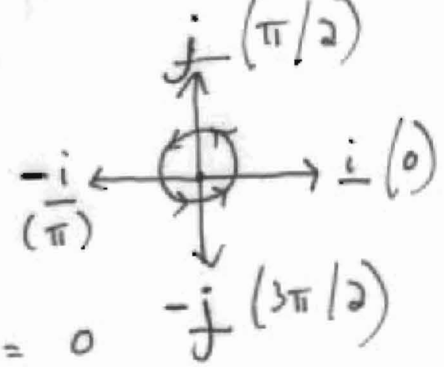
2) This is the electromagnetic potential associated with a light beam going around in a circle. From eq. (5):

$$\text{Real } \underline{A}^{(1)} = \text{Real}\left(\frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right)\left(\cos\omega t + i\sin\omega t\right)\right)$$

$$= \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i}\cos\omega t + \underline{j}\sin\omega t\right) \quad -(6)$$

Figure 1





<u>The frame</u> is <u>rotating</u> <u>counter-clockwise</u>. This generates the potential (6) and a beam of light travelling in a circle counter-clockwise.

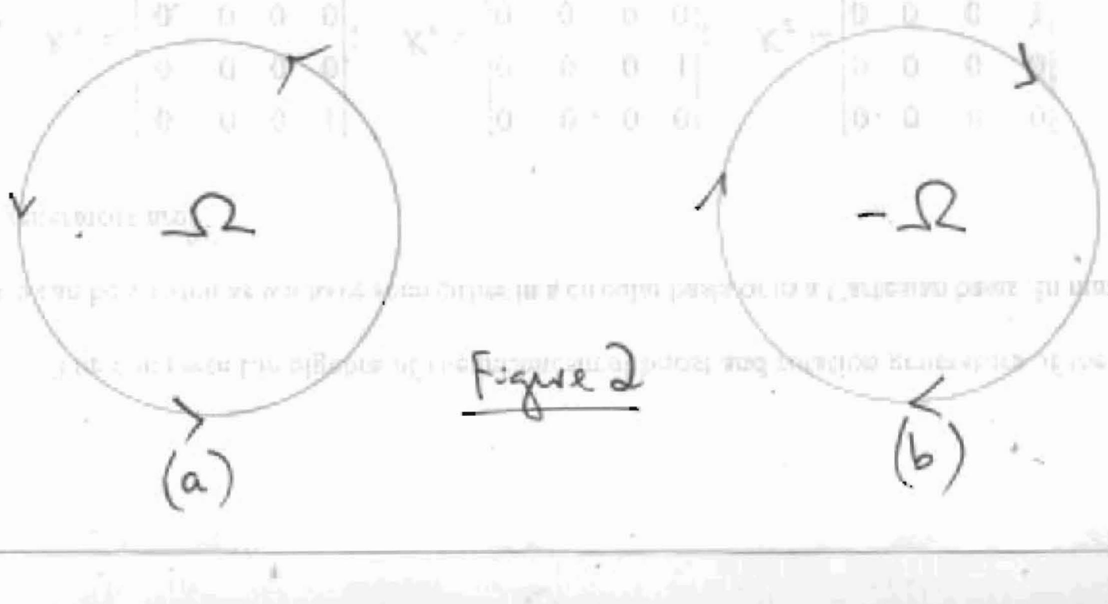


Figure 2

3) Now increase the angular frequency of the rotating frame as in Fig. 2a. The potential is ~~increased~~ changed to:

a) $$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \exp \left(i (\omega + \Omega) t \right) \quad -(7)$$

Similarly decrease the angular frequency of the rotating frame as in Fig. 2b. The potential becomes:

b) $$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \exp \left(i (\omega - \Omega) t \right) \quad -(8).$$

The Sagnac effect or ring laser gyro effect is:

$$\boxed{\Delta t = 2\pi \left(\frac{1}{\omega - \Omega} - \frac{1}{\omega + \Omega} \right)} \quad -(9)$$

It is a phase effect caused by the difference between the angular frequencies of two rotating frames.

