PREFACE

This volume, produced in three parts, is the Second Edition of Volume 85 of the series, *Modern Nonlinear Optics*, edited by M. W. Evans and S. Kielich. Volume 119 is largely a dialogue between two schools of thought, one school concerned with quantum optics and Abelian electrodynamics, the other with the emerging subject of non-Abelian electrodynamics and unified field theory. In one of the review articles in the third part of this volume, the Royal Swedish Academy endorses the complete works of Jean-Pierre Vigier, works that represent a view of quantum mechanics opposite that proposed by the Copenhagen School. The formal structure of quantum mechanics is derived as a linear approximation for a generally covariant field theory of inertia by Sachs, as reviewed in his article. This also opposes the Copenhagen interpretation. Another review provides reproducible and repeatable empirical evidence to show that the Heisenberg uncertainty principle can be violated. Several of the reviews in Part 1 contain developments in conventional, or Abelian, quantum optics, with applications.

In Part 2, the articles are concerned largely with electrodynamical theories distinct from the Maxwell–Heaviside theory, the predominant paradigm at this stage in the development of science. Other review articles develop electrodynamics from a topological basis, and other articles develop conventional or U(1) electrodynamics in the fields of antenna theory and holography. There are also articles on the possibility of extracting electromagnetic energy from Riemannian spacetime, on superluminal effects in electrodynamics, and on unified field theory based on an SU(2) sector for electrodynamics rather than a U(1) sector, which is based on the Maxwell–Heaviside theory. Several effects that cannot be explained by the Maxwell–Heaviside theory are developed using various proposals for a higher-symmetry electrodynamical theory. The volume is therefore typical of the second stage of a paradigm shift, where the prevailing paradigm has been challenged and various new theories are being proposed. In this case the prevailing paradigm is the great Maxwell–Heaviside theory and its quantization. Both schools of thought are represented approximately to the same extent in the three parts of Volume 119.

As usual in the *Advances in Chemical Physics* series, a wide spectrum of opinion is represented so that a consensus will eventually emerge. The prevailing paradigm (Maxwell–Heaviside theory) is ably developed by several groups in the field of quantum optics, antenna theory, holography, and so on, but the paradigm is also challenged in several ways: for example, using general relativity, using O(3) electrodynamics, using superluminal effects, using an extended electrodynamics based on a vacuum current, using the fact that longitudinal waves may appear in vacuo on the U(1) level, using a reproducible and repeatable device, known as the *motionless electromagnetic generator*, which extracts electromagnetic energy from Riemannian spacetime, and in several other ways. There is also a review on new energy sources. Unlike Volume 85, Volume 119 is almost exclusively dedicated to electrodynamics, and many thousands of papers are reviewed by both schools of thought. Much of the evidence for challenging the prevailing paradigm is based on empirical data, data that are reproducible and repeatable and cannot be explained by the Maxwell–Heaviside theory. Perhaps the simplest, and therefore the most powerful, challenge to the prevailing paradigm is that it cannot explain interferometric and simple optical effects. A non-Abelian theory with a Yang–Mills structure is proposed in Part 2 to explain these effects. This theory is known as O(3) *electrodynamics* and stems from proposals made in the first edition, Volume 85.

As Editor I am particularly indebted to Alain Beaulieu for meticulous logistical support and to the Fellows and Emeriti of the Alpha Foundation's Institute for Advanced Studies for extensive discussion. Dr. David Hamilton at the U.S. Department of Energy is thanked for a Website reserved for some of this material in preprint form.

Finally, I would like to dedicate the volume to my wife, Dr. Laura J. Evans.

MYRON W. EVANS

Ithaca, New York

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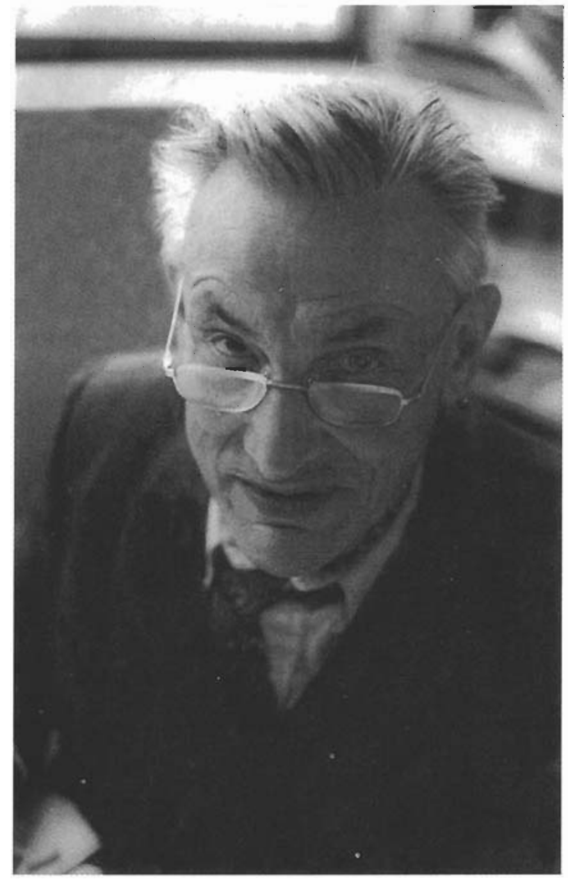

The Royal Swedish Academy of Sciences (KVA) hereby confirms its support to a joint production with John Wiley and Sons, Inc. of the collective scientific works by Professor Jean-Pierre Vigier. It is thereby understood that this endorsement only concerns and objective estimate of these works, and implying no economic obligation from the side of the Academy.

