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THE $\boldsymbol{B}^{(3)}$ FIELD AS A LINK BETWEEN GRAVITATION AND ELECTROMAGNETISM IN THE VACUUM

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The emergence of the $\boldsymbol{B}^{(3)}$ field in vacuo has shown that electromagnetism is non-Abelian and similar in structure to gravitation. In this paper the Christoffel symbol used in general relativity is developed for electromagnetism in curvilinear coordinates: the former becomes describable as the antisymmetric part of the gravitational Ricci tensor. Therefore gravitation and electromagnetism are respectively the symmetric and antisymmetric parts of the *same* Ricci tensor within a proportionality factor. Both fields are obtained from the Riemann curvature tensor, both are expressions of curvature in spacetime.

Key words: $\boldsymbol{B}^{(3)}$ field, gravitation, electromagnetism, unification.

1. INTRODUCTION

A new paradigm of electromagnetism in vacuo has recently been developed [1–8] through the cyclically symmetric equations (B cyclics),

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum.} \tag{1}$$

Here $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is the usual transverse, magnetic plane waves of scalar

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amplitude $B^{(0)}$, while $\boldsymbol{B}^{(3)}$ is a novel phase free magnetic flux density that propagates at c for all practical purposes (F.A.P.P.). Equations (1) show that electromagnetism is a property of spacetime itself, because the magnetic field components $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are rotation generators of the Poincaré group [1–4]. Fundamental tensorial duality in vacuo generates the equivalents of Eqs. (1) for electric components of vacuum electromagnetism (E cyclics),

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = -E^{(0)}(i\boldsymbol{E}^{(3)})^*, \text{ et cyclicum,} \tag{2}$$

in which the component $i\boldsymbol{E}^{(3)}$ is formally imaginary and unphysical in the classical theory of the electromagnetic field. The set of cyclic relations is completed by those for vector potential components (A cyclics),

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = -A^{(0)}(i\boldsymbol{A}^{(3)})^*, \text{ et cyclicum,} \tag{3}$$

Equations (1) to (3) are each written in the complex circular representation of three dimensional space [1–10],

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}, \text{ et cyclicum,} \tag{4}$$

where the $\boldsymbol{e}$'s are unit vectors related to Cartesian unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ by,

$$\boldsymbol{e}^{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}), \qquad \boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j}), \qquad \boldsymbol{e}^{(3)} = \boldsymbol{k}. \tag{5}$$

The existence in vacuo of the B, E, and A cyclics means that the Maxwell equations are incomplete, or self-inconsistent, because they are written in an O(2) = U(1) gauge group which allows no longitudinal field component in vacuo. Equations (1), on the other hand, reveal the presence of the longitudinal magnetic component $\boldsymbol{B}^{(3)}$ in vacuo. Moreover, the latter is observable in routine magneto-optics [11–14] in the second order form $iB^{(0)}\boldsymbol{B}^{(3)*}$. In orthodox theory [15] this is, from Eqs. (1), the conjugate product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ of non-linear optics, a quantity through which phenomena such as the inverse Faraday effect [11] are explained. Under the right conditions, $\boldsymbol{B}^{(3)}$ also acts at first order, generating an observable $I^{1/2}$ profile of magnetization in a sample, where I is beam power density in W m^{-2}. There is no reasonable doubt therefore that $\boldsymbol{B}^{(3)}$ is an experimental observable if the existence of magneto-optic effects is accepted, and these effects are well documented and reviewed [11–14].