The rotation of a frame of reference is caused by a spin connection of differential geometry. To avoid notational confusion write the spin connection as:

$$\alpha^\mu = (\alpha_0, \underline{\alpha}) \quad -(10)$$

The ~~only~~ frame in eqns (1) to (8) is rotated by the phase factors. Otherwise the frame is static. It follows that the phase is the spin connection.

$$\boxed{\omega^\mu = \left(\omega_0, \underline{\omega}\right) = \left(\frac{\omega}{c}, \underline{\kappa}\right)} \quad -(11)$$

where ω is the angular frequency and $\underline{\kappa}$ the wave-vector. So the phase is:

$$\phi = \exp\left(i\,\omega^\mu x_\mu\right) = \exp\left(i\left(\omega t - \underline{\kappa}\cdot\underline{r}\right)\right) \quad -(12)$$

Notably:

$$\boxed{\omega_0 = \frac{\omega}{c}} \quad -(13)$$

is the timelike part of the ω^μ four-vector.

Finally use:

$$\frac{\omega}{c} = \frac{1}{r}, \quad \frac{\Omega}{v} = \frac{1}{r} \quad -(14)$$

i.e.

$$\omega = \frac{c}{r}, \quad \Omega = \frac{v}{r} \quad -(15)$$

where r is the radius of the circle and v the tangential velocity generated by Ω. It follows that

$$\boxed{\omega_0 = \frac{1}{r}\left(1 \pm \frac{v}{c}\right)} \quad -(16)$$

The Sagnac effect is a direct observation of the spin connection.

5) If we write:

$$\alpha_L = \frac{1}{c}\left(\omega - \Omega\right) \quad -(17)$$

$$\alpha_R = \frac{1}{c}\left(\omega + \Omega\right) \quad -(18)$$

the Sagnac effect is:

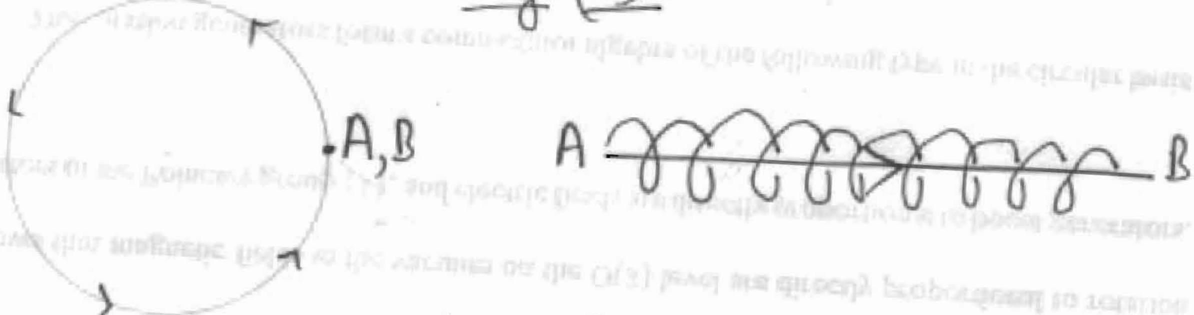
$$\boxed{\Delta t = \frac{2\pi}{c}\left(\frac{1}{\alpha_L} - \frac{1}{\alpha_R}\right)} \quad -(19)$$

As:

$$\Omega \to 0, \quad \alpha_L = \alpha_R = \alpha_0 \quad -(20)$$

Special Relativity

Fig. (3)



In special relativity, the light propagates around a circle from A to B at the speed of light c. For simplicity this path is drawn out into a straight line along the X in Fig (3). The potential is:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \exp\left(i\left(\omega t - \kappa X\right)\right) \quad -(21)$$

b)

The phase is therefore:

$$\phi = \exp\left(i\left(\omega t - \frac{\omega X}{c}\right)\right) \quad -(21)$$

because

$$\kappa = \frac{\omega}{c}. \quad -(22)$$

This is a Lorentz covariant theory, more accurately it is covariant under the Lorentz boost. The speed of the light along X is c. The latter is the same in a frame moving at v or -v with respect to the observer frame. Therefore in this theory there is no Sagnac effect, contrary to observation.

In the Sagnac effect there are equations such as:

$$\omega + \Omega = \frac{1}{c}\left(c + v\right) \quad -(23)$$

$$\omega - \Omega = \frac{1}{c}\left(c - v\right) \quad -(24)$$

but in a Lorentz boost covariant theory it is not possible to add v to c, or subtract v from c for light travelling in a vacuum, as in the Sagnac effect. The Maxwell Heaviside theory is Lorentz boost covariant by definition, and so cannot describe the Sagnac effect, or ring laser gyro.