THE PRESENT STATUS OF THE QUANTUM THEORY OF LIGHT

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I. INTRODUCTION

If one takes as the birth of the quantum theory of light, the publication of Planck's famous paper solving the difficulties inherent in the blackbody spectrum [1], then we are currently marking its centenary. Many developments have occurred since 1900 or so and are briefly reviewed below. (See Selleri [27] or Milloni [6] for a more comprehensive historical review). The debates concerning wave–particle duality are historically rooted in the seventeenth century with the publication of Newton's *Optiks* [2] and the *Treatise on Light* by Christian Huygens [3]. For Huygens, light was a form of wave motion propagating through an ether that was conceived as a substance that was "as nearly approaching to perfect hardness and possessing a springiness as prompt as we choose." For Newton, however, light comprised material particles and he argues, contra Huygens, "Are not all hypotheses erroneous, in which Light is supposed to consist of Pression, or Motion propagated through a Fluid medium?" (see Newton [2], Query 28). Newton attempts to refute Huygens' approach by pointing to the difficulties in explaining double refraction if light is simply a form of wave motion and asks, "Are not the Rays of Light very small bodies emitted from shining substances? For such bodies will pass through uniform Mediums in right Lines without bending into Shadow, which is the Nature of the Rays of Light?" (Ref. 2, Query 29). The corpuscular theory received a major blow in the nineteenth century with the publication of Fresnel's essay [4] on the diffraction of light. Poisson argued on the basis of Fresnel's analysis that a perfectly round object should diffract so as to produce a bright spot on the axis behind it. This was offered as a reductio ad absurdum argument against wave theory. However, Fresnel and Arago carried out the actual experiment and found that there is indeed a diffracted bright spot. The nineteenth century also saw the advent of accurate methods for the determination of the speed of light by Fizeau and Foucault that were used to verify the prediction from Maxwell's theory relating the velocity of light to known electric and magnetic constants. Maxwell's magnificent theory of electromagnetic waves arose from the work of Oersted, Ampère, and Faraday, which proved the intimate interconnection between electric and magnetic phenomena.

This volume discusses the consequences of modifying the traditional, classical view of light as a transverse electromagnetic wave whose electric and magnetic field components exist only in a plane perpendicular to the axis of propagation, and posits the existence of a longitudinal magnetic field component. These considerations are of relatively recent vintage, however [5].

The corpuscular view was revived in a different form early in twentieth century with Planck's solution of the blackbody problem and Einstein's adoption of the photon model in 1905. Milloni [6] has emphasized the fact that Einstein's famous 1905 paper [7] "Concerning a heuristic point of view toward the

emission and transformation of light" argues strongly for a model of light that *simultaneously* displays the properties of waves and particles. He quotes Einstein:

> The wave theory of light, which operates with continuous spatial functions, has worked well in the representation of purely optical phenomena and will probably never be replaced by another theory. It should be kept in mind, however, that the optical observations refer to time averages rather than instantaneous values. In spite of the complete experimental confirmation of the theory as applied to diffraction, reflection, refraction, dispersion, etc., it is still conceivable that the theory of light which operates with continuous spatial functions may lead to contradictions with experience when it is applied to the phenomena of emission and transformation of light.
>
> According to the hypothesis that I want here to propose, when a ray of light expands starting from a point, the energy does not distribute on ever increasing volumes, but remains constituted of a finite number of energy quanta localized in space and moving without subdividing themselves, and unable to be absorbed or emitted partially.

This is the famous paper where Einstein, adopting Planck's idea of light quanta, gives a complete account of the photoelectric effect. He predicts the linear relationship between radiation frequency and stopping potential: "As far as I can see, there is no contradiction between these conceptions and the properties of the photoelectric effect observed by Herr Lenard. If each energy quantum of the incident light, independently of everything else, delivers its energy to electrons, then the velocity distribution of the ejected electrons will be independent of the intensity of the incident light. On the other hand the number of electrons leaving the body will, if other conditions are kept constant, be proportional to the intensity of the incident light."

Textbooks frequently cite this work as strong empirical evidence for the existence of photons as quanta of electromagnetic energy localized in space and time. However, it has been shown that [8] a complete account of the photoelectric effect can be obtained by treating the electromagnetic field as a classical Maxwellian field and the detector is treated according to the laws of quantum mechanics.

In view of his subsequent discomfort with dualism in physics, it is ironic that Einstein [9] gave a treatment of the fluctuations in the energy of electromagnetic waves that is fundamentally dualistic insofar that, if the Rayleigh–Jeans formula is adopted, the fluctuations are characteristic of electromagnetic waves. However, if the Wien law is used, the fluctuations are characteristic of particles. Einstein made several attempts to derive the Planck radiation law without invoking quantization of the radiation but without success. There was no alternative but to accept the quantum. This raised immediately the difficult question as to how such quanta gave rise to interference phenomena. Einstein suggested that perhaps light quanta need not interfere with themselves, but might interfere with other quanta as they propagated. This suggestion was soon ruled out by interference experiments conduced at extremely low light levels. Dirac, in his well-known textbook [10] on quantum mechanics, stated "Each photon interferes only with itself. Interference between two different photons never occurs." The latter part of this statement is now known to be wrong [11]. The advent of highly coherent sources has enabled two-beam interference with two separate sources. In these experiments, the classic interference pattern is not observed but rather intensity correlations between the two beams are measured [12]. The recording of these intensity correlations is proof that the electromagnetic fields from the two lasers have superposed. As Paul [11] argues, any experiment that indicates that such a superposition has occurred should be called an interference experiment.

Taylor [13] was the first to report on two-beam interference experiments undertaken at extremely low light levels such that one can assert that, on average, there is never more than one photon in the apparatus at any given time. Such experiments have been repeated many times. However, given that the sources used in these experiments generated light beams that exhibited photon bunching [14], the basic assumption that there is only ever one photon in the apparatus at any given time is not sound. More recent experiments using sources that emit single-photon states have been performed [15–17].

In 1917 Einstein [18] wrote a paper on the dualistic nature of light in which he discusses emission "without excitation from external causes," in other words stimulated emission and also spontaneous absorption and emission. He derives Planck's formula but also discusses the recoil of molecules when they emit photons. It is the latter discussion that Einstein regarded as the most significant aspect of the paper: "If a radiation bundle has the effect that a molecule struck by it absorbs or emits a quantity of energy $h\nu$ in the form of radiation (ingoing radiation), then a momentum $h\nu/c$ is always transferred to the molecule. For an absorption of energy, this takes place in the direction of propagation of the radiation bundle; for an emission, in the opposite direction."