If so, the gauge group of vacuum electromagnetism becomes the group of rotations, O(3) [2], meaning that derivatives in vacuum electromagnetism become covariant in an O(3) gauge group. In this Letter, the consequences of this logic are worked out in curvilinear coordinates. In Sec. 2, the Christoffel symbol of general relativity, usually used to describe gravitation, is developed for use with O(3) electromagnetism, and is equated to a product of the Poincaré group's rotation generators with a canonical momentum -energy tensor eA_{μ}^{ν}. This is possible if and only if the $\boldsymbol{B}^{(3)}$ field is non-zero as observed experimentally in magneto-optics. Section 3 expresses O(3) vacuum electromagnetism as an Einstein equation, usually used for gravitation in curvilinear coordinates. The tensor eA_{μ}^{ν} is expressed therefore as a curvature tensor of curved spacetime, showing that electromagnetism curves spacetime in a manner exactly analogous with gravitation.

2. THE CHRISTOFFEL SYMBOL FOR O(3) VACUUM ELECTROMAGNETISM.

In the general gauge theory of fields, the derivative with respect to the spacetime four-vector x_{μ} of an n dimensional field $\psi(x)$ is not covariant. The fields $\psi(x)$ and $\psi(x+dx)$ are measured in different coordinate systems because [16,17] the quantity $d\psi$ carries information about the variation of the field ψ itself with distance, but also about the rotation of the axes themselves in curvilinear coordinates on moving from x to $x+dx$. The concept of parallel transport [16,17], as used in general relativity, is introduced into the theory of electromagnetism to describe the curvature of spacetime. As described by Landau and Lifshitz [17], the change, δA^{ν}, in the components of a vector under an infinitesimal parallel displacement depends linearly on the values of the components themselves. Thus,

$$\delta A^{\mu} = -\Gamma^{\mu}_{\lambda\nu} A^{\lambda} dx^{\nu}, \tag{6}$$

where $\Gamma^{\mu}_{\lambda\nu}$ is the Christoffel symbol [17,18]. In flat spacetime (special relativity) the symbol $\Gamma^{\mu}_{\lambda\nu}$ is zero. The analogy to Eq. (6) in the theory of electromagnetic fields for arbitrary gauge group symmetry is [2,16],

$$\delta\psi = igM^{a}A_{\mu}^{a}dx^{\mu}\psi, \tag{7}$$

where M^a is the *a'th* rotation generator matrix of the gauge group and g is a dimensionality adjustment [16]. Here A_{μ}^{a} is Feynman's *universal influence*, an additional field potential necessitated by the structure of spacetime. For an O(3) gauge group symmetry, M^a are identical with the three infinitesimal rotation generators of three dimensional space. These are, within a factor $\hbar$, angular momentum operators of quantum mechanics, and are in general $n \times n$ matrices where n is the spacetime dimension being considered. For the Poincaré group, $n = 4$, and the indices a run from 1 to 4. All group generators obey the Jacobi identity,

$$[[M_i, M_j], M_k] + [[M_j, M_k], M_i] + [[M_k, M_i], M_j] = 0. \tag{8}$$

In the O(3) theory of vacuum electromagnetism [2], using the circular basis ((1), (2), (3)) defined in Eqs. (4), the constant g is identified with the elementary charge, e, divided by the elementary unit of action. The latter is defined in the quantum hypothesis [1–3,16] as $\hbar$. The dimensions of eA_{μ}^{a} are those of an energy-momentum tensor. It is known that in the O(3) theory [2], there exists an equivalence between the quantized momentum of a photon, $\hbar\kappa$, where κ is the wave-vector magnitude, and $eA^{(0)}$, where $A^{(0)}$ is the magnitude of the wave-vector in vacuo. Thus,

$$p := \hbar\kappa = eA^{(0)}. \tag{9}$$

his result can be generalized by replacing O(3) by the Poincaré group,

$$p_\mu^\nu = eA_\mu^\nu, \tag{10}$$

here p_μ^ν is the canonical energy-momentum tensor of free space electromagne-ism. The link (10) between p_μ^ν and eA_μ^ν allows the Christoffel symbol to be lefined for free space electromagnetism in an evidently *curved* spacetime,

