1) 115(8): Lorentz Transform of the Faraday Law of Induction

We wish to consider the transform from frame K to K':

$$\partial_\mu \tilde{F}^{\mu\nu} \rightarrow \left(\partial_\mu \tilde{F}^{\mu\nu}\right)' = 0 \quad -(1)$$

using the well known rules of the Lorentz transformation. These rules are given in any good textbook such as Jackson, third edition or Carroll. They are:

$$\partial'_\mu = \frac{\partial}{\partial x^{\mu'}} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\frac{\partial}{\partial x^\mu}, \quad -(2)$$

$$\tilde{F}^{\mu\nu'} = \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)\tilde{F}^{\mu\nu}, \quad -(3)$$

so:

$$\left(\partial_\mu F^{\mu\nu}\right)' = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)\partial_\mu \tilde{F}^{\mu\nu}$$

$$\boxed{\left(\partial_\mu F^{\mu\nu}\right)' = \left(\frac{\partial x^{\nu'}}{\partial x^\nu}\right)\partial_\mu \tilde{F}^{\mu\nu} = 0} \quad -(4)$$

More generally:

$$\left(\partial_\alpha \tilde{F}^{\alpha\beta}\right)' = \left(\frac{\partial x^{\beta'}}{\partial x^\nu}\right)\partial_\mu \tilde{F}^{\mu\nu} = 0. \quad -(5)$$

The transform (5) is the same as for a vector:

$$V^{\beta'} = \left(\frac{\partial x^{\beta'}}{\partial x^\nu}\right)V^\nu. \quad -(6)$$

In the special case of the Lorentz transform of

2) Special relativity:

$$\frac{\partial x^{\mu'}}{\partial x^{\nu}} \rightarrow \Lambda^{\mu'}{}_{\nu} . \quad -(7)$$

The vector is:

$$\boxed{V^{\mu} = \left(c\,\underline{\nabla} \cdot \underline{B} \, , \, \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} \right) = 0} \quad -(8)$$

$$= \left(V^0, V^1, V^2, V^3 \right)$$
$$= \left(V^0, V_X, V_Y, V_Z \right).$$

So:
$$V^0 = c\,\underline{\nabla} \cdot \underline{B} , \quad -(9)$$
$$V_X = \left(\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} \right)_X \quad \text{etc.} \quad -(10)$$

and
$$\underline{V} = V_X \, \underline{i} + V_Y \, \underline{j} + V_Z \, \underline{k} . \quad -(11)$$

So
$$V^{\mu} = \left(V^0, \underline{V} \right). \quad -(12)$$

<u>Lorentz Boost in X Axis</u>

$$\begin{bmatrix} V^{0'} \\ V^{1'} \\ V^{2'} \\ V^{3'} \end{bmatrix} = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} \quad -(13)$$

where
$$\cosh\phi = \gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2} \quad -(14)$$

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$$\sinh\phi = \gamma\beta = \frac{v}{c}\gamma. \quad -(15)$$

Rotation about Z Axis

$$\begin{bmatrix} V^{0'} \\ V^{1'} \\ V^{2'} \\ V^{3'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^{0} \\ V^{1} \\ V^{2} \\ V^{3} \end{bmatrix}. \quad -(16)$$

Transformation of the Gauss Law

$$V^{0'} = V^{0}\cosh\phi - V^{1}\sinh\phi \quad -(17)$$

i.e.

$$\boxed{(\underline{\nabla}\cdot\underline{B})' = \left(1-\frac{v^2}{c^2}\right)^{-1/2}\left(\underline{\nabla}\cdot\underline{B} - \frac{v}{c}\left(\underline{\nabla}\times\underline{E} + \frac{\partial\underline{B}}{\partial t}\right)_x\right)} \quad -(18)$$

Thus :

$$\boxed{(\underline{\nabla}\cdot\underline{B})' = 0} \quad -(19)$$

because $\underline{\nabla}\cdot\underline{B} = 0 \quad -(20)$

and $\left(\underline{\nabla}\times\underline{E} + \frac{\partial\underline{B}}{\partial t}\right)_x = 0. \quad -(21)$

The last result follows from :

$$\underline{V} = V_x\,\underline{i} + V_y\,\underline{j} + V_z\,\underline{k} \quad -(22)$$
$$= \underline{0}$$

where:

$$\underline{V} := \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} \quad -(23)$$

so:

$$V_X = V_Y = V_Z = 0 . \quad -(24)$$

Transformation of the Faraday Law

$$\left.\begin{aligned} V^{1'} &= -V^0 \sinh\phi + V^1 \cosh\phi \\ V^{2'} &= V^2 \\ V^{3'} &= V^3 \end{aligned}\right\} \quad -(25)$$

So:

$$\boxed{\left(\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t}\right)'_X = \gamma\left(\left(\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t}\right)_X - v\, \underline{\nabla} \cdot \underline{B}\right)} \quad -(26)$$

i.e.:

$$\boxed{\left(\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t}\right)' = \underline{0}} \quad -(27)$$

Faraday Paradox of the Standard Model

In the Faraday disk there can be no induction of $\underline{E}$ by $\partial \underline{B} / \partial t = \underline{0}$ in frame K or K', contrary to observation.