In 1923, Compton [19] gave convincing experimental evidence for this process: "The experimental support of the theory indicates very convincingly that a radiation quantum carries with itself, directed momentum as well as energy." Einstein's dualism raises the following difficult question: If the particle carries all the energy and momentum then, in what sense can the wave be regarded as real? Einstein's response was to refer to such waves as "ghost fields" (Gespensterfelder). Such waves are also referred to as "empty" - a wave propagating in space and time but (virtually) devoid of energy and momentum. If described literally, then such waves could not induce any physical changes in matter. Nevertheless, there have been serious proposals for experiments that might lead to the detection of "empty" waves associated with either photons [20] or neutrons [21]. However, by making additional assumptions about the nature of such "empty" waves [22], experiments have been proposed that might reveal their actual existence. One such experiment [23] has not yielded any such definitive evidence. Other experiments designed to determine whether empty waves can induce coherence in a two-beam interference experiment have not revealed any evidence for their existence [24], although Croca [25] now argues that this experiment should be regarded as inconclusive as the count rates were very low.

Controversies still persist in the interpretation of the quantum theory of light and indeed more generally in quantum mechanics itself. This happens notwithstanding the widely held view that all the difficult problems concerning the correct interpretation of quantum mechanics were resolved a long time ago in the famous encounters between Einstein and Bohr. Recent books have been devoted to foundational issues [26] in quantum mechanics, and some seriously question Bohrian orthodoxy [27,28]. There is at least one experiment described in the literature [29] that purports to do what Bohr prohibits: demonstrate the simultaneous existence of wave and particle-like properties of light.

Einstein's dualistic approach to electromagnetic radiation was generalized by de Broglie [30] to electrons when he combined results from the special theory of relativity (STR) and Planck's formula for the energy of a quantum to produce his famous formula relating wavelength to particle momentum. His model of a particle was one that contained an internal periodic motion plus an external wave of different frequency that acts to guide the particle. In this model, we have a wave–particle unity—both objectively exist. To quote de Broglie [31]: "The electron ... must be associated with a wave, and this wave is no myth; its wavelength can be measured and its interferences predicted." De Broglie's approach to physics has been described by Lochak [32] as quoted in Selleri [27]:

> Louis de Broglie is an intuitive spirit, concrete and realist, in love with simple images in three-dimensional space. He does not grant ontological value to mathematical models, in particular to geometrical representations in abstract spaces; he does not consider and does not use them other than as convenient mathematical instruments, among others, and it is not in their handling that his physical intuition is directly applied; faced with these abstract representations, he always keeps in mind the idea of all phenomena actually taking place in physical space, so that these mathematical modes of reasoning have a true meaning in his eyes only insofar as he perceives at all times what physical laws they correspond to in usual space.

De Broglie's views are not widely subscribed to today since as with "empty" waves, there is no compelling experimental evidence for the existence of physical waves accompanying the particle's motion (see, however, the discussion in Selleri [27]). Models of particles based on de Broglian ideas are still advanced by Vigier, for example [33].

As is well known, de Broglie abandoned his attempts at a realistic account of quantum phenomena for many years until David Bohm's discovery of a solution of Schrödinger's equation that lends itself to an interpretation involving a physical particle traveling under the influence of a so-called quantum potential.

As de Broglie stated:

> For nearly twenty-five years, I remained loyal to the Bohr-Heisenberg view, which has been adopted almost unanimously by theorists, and I have adhered to it in my teaching, my lectures and my books. In the summer of 1951, I was sent the preprint of a paper by a young American physicist David Bohm, which was subsequently published in the January 15, 1952 issue of the Physical Review. In this paper, Mr. Bohm takes up the ideas I had put forward in 1927, at least in one of the forms I had proposed, and extends them in an interesting way on some points. Later, J.P. Vigier called my attention to the resemblance between a demonstration given by Einstein regarding the motion of particles in General Relativity and a completely independent demonstration I had given in 1927 in an exercise I called the "theory of the double solution."

A comprehensive account of the views of de Broglie, Bohm, and Vigier is given in Jeffers et al. [34]. In these models, contra Bohr particles actually do have trajectories. Trajectories computed for the double-slit experiment show patterns that reproduce the interference pattern observed experimentally [35]. Furthermore, the trajectories so computed never cross the plane of symmetry so that one can assert *with certainty* through which the particles traveled. This conclusion was also reached by Prosser [36,37] in his study of the double-slit experiment from a strictly Maxwellian point of view. Poynting vectors were computed whose distribution mirrors the interference pattern, and these never cross the symmetry plane as in the case of the de Broglie–Bohm–Vigier models. Prosser actually suggested an experimental test of this feature of his calculations. The idea was to illuminate a double-slit apparatus with very short microwave pulses and examine the received radiation at a suitable point off-axis behind the double slits. Calculations showed that for achievable experimental parameters, one could detect either two pulses if the orthodox view were correct, or only one pulse if the Prosser interpretation were correct. However, further investigation [38] showed that the latter conclusion was not correct. Two pulses would be observed, and their degree of separation (i.e., distinguishability) would be inversely related to the degree of contrast in the interference fringes.

Contemporary developments include John Bell's [39] discovery of his famous inequality that is predicated on the assumptions of both locality and realism. Bell's inequality is violated by quantum mechanics, and consequently, it is frequently argued, one cannot accept quantum mechanics, realism, and locality. Experiments on correlated particles appear to demonstrate that the Bell inequalities are indeed violated. Of the three choices, the most acceptable one is to abandon locality. However, Afriat and Selleri [40] have extensively reviewed both the current theoretical and experimental situation regarding the status of Bell's inequalities. They conclude, contrary to accepted wisdom, that one can construct local and realistic accounts of quantum mechanics that violate Bell's inequalities, and furthermore, there remain several loopholes in the experiments that have not yet been closed that allow for local and realist interpretations. No actual experiment that has been performed to date has *conclusively* demonstrated that locality has to be abandoned. However, experiments that approximate to a high degree the original gedanken experiment discussed by David Bohm, and that potentially close all known loopholes, will soon be undertaken. See the review article by Fry and Walther [41]. To quote these authors: "Quantum mechanics, even 50 years after its formulation, is still full of surprises." This underscores Einstein's famous remark: "All these years of conscious brooding have brought me no nearer to the answer to the question "What are light quanta?" Nowadays, every Tom, Dick, and Harry thinks he knows it, but he is mistaken."