$$\Gamma_{\lambda\nu}^\mu = -i\frac{e}{\hbar}M_\lambda A_\nu^\mu. \tag{11}$$

herefore p_μ^ν is expressible in terms of components of the Riemann curvature ensor [16–18] $R_{\lambda\mu\nu}^\kappa$, and, Einstein tensor $G_{\mu\nu}$. These considerations, in turn, llow electromagnetism to be expressed in terms of an Einstein equation, isually used, of course, for gravitation. The mathematical structure of lectromagnetism and gravitation becomes the same, provided that the gauge roup of the electromagnetic field is the Poincaré group, the group of pacetime, rather than O(3), the group of space, or the orthodox flat group (2) = U(1) [16]. This is possible if and only if there exist the B, E and A yclic equations of the introduction. In the orthodox theory of vacuum lectromagnetism [16], the gauge group is O(2) = U(1) and the $\boldsymbol{B}^{(3)}$ field is nconsidered. The $\boldsymbol{B}^{(3)}$ field is, however, defined by an experimental bservable, $iB^{(0)}\boldsymbol{B}^{(3)*}$, which is *geometrically* equal to $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ irrespec-ive of equations such as those of Maxwell, which are equations in spacetime erivatives. Equations (1) to (3) are evidently geometrical equations nalogous with Eqs. (4), and for this reason are as fundamental as the tructure of spacetime itself. The basic ingredients needed for free space lectromagnetism are: 1) charge, e; 2) spacetime geometry curved by charge.

The Christoffel symbol is a connection coefficient that is a property of curved spacetime. It is zero in flat spacetime. Equation (11) therefore shows that in the presence of electromagnetism in vacuo, spacetime becomes curved. In the theory of curvilinear coordinates, the connection coefficient must, furthermore, be symmetric in its lower indices, and in general is not a tensor, because it can exist in one frame of reference and vanish in another. The canonical energy-momentum tensor must also be symmetric in its indices [17,18] and taking the diagonal component we obtain

$$\Gamma^{\kappa}_{\lambda\nu} = \frac{e}{\hbar}A^{(0)}\delta^{\kappa}_{\lambda}M_{\nu}, \tag{12}$$

giving

$$\Gamma^{\lambda}_{\lambda\nu} = \frac{e}{\hbar}A^{(0)}M_{\nu} = \kappa M_{\nu}. \tag{13}$$

These are evidently equations of vacuum electromagnetism in curved spacetime. In the usual notation of general relativity applied to gravitation,

$$T^{\nu}_{\mu} = eA^{\nu}_{\mu}, \qquad T^{\mu\nu} = g^{\mu\sigma}T^{\nu}_{\sigma}, \tag{14}$$

where $g^{\mu\sigma}$ is the metric tensor. Finally, if J_{λ} are 4 x 4 angular momentum matrices, then,

$$\Gamma^{\mu}_{\lambda\nu} = \frac{J_{\lambda}}{\hbar}\frac{e}{\hbar}A^{\mu}_{\nu}, \tag{15}$$

and the quantity $-eA^{\mu}_{\nu}$ is a conserved current according to Noether's theorem. The integral over isospace,

$$Q_{\nu} = -\int_{\sigma} eA^{\mu}_{\nu}\, d\sigma_{\mu}, \tag{16}$$

is therefore a conserved charge.

In summary, the equations of the electromagnetic field can be unified with those of the gravitational field provided that the gauge group symmetry of electromagnetism is that of the Poincaré group. This implies that the $\boldsymbol{B}^{(3)}$ field is non-zero, and conversely, that the $\boldsymbol{B}^{(3)}$ field indicates a crucial link between the electromagnetic and gravitational fields.

