1) 117(5): Potentials due to Localized Charge Distribution

The localized distribution of charge is described by the charge density $\rho(\underline{r}')$, which exists only inside a sphere of radius R around some origin. The potential outside the sphere is written in terms of an expansion in spherical harmonics:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \quad -(1)$$

This is a multipole expansion. The potential is given in terms of the charge density by:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x' \quad -(2)$$

where the integration is over all charges in the universe. The electric field strength at $\underline{x}$ due to a system of point charges q_i located at $\underline{x}_i$, $i = 1, 2, \ldots, n$ is the vector sum:

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^3} \quad -(3)$$

If the charges are so small and numerous that they can be replaced by a charge density $\rho(\underline{x}')$ then

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\underline{x}') \frac{\underline{x} - \underline{x}'}{|\underline{x} - \underline{x}'|^3} d^3x' \quad -(4)$$

where:

$$\rho(\underline{x}) = \sum_{i=1}^{n} q_i \, \delta(\underline{x} - \underline{x}_i) \quad -(5)$$

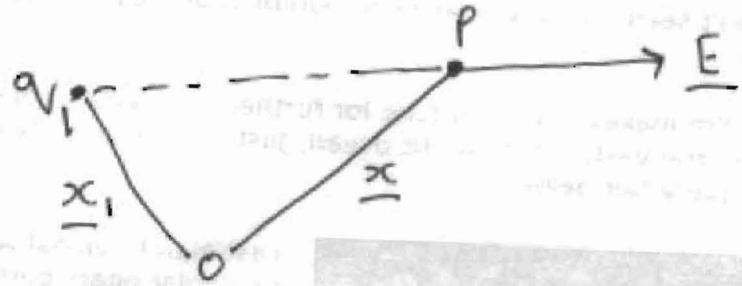
If $\underline{F}$ is the force on a point charge q_1 located at $\underline{x}_1$, due to another point charge q_2, located at $\underline{x}_2$, then Coulomb's law is:

2)

$$\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{(\underline{x}_1 - \underline{x}_2)}{|\underline{x}_1 - \underline{x}_2|^3} \quad — (6)$$

where q_1 and q_2 can be positive or negative. The electric field strength $\underline{E}$ (volts per metre) at the point $\underline{x}$ due to the point charge q_1 at the point $\underline{x}_1$ is:

$$\underline{E}(\underline{x}) = \frac{q_1}{4\pi\epsilon_0} \frac{(\underline{x} - \underline{x}_1)}{|\underline{x} - \underline{x}_1|^3} \quad — (7)$$



The Coulomb law in its continuous form is:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad — (8)$$

The potential at $\underline{x}$ due to a unit point charge at $\underline{x}'$ is expanded in the terms of spherical harmonics as follows:

$$\frac{1}{|\underline{x} - \underline{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \quad — (9)$$

where $r_<$ $(r_>)$ is the smaller (larger) of $|\underline{x}|$ and $|\underline{x}'|$ and γ is the angle between $\underline{x}$ and $\underline{x}'$.

3) This expansion can be generalized to:

$$\frac{1}{|\underline{x}-\underline{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{lm}^{*}(\theta', \phi') Y_{lm}(\theta, \phi) \qquad — (10)$$

This expression is used for integration over charge densities etc. where one variable is the variable of integration and the other is the coordinate of the observation point.

The potential outside the charge distribution is given by:

$$r_< = r', \quad r_> = r \qquad — (11)$$

and:

$$\phi(\underline{x}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} \left(\int Y_{lm}^{*}(\theta', \phi') r'^{l} \rho(\underline{x}') d^3x' \right) Y_{lm}(\theta, \phi) / r^{l+1} \qquad — (12)$$

The multipole moments are:

$$q_{lm} = \int Y_{lm}^{*}(\theta', \phi') r'^{l} \rho(\underline{x}') d^3x' \qquad — (13)$$

The total charge q is the monopole moment, given by:

$$q_{00} = \frac{1}{(4\pi)^{1/2}} \int \rho(\underline{x}') d^3x' = \frac{1}{(4\pi)^{1/2}} q$$

The electric dipole moment is:

$$\boxed{\underline{p} = \int \underline{x}' \rho(\underline{x}') d^3x'} \qquad — (14)$$