II. THE PROCA EQUATION

The first inference of photon mass was made by Einstein and de Broglie on the assumption that the photon is a particle, and behaves as a particle in, for example, the Compton and photoelectric effects. The wave–particle duality of de Broglie is essentially an extension of the photon, as the quantum of energy, to the photon, as a particle with quantized momentum. The Beth experiment in 1936 showed that the photon has angular momentum, whose quantum is $\hbar$. Other fundamental quanta of the photon are inferred in Ref. 42. In 1930, Proca [43] extended the Maxwell–Heaviside theory using the de Broglie guidance theorem:

$$\hbar\omega_0 = m_0 c^2 \tag{1}$$

where m_0 is the rest mass of the photon and m_0c^2 is its rest energy, equated to the quantum of rest energy $\hbar\omega_0$. The original derivation of the Proca equation therefore starts from the Einstein equation of special relativity:

$$p^\mu p_\mu = m_0^2 c^2 \tag{2a}$$

The usual quantum ansatz is applied to this equation to obtain a wave equation:

$$En = i\hbar\frac{\partial}{\partial t}; \qquad \boldsymbol{p} = -i\hbar\nabla \tag{2b}$$

This is an example of the de Broglie wave–particle duality. The resulting wave equation is

$$\left(\Box + \frac{m_0^2 c^4}{\hbar^2}\right)\psi = 0 \tag{3}$$

where ψ is a wave function, whose meaning was first inferred by Born in 1926. If the wave function is a scalar, Eq. (3) becomes the Klein–Gordon equation. If ψ is a 2-spinor, Eq. (3) becomes the van der Waerden equation, which can be related analytically to the Dirac equation, and if ψ is the electromagnetic 4-potential A^μ, Eq. (3) becomes the Proca equation:

$$\Box A^\mu = -\left(\frac{m_0 c^2}{\hbar}\right)^2 A^\mu \tag{4}$$

So A^μ can act as a wave function and the Proca equation can be regarded as a quantum equation if A^μ is a wave function in configuration space, and as a classical equation in momentum space.

It is customary to develop the Proca equation in terms of the vacuum charge current density

$$\Box A^\mu = -\left(\frac{m_0 c^2}{\hbar}\right)^2 A^\mu = -\kappa^2 A^\mu = \frac{1}{\varepsilon_0} J^\mu(\text{vac}) \tag{5}$$

The potential A^μ therefore has a physical meaning in the Proca equation because it is directly proportional to $J^\mu(\text{vac})$. The Proca equations in the vacuum are therefore

$$\partial_\mu F^{\mu\nu} + \left(\frac{m_0 c^2}{\hbar}\right)^2 A^\nu = 0 \tag{6}$$

$$\partial_\mu A^\mu = 0 \tag{7}$$

and, as described in the review by Evans in Part 2 of this compilation [44], these have the structure of the Panofsky, Phillips, Lehnert, Barrett, and O(3) equations, a structure that can also be inferred from the symmetry of the Poincaré group [44]. Lehnert and Roy [45] self-consistently infer the structure of the Proca equations from their own equations, which use a vacuum charge and current.

The problem with the Proca equation, as derived originally, is that it is not gauge-invariant because, under the U(1) gauge transform [46]

$$A^\mu \rightarrow A^\mu + \frac{1}{g}\partial^\mu \Lambda \tag{8}$$

the left-hand side of Eq. (4) is invariant but an arbitrary quantity $\frac{1}{g}\partial^\mu\Lambda$ is added to the right-hand side. This is paradoxical because the Proca equation is well founded in the quantum ansatz and the Einstein equation, yet violates the fundamental principle of gauge invariance. The usual resolution of this paradox is to assume that the mass of the photon is identically zero, but this assumption leads to another paradox, because a particle must have mass by definition, and the wave-particle dualism of de Broglie becomes paradoxical, and with it, the basis of quantum mechanics.

In this section, we suggest a resolution of this >70-year-old paradox using O(3) electrodynamics [44]. The new method is based on the use of covariant derivatives combined with the first Casimir invariant of the Poincaré group. The latter is usually written in operator notation [42,46] as the invariant $P_\mu P^\mu$, where P^μ is the generator of spacetime translation:

$$P^\mu = i\partial^\mu = \frac{p^\mu}{\hbar} \tag{9}$$

The ordinary derivative in gauge theory becomes the covariant derivative

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - igA_\mu \tag{10}$$

for all gauge groups. The generator D_μ is a generator of the Poincaré group because it obeys the Jacobi identity

$$\sum_{\sigma,\nu,\mu} [D_\sigma, [D_\nu, D_\mu]] \equiv 0 \tag{11}$$

and the covariant derivative (10) can be regarded as a sum of spacetime translation generators.

The basic assumption is that the photon acquires mass through the invariant

$$D_\mu D^{\mu *}\psi = 0 \tag{12}$$

for any gauge group. This equation can be developed for any gauge group as

$$(\partial_\mu - igA_\mu)(\partial^\mu + igA^{\mu *})\psi = 0 \tag{13}$$

and can be expressed as

$$\begin{aligned}
&\Box\psi - igA_\mu\partial^\mu\psi + ig\partial_\mu(A^\mu\psi) + g^2 A_\mu A^\mu\psi \\
&= 0 \\
&= \Box\psi - igA_\mu\partial^\mu\psi + ig\psi\partial_\mu A^\mu + igA^\mu\partial_\mu\psi + g^2 A_\mu A^\mu\psi \\
&= (\Box + ig\partial_\mu A^\mu + g^2 A_\mu A^\mu)\psi \\
&= 0
\end{aligned} \tag{14}$$

This equation reduces to

$$(\square + \kappa^2)\psi = -ig\partial_\mu A^\mu \psi \tag{15}$$

for any gauge group because

$$g = \frac{\kappa}{A^{(0)}}; \qquad A_\mu A^\mu = A^{(0)2} \tag{16}$$

In the plane-wave approximation:

$$\partial_\mu A^\mu = 0 \tag{17a}$$

and the Proca equation for any gauge group becomes

$$(\square + \kappa^2)\psi = 0 \tag{17b}$$

for any gauge group.