3. RIEMANN TENSOR AND EQUIVALENCE PRINCIPLE FOR ELECTROMAGNETISM

The Riemann tensor is defined in terms of connection coefficients as follows [17,18],

$$R^{\kappa}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\kappa}_{\lambda\nu} - \partial_{\nu}\Gamma^{\kappa}_{\lambda\mu} + \Gamma^{\rho}_{\lambda\nu}\Gamma^{\kappa}_{\rho\mu} - \Gamma^{\rho}_{\lambda\mu}\Gamma^{\kappa}_{\rho\nu} . \tag{17}$$

If we take the diagonal components of A^{κ}_{λ} to be the only non-zero components then

$$\Gamma^{\kappa}_{\lambda\nu} := \frac{e}{\hbar} A \delta^{\kappa}_{\lambda} M_{\nu} . \tag{18}$$

The Riemann tensor (17), contracted with $\kappa = \lambda$ then becomes *the antisymmetric Ricci tensor*,

$$\begin{aligned} R^{(A)}_{\mu\nu} = R^{\lambda}_{\lambda\mu\nu} &= \frac{e}{\hbar}\partial_{\mu}(AM_{\nu}) - \frac{e}{\hbar}\partial_{\nu}(AM_{\mu}) \\ &+ \left(\frac{e}{\hbar}A\right)^2 (M_{\nu}M_{\mu} - M_{\mu}M_{\nu}) . \end{aligned} \tag{19}$$

This has the structure of the electromagnetic field strength tensor $G_{\mu\nu}$ in gauge theory [16] determined by the rotation generators M_{μ} of the Poincaré group. This logic leads to the novel field-frame equivalence principle,

$$G_{\mu\nu} = \frac{\hbar}{e} R^{(A)}_{\mu\nu} , \tag{20}$$

between the electromagnetic field strength tensor and the antisymmetric Ricci tensor. Using Eqs. (19) and (20) we obtain

$$G_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + \frac{e}{\hbar}A^2 [M_{\nu}, M_{\mu}] , \tag{21}$$

where

$$A_{\nu} := AM_{\nu} , \qquad A_{\mu} := AM_{\mu} , \tag{22}$$

are potential four-vectors. In general the scalar A is phase dependent because

$$\Box A_\mu = \Box A_\nu = 0, \tag{23}$$

is the d'Alembert wave equation in vacuo. Therefore A_μ and A_ν are complex solutions of Eq. (23) and are rotational in nature. Equation (21) is the standard definition of electromagnetic field strength [16] in a group which allows the commutator $[M_\mu, M_\nu]$ to be different from zero. If A_μ and A_ν are considered to be rotational in nature, we can consider $[M_\mu, M_\nu]$ to be a commutator of infinitesimal rotation generators [16] of the O(3) subgroup of the Poincaré group of spacetime (the ten parameter inhomogeneous Lorentz group). The O(3) group is that of rotations in 3-D space, a valid basis for which is the complex circular basis ((1), (2), (3)) [1–8]. In this basis Eq. (21) gives the (3) component of $G_{\mu\nu}$ as [1–8],

$$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \frac{e}{\hbar}A^2\boldsymbol{e}^{(3)*}. \tag{24}$$

In terms of the fluxon $\Phi^{(0)} := \hbar/e$ and the scalar curvature $R = \kappa^2$,

$$\boldsymbol{B}^{(3)*} = \Phi^{(0)}R\boldsymbol{e}^{(3)*}, \tag{25}$$

for one photon. Equation (21) means that the complete three dimensional magnetic flux density vector for one photon is

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)}, \tag{26}$$

with [1–8],

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \text{ et cyclicum}, \tag{27}$$

and photon energy,

$$\hbar\omega := \frac{1}{\mu_0}\int \boldsymbol{B}\cdot\boldsymbol{B}\,dV, \tag{28}$$

where μ_0 is vacuum permeability and V the photon volume. Equation (24) is seen to be a type of Einstein equation, because $\boldsymbol{B}^{(3)}$ is an element of the