145(4): Relation between the Sagnac Effect and the Thomas Precession

In paper 110 the Thomas precession was derived from a rotation of the Minkowski metric. The same rotation applies to the Sagnac effect. So the two effects are related, and related to the spin connection. In cylindrical coordinates the Minkowski line element is:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(1)$$

The rotation is parameterized by

$$d\phi' = d\phi + \omega dt \quad -(2)$$

where

$$\omega = \frac{v}{r} \quad -(3)$$

The Sagnac time difference is:

$$\Delta t = 2\pi \left(\frac{1}{\omega_0 - \omega} - \frac{1}{\omega_0 + \omega} \right) \quad -(4)$$

Here:

$$v << c. \quad -(5)$$

The rotating line element is:

$$ds'^2 = c^2 dt^2 - dr^2 - r^2 d\phi'^2 - dZ^2 \quad -(6)$$

$$= \left(1 - \frac{v^2}{c^2}\right)\left(c^2 dt^2 - 2r^2 \omega' d\phi\, dt\right) - dr^2 - r^2 d\phi^2 - dZ^2 \quad -(7)$$

where

2)

$$\omega' = \omega \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad - (8)$$

is the Thomas angular velocity, or relativistic angular velocity. From eq. (7)

$$c^2 dt'^2 = \left(c^2 - v^2\right) dt^2 \quad - (9)$$

So

$$dt' = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt. \quad - (10)$$

For a rotation of 2π radians:

$$\omega \, dt = 2\pi, \quad dt = \frac{2\pi}{\omega} \quad - (11)$$

and

$$\omega' dt' = 2\pi \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (12)$$

$$\omega' dt' = \frac{2\pi}{\omega}\left(1 - \frac{v^2}{c^2}\right)^{1/2} := \frac{2\pi}{\omega + \omega_1} \quad - (13)$$

So

$$\boxed{\omega_1 = \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)\omega} \quad - (14)$$

This is the frequency shift of the Thomas Precession observed in a high accuracy pendulum or in spin-orbit coupling in spectra.

For $v << c$ — (15)

3)

$$\omega_1 \sim \frac{1}{2}\left(\frac{v}{c}\right)^2 \omega . \quad - (16)$$

The spin connection from (16) is:

$$\boxed{\alpha_0 = \frac{\omega_1}{c} = \frac{1}{2}\left(\frac{v}{c}\right)^2 \frac{\omega}{c}} . \quad - (17)$$

<u>The</u> <u>relativistic</u> <u>correction</u> <u>to the Sagnac</u> <u>effect</u> is <u>therefore</u>:

$$\boxed{(\Delta t)_1 = 2\pi \left(\frac{1}{\omega_0 - \omega_1} - \frac{1}{\omega_0 + \omega_1} \right)} - (18)$$

This relativistic correction is observable when the platform of the Sagnac interferometer is rotated very quickly, so v approaches c.

The spin connection of both effects is:

$$\alpha_0 = \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right) \frac{\omega}{c} \quad - (19)$$

with

$$\alpha^\mu = (\alpha_0, \underline{\alpha}) . \quad - (20)$$

145(5): Effect of Gravitation on the Thomas Precession of the Earth.

The method of calculation is to start with the Frobenius Theorem defining the most general line element. As in ECE UFT 6, p. 374 ff., the theorem of orbits is a special case of the Frobenius Theorem for the spherically symmetric spacetime. The theorem of orbits gives the line element:

$$ds^2 = n(r)c^2 dt^2 - m(r)dr^2 - r^2 d\phi^2 - dZ^2 \quad -(1)$$

where $n(r) = 1 + \frac{\mu}{r}$, $m(r) = \left(1 + \frac{\mu}{r}\right)^{-1}$ $-(2)$

For gravitation: $\mu = -\frac{2MG}{c^2}$ $-(3)$

where M is mass, G is Newton's constant:

$$G = 6.6726 \times 10^{-11} m^3 kg^{-1} s^{-2} \quad -(4)$$

and $c \sim 3 \times 10^8 ms^{-1}$ $-(5)$

The Thomas precession of the Earth is described by

$$d\phi' = d\phi + \omega dt \quad -(6)$$

where ω is the angular velocity of the Earth:

