Supplementary Notes on the Numerical Problem of the Binary Pulsar.

The basic problem is to integrate the following equation to show how that r decreases by 3.1 mm a year for the Hulse Taylor binary pulsar:

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad - (1)$$

where:

$$r_s = \frac{T}{R}(r, \phi) \quad - (2)$$

in general. In the solar system:

$$r_s \rightarrow \frac{2mG}{c^2}. \quad - (3)$$

From eq. (1) the general orbital equation is obtained in a book such as that by Carroll by defining the constants of motion:

$$E = mc^2 \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau}, \quad - (4)$$

$$L = mr^2 \frac{d\phi}{d\tau}. \quad - (5)$$

The orbital equation is then:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m^2 c^2} - \left(1 - \frac{r_s}{r}\right)\left(c^2 + \frac{L^2}{m^2 r^2}\right) - (6)$$

Eq. (6) must be integrated numerically for r, using the definitions in eqs. (4) and (5).

2)

The potential energy of the system is:

$$V(r) = \frac{I}{R}\left(\frac{c^2}{r} + rm^2\left(\frac{d\phi}{d\tau}\right)^2\right) + \frac{1}{2}mr^2\left(\frac{d\phi}{d\tau}\right)^2 \quad -(7)$$

In general:

$$\frac{I}{R} = \frac{I}{R}(r, \phi) \quad -(8)$$

It is important to note that in a binary pulsar r is not constant, so in this sense the orbit is not stable. It decreases by 3.1 mm a year. The centripetal part of $V(r)$ is:

$$V(r) = \frac{1}{2}mr^2\left(\frac{d\phi}{d\tau}\right)^2 \quad -(9)$$

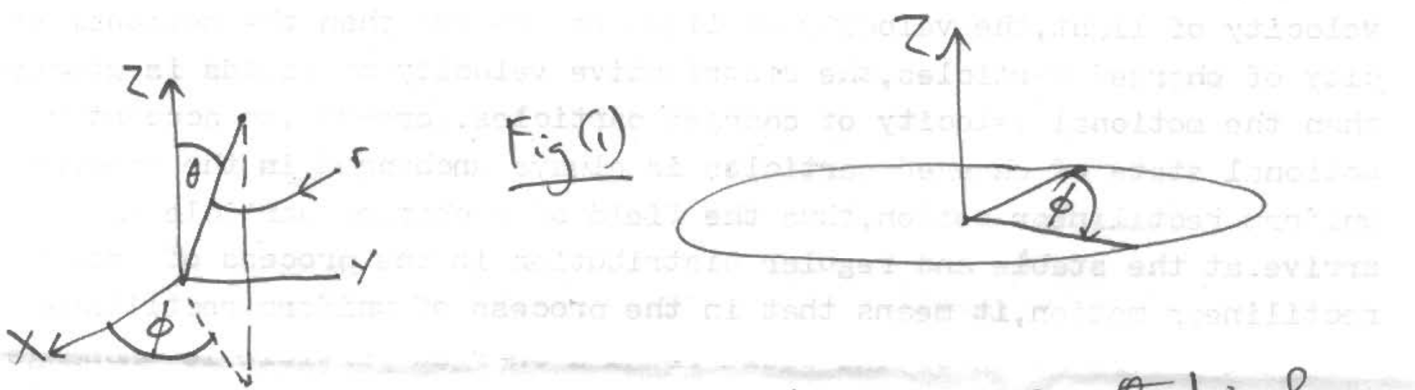

Fig (1)

By reference to Fig (1) it is seen that for constant r, (net r not decreasing per revolution) eqs. (5) and (9) represent the Kepler law that equal areas are swept out in equal times.

3)

From eq. (7) it is seen that as:

$$r \to 0 \quad -(10)$$

then

$$V(r) \to -\infty, \quad -(11)$$

i.e. the two objects of the pulsar will collide. It is seen from eq. (1) that r and ϕ are functions of t and τ. From eq. (1):

$$c^2\left(\frac{d\tau}{dt}\right)^2 = \left(1-\frac{r_s}{r}\right)c^2 - \left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2 \quad -(12).$$

where:

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau}. \quad -(13)$$

Also from eq. (1):

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\left(1-\frac{r_s}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{r_s}{r}\right)\left(c^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2\right)$$

$-(14)$

so:

$$\boxed{r = \int\left(c^2\left(\left(1-\frac{r_s}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{r_s}{r}\right)\right) + r^2\left(1-\frac{r_s}{r}\right)\left(\frac{d\phi}{d\tau}\right)^2\right)^{1/2} d\tau}$$

$-(15)$

4)

where in general:

$$r_s = -\frac{I}{R}(r, \phi) \quad - (16)$$

The EH Limit

This is defined by:

$$r_s \rightarrow \frac{2MG}{c^2}, \quad - (17)$$

$$\frac{dt}{d\tau} = \text{constant} \quad - (18)$$

$$\frac{d\phi}{d\tau} = \text{constant} \quad - (19)$$

Suggested Numerical Procedure

The orbit of the binary pulsar is decreasing by 3.1 mm a year. If this decrease is denoted δr, then $\delta r << r$. Therefore the conditions (18) and (19) are true to an excellent approximation. So:

$$r = \int \left(c^2 \left(\left(1 - \frac{r_s}{r}\right)^2 A - \left(1 - \frac{r_s}{r}\right) \right) + r^2 \left(1 - \frac{r_s}{r}\right) B \right)^{1/2} d\tau \quad - (20)$$

where A and B are constants.

5)

Now assume that:

$$r_s = \frac{I}{R}(\tau) \quad -(21)$$

and numerically integrate eq. (20) to give an equation for r in terms of A and B and $\frac{I}{R}(\tau)$. If the motion is approximately non-relativistic:

$$\frac{dt}{d\tau} \sim 1 \quad -(22)$$

and $d\phi / d\tau$ may be found experimentally. Therefore A and B are known, and so from eq. (20) r may be found as a function of $\tau \sim t$. The function r_s is modelled to give an orbit that gradually results in a collision of the two component stars of the binary pulsar.