Therefore Eq. (18) has been shown to be an invariant of the Poincaré group, Eq. (12), and a product of two Poincaré covariant derivatives. In momentum space, this operator is equivalent to the Einstein equation under any condition. The conclusion is reached that the factor g is nonzero in the vacuum.

In gauge theory, for any gauge group, however, a rotation

$$\psi' = e^{i\Lambda}\psi \equiv S\psi \tag{18}$$

in the internal gauge space results in the gauge transformation of A_μ as follows

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \tag{19}$$

and to construct a gauge-invariant Proca equation from the operator (16), a search must be made for a potential A_μ that is invariant under gauge transformation. It is not possible to find such a potential on the U(1) level because the inhomogeneous term is always arbitrary. On the O(3) level, however, the potential can be expressed as

$$\boldsymbol{A}_\mu = A_\mu^{(2)}\boldsymbol{e}^{(1)} + A_\mu^{(1)}\boldsymbol{e}^{(2)} + A_\mu^{(3)}\boldsymbol{e}^{(3)} \tag{20}$$

if the internal gauge space is a physical space with O(3) symmetry described in the complex circular basis ((1),(2),(3)) [3]. A rotation in this physical gauge space can be expressed in general as

$$\psi' = \exp(i\mathrm{M}^a\Lambda^a(x^\mu))\psi \tag{21}$$

where M^a are the rotation generators of O(3) and where $\Lambda^{(1)}, \Lambda^{(2)}$, and $\Lambda^{(3)}$ are angles.

Developing Eq. (13), we obtain

$$\begin{aligned}(\partial_\mu - igA_\mu^{(1)})(\partial^\mu + igA^{\mu(2)})\psi &= 0\\ (\partial_\mu - igA_\mu^{(2)})(\partial^\mu + igA^{\mu(1)})\psi &= 0\\ (\partial_\mu - igA_\mu^{(3)})(\partial^\mu + igA^{\mu(3)})\psi &= 0\end{aligned} \tag{22}$$

The eigenfunction ψ may be written in general as the O(3) vector

$$\psi \equiv \boldsymbol{A}^\nu \tag{23}$$

and under gauge transformation

$$\boldsymbol{A}^{\nu'} = \exp(i\mathrm{M}^a\Lambda^a(x^\mu))\boldsymbol{A}^\nu \tag{24}$$

from Eq. (21). Here, $\Lambda^{(1)}, \Lambda^{(2)}$, and $\Lambda^{(3)}$ are angles in the physical internal gauge space of O(3) symmetry.

Therefore Eqs. (22) become

$$\square^2\boldsymbol{A}^\nu = -\kappa^2\boldsymbol{A}^\nu = \frac{1}{\varepsilon_0}\boldsymbol{J}^\nu(\text{vac}) \tag{25}$$

where

$$\boldsymbol{J}^\nu = \left(\rho^{(i)}, \frac{\boldsymbol{J}^{(i)}}{c}\right) \qquad i = 1, 2, 3 \tag{26}$$

and Eqs. (25) become

$$\square A^{\nu(1)} = -\kappa^2 A^{\nu(1)} = \frac{J^{\nu(1)}}{\varepsilon_0} \tag{27}$$

$$\square A^{\nu(2)} = -\kappa^2 A^{\nu(2)} = \frac{J^{\nu(2)}}{\varepsilon_0} \tag{28}$$

$$\square A^{\nu(3)} = 0 \tag{29}$$

It can be seen that the photon mass is carried by $A^{\nu(1)}$ and $A^{\nu(2)}$, but not by $A^{\nu(3)}$. This result is also obtained by a different route using the Higgs mechanism in Ref. 42, and is also consistent with the fact that the mass associated with $A^{\nu(3)}$ corresponds with the superheavy boson inferred by Crowell [42], reviewed in

Ref. 42 and observed in a LEP collaboration [42]. The effect of a gauge transformation on Eqs. (27)–(29) is as follows:

$$\square\left(A_\mu^{(1)} + \frac{1}{g}\partial_\mu\Lambda^{(1)}\right) = -\kappa^2\left(A_\mu^{(1)} + \frac{1}{g}\partial_\mu\Lambda^{(1)}\right) \tag{30}$$

$$\square\left(A_\mu^{(2)} + \frac{1}{g}\partial_\mu\Lambda^{(2)}\right) = -\kappa^2\left(A_\mu^{(2)} + \frac{1}{g}\partial_\mu\Lambda^{(2)}\right) \tag{31}$$

$$\square\left(A_\mu^{(3)} + \frac{1}{g}\partial_\mu\Lambda^{(3)}\right) = 0 \tag{32}$$

Equations (30) and (31) are eigenequations with the same eigenvalue, $-\kappa^2$, as Eqs. (27) and (28). On the O(3) level, the eigenfunctions $A_\mu^{(1)} + \frac{1}{g}\partial_\mu\Lambda^{(1)}$ are not arbitrary because $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are angles in a physical internal gauge space. The original Eq. (12) is gauge-invariant, however, because on gauge transformation

$$g^2 A_\mu A^{\mu*} \rightarrow g^2 A'_\mu A^{\mu*'}; \qquad g' = \frac{\kappa}{A^{(0)'}} \tag{33}$$

and

$$D_\mu D^{\mu*}\psi \rightarrow D_\mu D^{\mu*}(S\psi) = \psi D_\mu D^{\mu*} S + S D_\mu D^{\mu*}\psi = 0 \tag{34}$$

because S must operate on ψ.