2)

$$\omega = 7.29 \times 10^{-5} \text{ rads}^{-1} \quad -(7)$$

The mass of the earth is:

$$m = 5.98 \times 10^{24} \text{ kgm} \quad -(8)$$

Thus:

$$ds'^2 = n(r)c^2 dt^2 - m(r)dr^2 - r^2 d\phi'^2 - dZ^2$$

$$= \left(1 - \frac{2mG}{c^2 r}\right)c^2 dt^2 - r^2\left(d\phi + \omega dt\right)^2 - m(r)dr^2 - dZ^2$$

$$= \left(1 - \frac{2mG}{c^2 r}\right)c^2 dt^2 - r^2\left(d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2\right) - m(r)dr^2 - dZ^2 .$$

$$= \left(\left(1 - \frac{2mG}{c^2 r}\right)c^2 - v^2\right)dt^2 - 2r^2\omega d\phi dt - m(r)dr^2 - dZ^2 - r^2 d\phi^2 \quad -(9)$$

Define:

$$dt' = \left(1 - \frac{2mG}{c^2 r} - \frac{v^2}{c^2}\right)^{1/2} dt \quad -(10)$$

$$\omega' = \left(1 - \frac{2mG}{c^2 r} - \frac{v^2}{c^2}\right)^{-1}\omega \quad -(11)$$

So in a 2π rotation:

3) $$\alpha = \int_0^{2\pi} (\omega' dt' - \omega dt) = 2\pi\left(\left(1-\frac{2MG}{c^2 r}-\frac{v^2}{c^2}\right)^{-1/2}-1\right) \quad -(12)$$

The Thomas precession is defined by the limit:

$$r \rightarrow \infty \quad -(13)$$

So $$\alpha \rightarrow 2\pi\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2}-1\right) \quad -(14)$$

$$\sim \pi \frac{v^2}{c^2} \quad -(15)$$

$$= \pi\left(\frac{\omega r}{c}\right)^2 \quad -(16)$$

This is the precession of Earth unaffected by the mass M of another planet or the sun. If:

$$\frac{2MG}{c^2 r} << 1, \quad \frac{v}{c} << 1 \quad -(17)$$

eq. (12) is:

$$\boxed{\alpha \sim \pi\left(\frac{v^2}{c^2}+\frac{2MG}{c^2 r}\right)} \quad -(18)$$

4) where M is the mass of another object such as the sun. Therefore the correction to the Earth's Thomas precession due to the gravitational pull of the sun is, in 2π radians (one day):

$$\boxed{\Delta\alpha = \frac{2\pi M G}{c^2 r}} \qquad (19)$$

where M is the mass of the sun, r is the distance between the earth and sun. So:

$$M = 1.9891 \times 10^{30} \text{ kgm}$$

$$r = 1.496 \times 10^{11} \text{ m}$$

$$\Delta\alpha = 6.19 \times 10^{-8} \text{ radians}$$

$$= 6.19 \times 10^{-8} \times 2.0626 \times 10^{5} \text{ arc seconds}$$

$$\Delta\alpha = 1.277 \times 10^{-2}$$

$$\boxed{\Delta\alpha = 0.01277 \text{ arc seconds}}$$

1) 145(6): Derivation of Sagnac Effect from the Thomas Precession.

The Thomas precession metric is:

$$\frac{ds'^2}{c^2} = \left(1 - \frac{v^2}{c^2}\right) dt^2 - \frac{2r^2\omega\, d\phi\, dt}{c^2} - \frac{r^2 d\phi^2}{c^2} - \frac{dr^2}{c^2} - \frac{dZ^2}{c^2} \quad -(1)$$

Consider the case of the null geodesic in a plane:

$$ds' = 0, \; dr = 0, \; dZ = 0. \quad -(2)$$

Then:

$$\left(1 - \frac{v^2}{c^2}\right) dt^2 = \left(\frac{r}{c}\right)^2 \left(2\omega\, d\phi\, dt + d\phi^2\right) \quad -(3)$$

i.e.

