1) 108(9): New Cubic Equation for the Relativistic Kepler Problem.

Consider again the potential energy of the system:

$$V = mc^2\left(\frac{1}{2} - \frac{r_s}{r}\right) + \frac{mL^2}{2r^2} - mL^2\frac{r_s}{r^3} \quad -(1)$$

In the standard model r_s is a constant but in ECE it may be regarded as a variable, so:

$$\frac{\partial V}{\partial r_s} = -\frac{mc^2}{r} - \frac{mL^2}{r^3} \quad -(2)$$

Therefore:

$$\boxed{\left(\frac{\partial V}{\partial r_s}\right) r^3 + mc^2 r^2 + mL^2 = 0} \quad -(3)$$

This is a cubic equation with three roots, in which L is a constant and where $(\partial V / \partial r_s)$ can be regarded as an adjustable parameter. In ECE:

$$|r_s| = T / R \quad -(4)$$

The rate of change of V with r is:

$$\frac{\partial V}{\partial r} = \frac{m}{r^2}\left(c^2 r_s - \frac{L^2}{r} + 3L^2\frac{r_s}{r^2}\right) \quad -(4)$$

Now use:

2)

$$\frac{\partial \bar{V}}{\partial r_s} = \frac{\partial \bar{V}}{\partial r}\frac{\partial r}{\partial r_s} \quad -(5)$$

So:

$$\frac{\partial r}{\partial r_s} = \left(\frac{\partial \bar{V}}{\partial r_s}\right) \Big/ \left(\frac{\partial \bar{V}}{\partial r}\right) \quad -(6)$$

$$\boxed{\frac{\partial r}{\partial r_s} = -\left(\frac{mc^2}{r} + \frac{mL^2}{r^3}\right)\left(\frac{m}{r^2}\left(c^2 r_s - \frac{L^2}{r} + 3L^2\frac{r_s}{r^2}\right)\right)^{-}}$$

$$-(7)$$

Thus:

$$\frac{\partial r}{\partial r_s} = -\left(rc^2 + \frac{L^2}{r}\right)\left(c^2 r_s - \frac{L^2}{r} + 3L^2\frac{r_s}{r^2}\right)^{-1}$$

and
$$\frac{\partial r_s}{\partial r} = -\left(c^2 r_s - \frac{L^2}{r} + 3L^2\frac{r_s}{r^2}\right) \Big/ \left(rc^2 + \frac{L^2}{r}\right)$$

$$\boxed{\frac{\partial r_s}{\partial r} = r\left(\frac{L^2}{r} - c^2 r_s - 3L^2\frac{r_s}{r^2}\right) \Big/ \left(r^2c^2 + L^2\right)}$$

$$r_s = \int \frac{r\left(\frac{L^2}{r} - c^2 r_s - 3L^2\frac{r_s}{r^2}\right)}{\left(r^2c^2 + L^2\right)}\,dr$$

3) Now use:

$$L = r^2 \frac{d\phi}{d\tau}, \quad - (9)$$

and:

$$r = \int \left(rc^2 + \frac{L^2}{r} \right) \left(\frac{L^2}{r} - c^2 r_s - 3 L^2 \frac{r_s}{r^2} \right)^{-1} dr_s$$

$$- (10)$$

Thus r_s may be integrated out from eq. (10) and r graphed as a function of L. For example if

$$|r_s| = T / R \quad - (11)$$

then for a given $(T / R)_0$: $- (12)$

$$r = \int_0^{(T/R)_0} \left(rc^2 + \frac{L^2}{r} \right) \left(\frac{L^2}{r} - c^2 r_s - 3 L^2 \frac{r_s}{r^2} \right)^{-1} dr_s$$

This gives an equation for r in terms of $(T/R)_0$ and L.

As $r_s \rightarrow 0$ in eq. (7):

$$\frac{\partial r}{\partial r_s} \rightarrow \frac{r}{L^2} \left(rc^2 + \frac{L^2}{r} \right) \quad - (13)$$

4)

Therefore as $r_S \longrightarrow 0$:

$$\boxed{r \longrightarrow \left(1 + \left(\frac{r c}{L}\right)^2\right) r_S} \quad -(14)$$

This gives the quadratic equation:

$$\left(\frac{c^2 r_S}{L^2}\right) r^2 - r + r_S = 0 \quad -(15)$$

So
$$r = \frac{2L^2}{c^2 r_S}\left(1 \pm \left(1 - 4\left(\frac{c r_S}{L}\right)^2\right)^{1/2}\right)$$

$$\rightarrow \frac{2L^2}{c^2 r_S} = \frac{2r^4}{c^2 r_S}\left(\frac{d\phi}{d\tau}\right)^2.$$

So:
$$r^3 \longrightarrow \frac{1}{2} c^2 r_S \left(\frac{d\phi}{d\tau}\right)^{-2} \quad -(16)$$

It is known that L is constant:

$$L = r^2 \frac{d\phi}{d\tau} = \text{constant} \quad -(17)$$

$$\text{so as } r \rightarrow 0, \ \frac{d\phi}{d\tau} \rightarrow \infty \quad -(18)$$

From (17) and (18):

$$\boxed{r \longrightarrow 0 \text{ as } r_S \longrightarrow 0} \quad -(19)$$