4) Therefore:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{\underline{p}\cdot\underline{x}}{r^3} + \cdots\right) \quad -(15)$$

The electric field components for a given multipole are found from:

$$\underline{E} = -\underline{\nabla}\phi \quad -(16)$$

For given l and m:

in the standard physics:

$$E_r = \frac{(l+1)}{(2l+1)\epsilon_0} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+2}} \quad -(17)$$

$$E_\theta = -\frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi) \quad -(18)$$

$$E_\phi = -\frac{1}{(2l+1)\epsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{im}{\sin\theta} Y_{lm}(\theta,\phi) \quad -(19)$$

For a dipole p along ~~ϕ~~ z axis:

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \quad -(20)$$

$$E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \quad -(21)$$

$$E_\phi = 0 \quad -(22)$$

5) The dipole field in vector form is found by recombining eqs. (20) to (22):

$$\boxed{\underline{E}(\underline{x}) = \frac{3\underline{n}(\underline{p}\cdot\underline{n}) - \underline{p}}{4\pi\epsilon_0 |\underline{x} - \underline{x}_0|^3}} \quad -(23)$$

This is the electric field strength at point $\underline{x}$ due to a dipole $\underline{p}$ at the point $\underline{x}_0$. In eq. (23) $\underline{n}$ is the unit vector directed from $\underline{x}_0$ to $\underline{x}$.

Eq. (23) is a term in the expansion of eq. (4), which in turn is a limit of the Coulomb laws (6). So the limits of validity of eq. (23) are the same as those of the Coulomb law itself.

The dipole moment coefficients in eq. (23) depend on the choice of origin. For example we can make the choice:

$$\underline{x}_0 = \underline{0} \quad -(24)$$

or we may define:

$$r = |\underline{x} - \underline{x}_0|. \quad -(25)$$

The dipole - dipole interaction energy is:

$$W_{12} = \frac{(\underline{p}_1\cdot\underline{p}_2 - 3(\underline{n}\cdot\underline{p}_1)(\underline{n}\cdot\underline{p}_2))}{4\pi\epsilon_0 |\underline{x}_1 - \underline{x}_2|^3} \quad -(26)$$

6) The basic Biot-Savart law of magnetic flux density (tesla or weber per square metre) is:

$$\underline{B}(\underline{x}) = \frac{\mu_0}{4\pi}\int \underline{J}(\underline{x}') \times \frac{(\underline{x}-\underline{x}')}{|\underline{x}-\underline{x}'|^3}\, d^3x' \quad -(27)$$

and this is the magnetic analogue of eq. (4). Here:

$$\frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3} = -\underline{\nabla}\left(\frac{1}{|\underline{x}-\underline{x}'|}\right) \quad -(28)$$

So:

$$\underline{B}(\underline{x}) = \frac{\mu_0}{4\pi}\underline{\nabla} \times \int \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\, d^3x' \quad -(29)$$

Therefore:

$$\boxed{\underline{\nabla} \cdot \underline{B} = 0} \quad -(30)$$

which is the first law of magnetostatics. Also:

$$\underline{\nabla} \times \underline{B} = \frac{\mu_0}{4\pi}\underline{\nabla} \times \underline{\nabla} \times \int \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\, d^3x' \quad -(31)$$

For steady state magnetic phenomena:

$$\underline{\nabla} \cdot \underline{J} = 0 \quad -(32)$$

So:

$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}} \quad -(33)$$

) which is the Ampère Law.

In the standard physics the vector potential is defined directly from eq. (29) as:

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(x')}{|\underline{x} - \underline{x}'|} d^3x' \quad - (34)$$

which is the magnetic analogue of eq. (2). The same denominator appears in both equations and can be expanded as in eq. (10) in multipoles. This denominator can be traced back to the Coulomb law as in these notes.

For a general current distribution localized in a region of charge, as in magnetostatics, the magnetic dipole moment is defined as:

$$\boxed{\underline{m} = \frac{1}{2} \int \underline{x}' \times \underline{J}(x') \, d^3x'} \quad - (35)$$

which is the analogue of the electric dipole moment:

$$\underline{p} = \int \underline{x}' \rho(\underline{x}') d^3x' \quad - (14)$$

In both cases the expansion (10) applies. In general there is no restriction on $\underline{x}$ and $\underline{x}'$.