$$A\, dt^2 - 2\omega\, d\phi\, dt - d\phi^2 = 0 \quad -(4)$$

where:

$$A = \left(\frac{c}{r}\right)^2 \left(1 - \frac{v^2}{c^2}\right). \quad -(5)$$

Eq. (4) is a quadratic of the type:

$$ax^2 + bx + c = 0 \quad -(6)$$

so:

$$x = \frac{1}{2a}\left(-b \pm \left(b^2 - 4ac\right)^{1/2}\right) \quad -(7)$$

Thus:

$$dt = \frac{1}{A}\left(\omega \pm \left(\omega^2 + A\right)^{1/2}\right) d\phi \quad -(8)$$

with

$$v = \omega r. \quad -(9)$$

2) Thus:

$$\omega^2 + A = \frac{v^2}{c^2} + \left(\frac{c}{r}\right)^2\left(1 - \frac{v^2}{c^2}\right) = \left(\frac{c}{r}\right)^2 \quad -(10)$$

and:

$$dt = \left(\frac{r}{c}\right)^2\left(\frac{1}{1 - v^2/c^2}\right)\left(\frac{v}{r} \pm \frac{c}{r}\right) d\phi \quad -(11)$$

$$= \frac{(1 \pm v/c)}{(1 - v^2/c^2)} \frac{r}{c}\, d\phi \quad -(12)$$

Finally we: $$\left(1 - \frac{v^2}{c^2}\right) = \left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right) \quad -(13)$$

so.. $$\boxed{dt = \frac{r/c}{1 \pm v/c}\, d\phi} \quad -(14)$$

From eq. (14):

$$dt = \frac{r/c}{1 \pm \frac{r}{c}\omega}\, d\phi \quad -(15)$$

$$= \frac{1}{\frac{c}{r} \pm \omega}\, d\phi \quad -(16)$$

Finally we: $$\omega_0 = \frac{c}{r} \quad -(17)$$

to obtain $$\boxed{dt = \frac{1}{\omega_0 \pm \omega}\, d\phi} \quad -(18)$$

3)

If $$\int_0^{2\pi} d\phi = 2\pi \quad -(19)$$

then the time taken to cover a rotation of 2π is:

$$\boxed{t = \frac{2\pi}{\omega_0 \pm \omega}} \quad -(20)$$

This is the Sagnac effect. It has been shown that it is the null geodesic in a plane of the metric of the Thomas precession. This means that it is the Thomas precession for the photon. The Thomas precession is

$$\theta' = \gamma\theta \quad -(21)$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2} \quad -(22)$$

so

$$\theta' - \theta = \theta(\gamma - 1) \quad -(23)$$

and for a 2π rotation:

$$\boxed{\Delta\theta = 2\pi(\gamma - 1)} \quad -(24)$$

The Thomas precession is the dilation of angle due to the Lorentz boost. We have:

$$\Omega\, d\tau = \gamma\, d\theta \quad -(25)$$

$$\Omega = \gamma \frac{d\theta}{d\tau} \quad -(26)$$

4) So the relativistic angular velocity is:

$$\Omega = \left(1 - \frac{v^2}{c^2}\right)^{-1} \omega \quad -(27)$$

because

$$d\tau = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt \quad -(28)$$

and

$$\omega = \frac{d\theta}{dt} \quad -(29)$$

In ECE theory the rotating metric becomes a rotating spacetime.

Effect of Gravitation on the Sagnac Effect.

The factor A of eq. (5) is changed to:

$$\boxed{A = \left(\frac{c}{r}\right)^2 \left(1 - \frac{v^2}{c^2} - \frac{2MG}{c^2 R}\right)} \quad -(30)$$

so

$$v \rightarrow v_1 = \left(v^2 + \frac{2MG}{R}\right)^{1/2} \quad -(31)$$

Here M is the mass of a gravitating object, R is the distance between the photon and the gravitating object. If the speed of rotation of the platform is considered fixed then the effect of gravitation on

5) The Sagnac effect is the effect of gravitation on the photon. From eqs. (8) and (30), the Sagnac effect is, for a rotation of 2π:

$$\boxed{\begin{aligned} t &= \frac{2\pi}{A}\left(\omega \pm \left(\omega^2 + A\right)^{1/2}\right) \quad &-(32) \\ \text{where} \quad A &= \left(\frac{c}{r}\right)^2 \left(1 - \frac{v^2}{c^2} - \frac{2MG}{c^2 R}\right) \quad &-(33) \\ \text{and} \quad v &= \omega r \quad &-(34) \end{aligned}}$$

The algebra in eq. (32) can be worked out by computer algebra or by hand. The Sagnac effect in the absence of gravitation is recovered when:

$$R \rightarrow \infty \quad -(35)$$

145(7): Effect of Gravity on the Sagnac Effect:

Use eqns. (32) to (34) of the previous note to find that:

$$dt = \left(\frac{r}{c}\right)^2 \left(1 - \frac{v^2}{c^2} - \frac{2mG}{c^2R}\right)^{-1} \left(\frac{v}{r} \pm \frac{c}{r}\left(1 - \frac{2mG}{c^2R}\right)^{1/2}\right) d\phi$$

$$= \frac{r^2}{c^2} \frac{\left(\omega \pm \frac{c}{r}\left(1 - \frac{2mG}{c^2R}\right)^{1/2}\right) d\phi}{\left(1 - \frac{v^2}{c^2} - \frac{2mG}{c^2R}\right)} \quad -(1)$$

$$= \frac{1}{\omega_0^2} \frac{\left(\omega \pm \omega_0\left(1 - \frac{2mG}{c^2R}\right)^{1/2}\right) d\phi}{\left(1 - \frac{2mG}{c^2R} - \frac{v^2}{c^2}\right)}$$

$$= \frac{1}{\omega_0^2} \left(\frac{\omega_0 x \pm \omega}{x^2 - \frac{\omega^2}{\omega_0^2}}\right) \quad -(2)$$

where

$$x = 1 - \frac{2mG}{c^2R} \quad -(3)$$

Eq. (2) is:

$$dt = \frac{d\phi}{x\omega_0 \pm \omega} \quad -(4)$$

so

$$\boxed{t = \frac{2\pi}{x\omega_0 \pm \omega}} \quad -(5)$$

for a 2π path.

2)

The effect is the gravitational red-shift:

$$\omega_0 \rightarrow \left(1 - \frac{2MG}{c^2R}\right)^{1/2} \omega_0 \quad -(6)$$

of the photon angular frequency ω_0.

Results

1) Sagnac effect in absence of gravitation $= \frac{2\pi}{\omega_0 \pm \omega}$

2) In presence of gravitation $= \frac{2\pi}{x\omega_0 \pm \omega}$

where $x = \left(1 - \frac{2MG}{c^2R}\right)^{1/2}$

The photon is always in the presence of the Earth's mass M, at a distance R from the centre of the Earth. So its frequency as we normally measure it is $x\omega_0$. So its frequency in free space is ω_0.

$$\omega_0 = \frac{\Omega}{x} \quad -(7)$$

where $\Omega = x\omega_0$. Eq. (7) shows that the earth's mass M causes a gravitational red-shift of ω_0.

3.) Practical Applications

The ring laser gyro can be used to measure differences in M on the surface of the Earth, and build up a map.

In pure physics, this shows that the rotating frame metric of ECE theory is precisely equivalent to the use of rotating cylindrical coordinates in the metric. Phase factors in ECE theory are always due to spin connections.

145(8): Effect of Gravitation on the Sagnac Effect.

The effect is:

$$t = \frac{2\pi}{x\omega_0 \pm \omega} \quad -(1)$$

where

$$x = \left(1 - \frac{2MG}{c^2 R}\right)^{1/2} \quad -(2)$$

For the Earth:

$$M = 5.98 \times 10^{24} \text{ kgm}$$
$$G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$
$$c = 3 \times 10^{8} \text{ m s}^{-1}$$
$$R = 6.37 \times 10^{6} \text{ m}$$

So:

$$x = 1.39 \times 10^{-35} \quad -(3)$$

However, the gravitational red shift has been observed in the Pound Rebka experiment and the Sagnac interferometer could be adapted to use it as a gravimeter.

On a personal note I much like the simplicity of the result (1), and the way in which it brings together many different concepts.

145(9): Practical Application of the Sagnac Gravimeter

The time delay between clockwise and anticlockwise loops is:

$$\Delta t = 2\pi \left(\frac{1}{x\omega_0 - \omega} - \frac{1}{x\omega_0 + \omega} \right) \quad -(1)$$

where

$$x = \left(1 - \frac{2GM}{c^2 R} \right)^{1/2} \quad -(2)$$

and

ω_0 = light angular frequency,
ω = platform angular frequency
M = mass of gravitating object
R = photon to mass distance
c = vacuum speed of light
G = Newton's constant.