In order for Eq. (34) to be compatible with Eqs. (30) and (31), we obtain

$$\square(\partial_\mu\Lambda^{(1)}) = -\kappa^2(\partial_\mu\Lambda^{(1)}) \tag{35}$$

$$\square(\partial_\mu\Lambda^{(2)}) = -\kappa^2(\partial_\mu\Lambda^{(2)}) \tag{36}$$

which are also Proca equations. So the >70-year-old problem of the lack of gauge invariance of the Proca equation is solved by going to the O(3) level.

The field equations of electrodynamics for any gauge group are obtained from the Jacobi identity of Poincaré group generators [42,46]:

$$\sum_{\sigma,\mu,\nu}\left[D_\sigma, \left[D_\mu, D_\nu\right]\right] \equiv 0 \tag{37}$$

If the potential is classical, the Jacobi identity (37) can be written out as

$$D_\sigma G_{\mu\nu} + D_\mu G_{\nu\sigma} + D_\nu G_{\sigma\mu} - G_{\mu\nu} D_\sigma - G_{\nu\sigma} D_\mu - G_{\sigma\mu} D_\nu \equiv 0 \tag{38}$$

This equation implies the Jacobi identity:

$$[A_\sigma, G_{\mu\nu}] + [A_\mu, G_{\nu\sigma}] + [A_\nu, G_{\sigma\mu}] \equiv 0 \tag{39}$$

which in vector form can be written as

$$\begin{aligned} \boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} &= \boldsymbol{A}^\sigma \times \boldsymbol{G}^{\mu\nu} + \boldsymbol{A}^\mu \times \boldsymbol{G}^{\nu\sigma} + \boldsymbol{A}^\nu \times \boldsymbol{G}^{\sigma\mu} \\ &\equiv \boldsymbol{0} \end{aligned} \tag{40}$$

As a result of this Jacobi identity, the homogeneous field equation

$$D_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{41}$$

reduces to

$$\partial_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{42}$$

for all gauge group symmetries. The implication is that instantons or pseudoparticles do not exist in Minkowski spacetime in a pure gauge theory, because magnetic monopoles and currents vanish for all internal gauge group symmetries. Therefore, the homogeneous field equation of electrodynamics, considered as a gauge theory of any internal symmetry, can be obtained from the Jacobi identity (42) of the Poincaré group of Minkowski spacetime. The homogeneous field equation is gauge-covariant for any internal symmetry. Analogously, the Proca equation is the mass Casimir invariant (12) of the Poincaré group of Minkowski spacetime.

There are several major implications of the Jacobi identity (40), so it is helpful to give some background for its derivation. On the U(1) level, consider the following field tensors in $c = 1$ units and contravariant covariant notation in Minkowski spacetime:

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{bmatrix}; \qquad \tilde{F}_{\mu\nu} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix}$$

$$F_{\rho\sigma} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}; \qquad F^{\rho\sigma} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} \tag{43}$$

These tensors are generated from the duality relations [47]

$$\begin{aligned} \tilde{G}^{\mu\nu} &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}; \qquad & G^{\mu\nu} &= -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\tilde{G}_{\rho\sigma} \\ \tilde{G}_{\mu\nu} &= \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}; \qquad & G_{\mu\nu} &= -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\tilde{G}^{\rho\sigma} \end{aligned} \tag{44}$$

where the totally antisymmetric unit tensor is defined as

$$\varepsilon^{0123} = 1 = -\varepsilon_{0123} \tag{45}$$

and result in the following Jacobi identity:

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} \equiv 0 \tag{46}$$

It also follows that

$$\partial_\mu F^{\mu\nu} = \partial_\sigma \tilde{F}^{\mu\nu} + \partial_\mu \tilde{F}^{\nu\sigma} + \partial_\nu \tilde{F}^{\sigma\mu} \tag{47}$$

The proof of the Jacobi identity (46) can be seen by considering a development such as

$$\begin{aligned}\partial_\mu \tilde{F}^{\mu\nu} &= \frac{1}{2}\partial_\mu(\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma})\\ &= \frac{1}{2}\partial_\mu\big(\varepsilon^{\mu\nu01}F_{01} + \varepsilon^{\mu\nu02}F_{02} + \varepsilon^{\mu\nu03}F_{03} + \varepsilon^{\mu\nu10}F_{10} + \varepsilon^{\mu\nu20}F_{20} + \varepsilon^{\mu\nu30}F_{30}\\ &\quad + \varepsilon^{\mu\nu12}F_{12} + \varepsilon^{\mu\nu13}F_{13} + \varepsilon^{\mu\nu21}F_{21} + \varepsilon^{\mu\nu31}F_{31} + \varepsilon^{\mu\nu23}F_{23} + \varepsilon^{\mu\nu32}F_{32}\big)\end{aligned} \tag{48}$$

If $\nu = 0$, then

$$\partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30} = -\partial_1 F^{23} - \partial_2 F^{13} - \partial_3 F^{12} \equiv 0 \tag{49}$$

Equation (47) may be proved similarly. On the O(3) level there exist the analogous equations (40) and

$$\boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} = \boldsymbol{A}_\sigma \times \tilde{\boldsymbol{G}}^{\mu\nu} + \boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\nu\sigma} + \boldsymbol{A}_\nu \times \tilde{\boldsymbol{G}}^{\sigma\mu} \tag{50}$$

which is not zero in general.