Riemann tensor, and the right hand side is proportional to an element of the energy momentum tensor cross multiplied into a factor $\boldsymbol{A}^{(2)}/\hbar$. Since the Riemann tensor is a measure of curvature, it is concluded that electromagnetism bends spacetime in direct analogy to gravitation, and in coming to this conclusion, the role of $\boldsymbol{B}^{(3)}$ is critically important. The analysis leading to the definition of $\boldsymbol{B}^{(3)}$ in Eq. (27) shows conclusively that $\boldsymbol{B}^{(3)}$ is non-zero in general relativity. It is inferred furthermore that there is a gravitational field akin to $\boldsymbol{B}^{(3)}$, which propagates with gravitational waves at the speed of light. Gravitation and electromagnetism can now be understood within the same curvilinear geometrical framework. For example the inhomogeneous free space electrodynamic equations become structured in the same way as the Bianchi identity [2]

$$D_\rho R^{\kappa}_{\lambda\mu\nu} + D_\mu R^{\kappa}_{\lambda\nu\rho} + D_\nu R^{\kappa}_{\lambda\rho\mu} = 0, \tag{29}$$

and the Riemann tensor contains electric and magnetic field components in curved spacetime. The curvature of spacetime due to the presence of electromagnetism is determined by the energy-momentum tensor through the free space minimal prescription,

$$T^{\nu}_{\mu} = eA^{\nu}_{\mu}, \tag{30}$$

and the partial derivatives in the theory of electromagnetism are replaced by covariant derivatives,

$$D_\nu V^\mu := \partial_\nu V^\mu + \Gamma^{\mu}_{\lambda\nu} V^\lambda, \tag{31}$$

defined through the Christoffel symbol (11).

DISCUSSION

The main result of general relativity, the Einstein equation, shows that the stress energy of matter generates a curvature [18],

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{32}$$

This equation is also a propagation equation for the anisotropic part of curvature. From the analogy drawn in Sec. 3 we can conclude that the stress energy of electromagnetism generates curvature in direct analogy with the stress energy of matter. Similarly, general relativity shows that curvature in geometry manifests itself as gravitation, and so it also manifests itself as electromagnetism. Again, the source of mass-energy causes curvature in geometry, and so the source of charge-energy also causes curvature, which manifests itself as electromagnetism. In a sense, this is simply an example of the general rule [18] that all laws of physics can be expressed geometrically. However, the introduction of $\boldsymbol{B}^{(3)}$ into electromagnetism makes the latter non-linear [1–3], and the covariant derivatives of electromagnetic theory become the SAME in structure as those of gravitation. Without $\boldsymbol{B}^{(3)}$ this is not true, because electromagnetism is then describable with ordinary partial derivatives. For example, in electromagnetism without $\boldsymbol{B}^{(3)}$ we have well known results such as [17,18],

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu;\mu} - A_{\mu;\nu}, \tag{33}$$

in which the connection coefficients have vanished, even in curved spacetime. Equation (33) shows that the field tensor $F_{\mu\nu}$ is the same in special and general relativity if $\boldsymbol{B}^{(3)}$ is missing from the analysis. Equation (24) however, shows that this is obviously not true if $\boldsymbol{B}^{(3)}$ is accounted for as it should.

Lastly, it is important to note that the expression of electromagnetism in terms of curvilinear coordinates developed in this paper does *not* mean that there is a gravitational field present, because electromagnetism itself is considered to bend spacetime. It has been established that the O(3) inhomogeneous electrodynamic equations [2] are described by a novel antisymmetric Ricci tensor obtained by contracting the Riemann tensor. The symmetric part of the *same* overall Ricci tensor is the Einstein tensor of gravitation within a proportionality factor. This results in the unification of electrodynamics and gravitation. It has been established that the O(3) inhomogeneous electrodynamic equations [2] are described by a novel anti-symmetric Ricci tensor obtained by contracting the Riemann tensor. The symmetric part of the *same* overall Ricci tensor is the Einstein tensor of gravitation within a proportionality factor. This results in the unification of electrodynamics and gravitation.

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