Here:

$$\omega_0 \gg \omega \quad -(3)$$

$$\frac{2GM}{c^2 R} \ll 1 \quad -(4)$$

Therefore

$$\Delta t = \frac{4\pi\omega}{x^2\omega_0^2 - \omega^2} \sim \frac{4\pi\omega}{x^2\omega_0^2}$$

$$= \frac{4\pi\omega}{\omega_0^2} \left(1 - \frac{2GM}{Rc^2} \right)^{-1}$$

$$\boxed{\Delta t \sim \frac{4\pi\omega}{\omega_0^2} \left(1 + \frac{2GM}{Rc^2} \right)} \quad -(5)$$

The relative shift is therefore:

2)

$$\boxed{1 : \frac{2GM}{Rc^2}} \quad -(6)$$

Simple Formula

Use $\frac{2G}{c^2} = 1.48 \times 10^{-27}$ $-(7)$

so the relative shift is:

$$\boxed{1 : 1.48 \times 10^{-27} \frac{M}{R}} \quad -(8)$$

If the Sagnac interferometer is placed one metre away from a one kilogram mass in the laboratory, the frequency shift is one part in 1.48×10^{-27}.

The instrument would need a frequency resolution of this accuracy.

Obviously, terrestrial masses such as a mountainside would have masses of order a million metric tonnes (10^9 kilograms). If the instrument were placed 100 m away from such a mass, the shift is 1.48×10^{-20}. This comes within range of a high accuracy ring laser gyro.

145(10): Doppler Effect and Gyro Gravimeter

! This is a very precise test of relativity and is based on the Pound Rebka experiment (Fig. (1)).

Photons of a precisely determined frequency are emitted from e to a receiver r, situated h away. The emitter is moved away at a velocity v. At resonance:

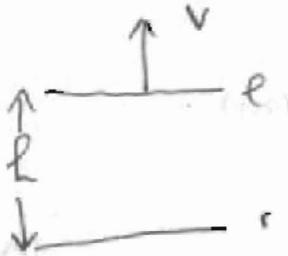


Fig. (1)

$$\frac{f_r}{f_e} = \left(\frac{1+v/c}{1\mp v/c}\right)^{1/2} = \left(\frac{1-r_0/(R+h)}{1-r_0/R}\right)^{1/2} \quad -(1)$$

where

$$r_0 = \frac{2mG}{c^2} \quad -(2)$$

and R = radius of earth

Eq. (1) is a balance of the relativistic Doppler effect and red shift.

Consider now two ring gyro's as in Fig. (2)

One gyro moves away from the other at a velocity v, and are situated a distance h apart:

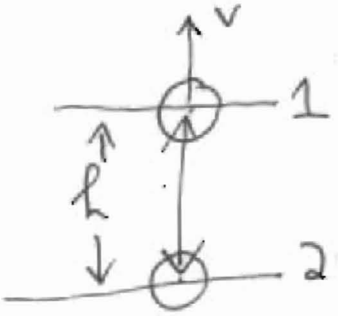


Fig. (2)

For gyro 1:

$$\Delta t_1 = 2\pi\left(\frac{1}{\omega_0 x_1 - \omega} - \frac{1}{\omega_0 x_1 + \omega}\right)$$

$$= \frac{4\pi\omega}{\omega_0^2 x_1^2 - \omega^2} \sim \left(\frac{4\pi\omega}{\omega_0^2}\right)\frac{1}{x_1^2} \quad -(3)$$

2)

where $$x_1 = \left(1 - \frac{2GM}{c^2(R+h)}\right)^{1/2} \quad -(4)$$

For gyro 2:
$$\Delta t_2 \sim \left(\frac{4\pi\omega}{\omega_0^2}\right)\frac{1}{x_2} \quad -(5)$$

where $$x_2 = \left(1 - \frac{2GM}{c^2 R}\right)^{1/2} \quad -(6)$$

So: $$\frac{\Delta t_1}{\Delta t_2} = \left(\frac{x_2}{x_1}\right) \quad -(7)$$

and
$$\left(\frac{\Delta t_2}{\Delta t_1}\right)^{1/2} = \left(\frac{1+v/c}{1+v/c}\right)^{1/2} = \left(\frac{1 - r_0/(R+h)}{1 - r_0/R}\right)^{1/2} \quad -(8)$$

Therefore:
$$\left(1+\frac{v}{c}\right)\left(1-\frac{r_0}{R}\right) = \left(1-\frac{v}{c}\right)\left(1-\frac{r_0}{R+h}\right) \quad -(9)$$

This equation can be solved to find m/R^2 in terms of v and h as follows. This gives a simple method of mapping m/R^2.