It follows from the Jacobi identity (40) that there also exist other Jacobi identities such as [42]

$$\boldsymbol{A}_\lambda^{(2)} \times (\boldsymbol{A}_\mu^{(1)} \times \boldsymbol{A}_\nu^{(2)}) + \boldsymbol{A}_\mu^{(2)} \times (\boldsymbol{A}_\nu^{(1)} \times \boldsymbol{A}_\lambda^{(2)}) + \boldsymbol{A}_\nu^{(2)} \times (\boldsymbol{A}_\lambda^{(1)} \times \boldsymbol{A}_\mu^{(2)}) \equiv \boldsymbol{0} \tag{51}$$

The Jacobi identity (40) means that the homogeneous field equation of electrodynamics for any gauge group is

$$\partial_\mu \tilde{G}^{\mu\nu} \equiv 0 \tag{52}$$

If the symmetry of the gauge group is O(3) in the complex basis ((1),(2),(3)) [42,47], Eq. (52) can be developed as three equations:

$$\partial_\mu \tilde{G}^{\mu\nu(1)} \equiv 0 \tag{53}$$

$$\partial_\mu \tilde{G}^{\mu\nu(2)} \equiv 0 \tag{54}$$

$$\partial_\mu \tilde{G}^{\mu\nu(3)} \equiv 0 \tag{55}$$

Now consider a component of the Jacobi identity (39)

$$\varepsilon_{(1)(2)(3)}A_\sigma^{(2)}G_{\mu\nu}^{(3)} + \varepsilon_{(1)(2)(3)}A_\mu^{(2)}G_{\nu\sigma}^{(3)} + \varepsilon_{(1)(2)(3)}A_\nu^{(2)}G_{\sigma\mu}^{(3)} \equiv 0 \tag{56}$$

and consider next the following cyclic permutation:

$$A_0^{(2)}G_{23}^{(3)} - A_0^{(3)}G_{23}^{(2)} + A_2^{(2)}G_{30}^{(3)} - A_2^{(3)}G_{30}^{(2)} + A_3^{(2)}G_{02}^{(3)} - A_3^{(3)}G_{02}^{(2)} \equiv 0 \tag{57}$$

This gives the result

$$B_X^{(2)} + \frac{E_Y^{(2)}}{c} - A_Y^{(2)}E_Z^{(3)} \equiv 0 \tag{58}$$

Using Eq. (54), we obtain the result

$$E_Z^{(3)} \equiv 0 \tag{59}$$

thus $\boldsymbol{E}^{(3)}$ vanishes identically in O(3) electrodynamics. The third equation (55) therefore becomes the following identity:

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} \equiv 0 \tag{60}$$

In other words, $\boldsymbol{B}^{(3)}$ is identically independent of time, a result that follows from its definition [42,47]

$$\boldsymbol{B}^{(3)} \equiv -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{61}$$

The ansatz, upon which these results are based, is that the configuration of the vacuum is described by the doubly connected group O(3), which supports the Aharonov–Bohm effect in Minkowski spacetime [46]. More generally, the vacuum configuration could be described by an internal gauge space more general than O(3), such as the Lorentz, Poincaré, or Einstein groups. The O(3) group is the little group of the Poincaré group for a particle with identically nonzero mass, such as the photon. If the internal space were extended from O(3) to the Poincaré group, there would appear boost and spacetime translation operators in the gauge transform (36), as well as rotation generators. The Poincaré group is the most general group of special relativity, and the Einstein group, that of general relativity. Both groups are defined in Minkowski spacetime. In all these groups, there would be no magnetic monopole or current in Minkowski spacetime because of the Jacobi identity (37) between any group generators. The superiority of O(3) over U(1) electrodynamics has been demonstrated in several ways using empirical data [42,47–61] such as those available in the Sagnac effect, so its seems logical to extend the internal space to the Poincaré group. The widespread use of a U(1) group for electrodynamics is a historical accident. The use of an O(3) group is an improvement, so it is expected that the use of a Poincaré group would be an improvement over O(3).

Meanwhile, the Jacobi identity (40) implies, in vector notation, the identities

$$\begin{aligned} \boldsymbol{A}^{(2)}\cdot\boldsymbol{B}^{(3)} - \boldsymbol{B}^{(2)}\cdot\boldsymbol{A}^{(3)} &\equiv \mathbf{0} \\ \boldsymbol{A}^{(3)}\cdot\boldsymbol{B}^{(1)} - \boldsymbol{B}^{(3)}\cdot\mathbf{A}^{(1)} &\equiv \mathbf{0} \\ \boldsymbol{A}^{(1)}\cdot\boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)}\cdot\boldsymbol{A}^{(2)} &\equiv \mathbf{0} \end{aligned} \tag{62}$$

and

$$\begin{aligned} cA_0^{(3)}\boldsymbol{B}^{(2)} - cA_0^{(2)}\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(2)} &\equiv \mathbf{0} \\ cA_0^{(1)}\boldsymbol{B}^{(3)} - cA_0^{(3)}\boldsymbol{B}^{(1)} + \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(1)} - \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(3)} &\equiv \mathbf{0} \\ cA_0^{(2)}\boldsymbol{B}^{(1)} - cA_0^{(1)}\boldsymbol{B}^{(2)} + \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(2)} - \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(1)} &\equiv \mathbf{0} \end{aligned} \tag{63}$$

It has been shown elsewhere [42] that the identities (63) correspond with the B cyclic theorem [42,47–61] of O(3) electrodynamics:

$$\begin{aligned} \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} &= iB^{(0)}\boldsymbol{B}^{(3)*} \\ &\cdots \end{aligned} \tag{64}$$

which is therefore also an identity of the Poincaré group. Within a factor, the B cyclic theorem is the rotation generator Lie algebra of the Poincaré group. In terms of the unit vectors of the basis ((1),(2),(3)), the B cyclic theorem reduces to

$$\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*} \\ &\cdots \end{aligned} \tag{65}$$

which is the frame relation itself. This relation is unaffected by a Lorentz boost and a spacetime translation. A rotation produces the same relation (65). So the B cyclic theorem is invariant under the most general type of Lorentz transformation, consisting of boosts, rotations, and spacetime translations. Similarly, the definition of $\boldsymbol{B}^{(3)}$, Eq. (61), is Lorentz-invariant.