From eq. (9):

3)

$$1 + \frac{v}{c} - \frac{r_0}{R} - \frac{v}{c}\left(\frac{r_0}{R}\right) = 1 - \frac{v}{c} - \frac{r_0}{R+h} + \frac{v}{c}\frac{r_0}{R+h},$$

$$2\frac{v}{c} = r_0\left(\frac{1}{R} - \frac{1}{R+h}\right) + \frac{v}{c}\left(\frac{r_0}{R+h} - \frac{r_0}{R}\right)$$

$$\boxed{2\frac{v}{c} = r_0\left(\frac{1}{R} - \frac{1}{R+h}\right)\left(1 - \frac{v}{c}\right)} \quad -(10)$$

If

$$v \ll c \quad -(11)$$

$$2\frac{v}{c} \sim \frac{h r_0}{R(R+h)}. \quad -(12)$$

If

$$h \ll R \quad -(13)$$

$$2\frac{v}{c} \sim \frac{h r_0}{R^2} \quad -(14)$$

i.e.

$$\frac{v}{c} = \frac{hmG}{c^2R^2} \quad -(15)$$

and

$$\frac{m}{R^2} = \frac{c}{G}\frac{v}{h}$$

$$\boxed{\frac{m}{R^2} = 4.5\times 10^{18}\,\frac{v}{h}} \quad -(16)$$

Note 145(11) : Practical Implementations of the Doppler Laser Gyro Gravimeter.

From eq. (8) of note 145(10) :

$$\frac{\Delta t_2}{\Delta t_1} = \frac{1+v/c}{1-v/c} = \frac{1 - r_0/(R+h)}{1 - r_0/R} \quad -(1)$$

1) Move the gyro away from the earth on an optical bench at a velocity v. Then:

$$\frac{\Delta t_2}{\Delta t_1} = \frac{1+v/c}{1-v/c} \sim \left(1+\frac{v}{c}\right)\left(1+\frac{v}{c}\right)$$

$$= 1 + \frac{2v}{c} + \frac{v^2}{c^2} \quad -(2)$$

if

$$v << c \quad -(3)$$

so

$$\boxed{\frac{\Delta t_2}{\Delta t_1} \sim 1 + \frac{2v}{c}} \quad -(4)$$

Since $v/c \sim 10^{-8}$ for $v = 1$ metre sec^{-1} this instrument may be used as a very accurate measure of velocity. The frequency resolution of 10^{-8} is well within the range of the ring laser gyro.

) Place the gyro a distance h above ~~away from~~ the

2) other gyro a vertical optical bench, then:

$$\frac{\Delta t_2}{\Delta t_1} = \left(1 - \frac{r_0}{R+h}\right)\left(1 - \frac{r_0}{R}\right)^{-1}$$

$$\sim \left(1 - \frac{r_0}{R+h}\right)\left(1 + \frac{r_0}{R}\right)$$

if

$$r_0 \ll R \quad — (5)$$

Here

$$r_0 = \frac{2MG}{c^2}. \quad — (6)$$

For the Earth:

$$M = 5.98 \times 10^{24}\ \mathrm{kgm}$$
$$G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$$
$$c = 3 \times 10^{8}\ m s^{-1}$$

so

$$r_0 = 8.86 \times 10^{-3}$$
$$R = 6.37 \times 10^{6}$$
$$r_0 / R = 1.39 \times 10^{-9} \ll 1 \quad — (7)$$

Therefore:

$$\frac{\Delta t_2}{\Delta t_1} \sim 1 - \frac{r_0}{R+h} + \frac{r_0}{R} - \frac{r_0^2}{(R+h)R} \quad — (8)$$

The relative shift is

$$1 : r_0 \left(\frac{1}{R} - \frac{1}{R+h}\right) \quad — (9)$$

$$= \frac{r_0 h}{R(R+h)}$$

3)

Since: $h << R$ — (10)

then the relative shift is, to an excellent approximation:

$$\boxed{1 : \left(\frac{r_0}{R^2}\right) h} \quad — (11)$$

We have:

$$r_0 = 8.86 \times 10^{-3} \text{ m}$$
$$R = 6.37 \times 10^{6} \text{ m}$$

so the shift is:

$$\boxed{1 : 2.18 \times 10^{-16} h} \quad — (12)$$

This is well within the frequency resolution of a Sagnac interferometer.

This instrument is a very accurate altimeter.