The Jacobi identities (63) reduce to the B cyclic theorem (64) because of Eqs. (53)–(55), and because $\boldsymbol{E}^{(3)}$ vanishes identically [42,47–61], and the B cyclic theorem is self-consistent with Eqs. (53)–(55). The identities (62) and (63) imply that there are no instantons or pseudoparticles in O(3) electrodynamics, which is a dynamics developed in Minkowski spacetime. If the pure gauge theory corresponding to O(3) electrodynamics is supplemented with a Higgs mechanism, then O(3) electrodynamics supports the ‘t Hooft–Polyakov magnetic monopole [46]. Therefore Ryder [46], for example, in his standard text, considers a form of O(3) electrodynamics [46, pp. 417ff.], and the ‘t Hooft–Polyakov magnetic monopole is a signature of an O(3) electrodynamics with its symmetry broken spontaneously with a Higgs mechanism. In the pure gauge theory, however, the magnetic monopole is identically zero. It is clear that the theory of ‘t Hooft and Polyakov is O(3) electrodynamics plus a Higgs mechanism, an important result.

In order to show that the Proca equation from gauge theory is gauge-invariant, it is convenient to consider the Jacobi identity

$$D_\mu \tilde{G}^{\mu\nu} \equiv 0 \tag{66}$$

which is gauge-invariant in all gauge groups. Now use

$$D_\mu G^{\mu\nu} = D_\sigma \tilde{G}^{\lambda\kappa} + D_\kappa \tilde{G}^{\sigma\lambda} + D_\lambda \tilde{G}^{\kappa\sigma} \tag{67}$$

and let two indices be the same on the right-hand side. This procedure produces

$$D_\mu G^{\mu\nu} = D_\sigma\left(\tilde{G}^{\sigma\kappa} + \tilde{G}^{\kappa\sigma}\right) = 0 \tag{68}$$

showing that:

$$D_\mu G^{\mu\nu} = 0 \tag{69}$$

is also gauge-invariant for all gauge groups. Finally, expand Eq. (69) as

$$D_\mu G^{\mu\nu} = D_\mu(D^\mu A^\nu - D^\nu A^\mu) = 0 \tag{70}$$

to obtain

$$D_\mu D^\mu A^\nu = 0 \tag{71}$$

which is also gauge-invariant for all gauge groups.

On the U(1) level, for example, the structure of the Lehnert [45] and gauge-invariant Proca equations is obtained as follows:

$$(\Box + \kappa^2)A^{\nu} = 0 \tag{72}$$

$$\left(\partial_{\mu} + igA_{\mu}^{*}\right)G^{\mu\nu} = 0 \tag{73}$$

These are regarded as eigenequations with eigenfunctions A^{ν} and $G^{\mu\nu}$ in configuration space. In this method, there is no need for the Lorenz condition. The equivalent of Eq. (72) in momentum space is the Einstein equation (2), and this statement is true for all gauge group symmetries. Comparing Eqs. (6) and (7) with Eqs. (72) and (73), the following equation is obtained on the U(1) level:

$$\kappa^2 A^{\nu} = igA_{\mu}^{*}G^{\mu\nu} \tag{74}$$

This equation may be developed as follows:

$$\kappa^2 A^{(0)} = ig\boldsymbol{A}^{*}\cdot\frac{\boldsymbol{E}}{c} \tag{75}$$

In the plane-wave approximation

$$\kappa A^{(0)} = \frac{E^{(0)}}{c} = B^{(0)} \tag{76}$$

and it is seen that condition (74) is true on the U(1) level. Equation (73) can be written as

$$\partial_{\mu}G^{\mu\nu} = -igA_{\mu}^{*}G^{\mu\nu} \equiv \frac{J^{\mu}}{\varepsilon_0} \tag{77}$$

in the vacuum, and this is the Lehnert equation [42,45]. The latter gives longitudinal or axisymmetric solutions and can describe physical situations that the Maxwell–Heaviside theory cannot.

On the O(3) level, one can write the Proca equation in the following form (22):

$$\begin{aligned}(\Box + g^2A_{\mu}^{(1)}A^{\mu(2)})A^{\nu(1)} &= 0\\ (\Box + g^2A_{\mu}^{(2)}A^{\mu(1)})A^{\nu(2)} &= 0\\ (\Box + g^2A_{\mu}^{(3)}A^{\mu(3)})A^{\nu(3)} &= 0\end{aligned} \tag{78}$$

The third equation of (22) reduces to a d'Alembert equation

$$\Box A^{\nu(3)} = 0 \tag{79}$$

because $A_{\mu}^{(3)}A^{\mu(3)} = 0$ in O(3) electrodynamics. Equation (79) is consistent with the fact that $A_{\mu}^{(3)}$ is phaseless by definition in O(3) electrodynamics. The first two equations of the triad (78) are complex conjugate Proca equations of the form

$$\begin{aligned}(\Box + \kappa^2)A^{\nu} &= 0\\ (\Box + \kappa^2)A^{\nu*} &= 0\end{aligned} \tag{80}$$

so we obtain the U(1) Proca equation, but with the advantages of O(3) electrodynamics inbuilt.

In summary, the structure of the Proca equation on the O(3) level is as follows:

$$D_{\mu}G^{\mu\nu} = 0 \tag{81}$$

which is equivalent to

$$\partial_{\mu}\boldsymbol{G}^{\mu\nu} = -g\boldsymbol{A}_{\mu} \times \boldsymbol{G}^{\mu\nu} \tag{82}$$

The latter equation can be expanded in the basis ((1),(2),(3)) as [42]

$$\begin{aligned}
\nabla\cdot\boldsymbol{D}^{(1)*} &= ig(\boldsymbol{A}^{(2)}\cdot\boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)}\cdot\boldsymbol{A}^{(3)})\\
\nabla\cdot\boldsymbol{D}^{(2)*} &= ig(\boldsymbol{A}^{(3)}\cdot\boldsymbol{D}^{(1)} - \boldsymbol{D}^{(3)}\cdot\boldsymbol{A}^{(1)})\\
\nabla\cdot\boldsymbol{D}^{(3)*} &= ig(\boldsymbol{A}^{(1)}\cdot\boldsymbol{D}^{(2)} - \boldsymbol{D}^{(1)}\cdot\boldsymbol{A}^{(2)})\\
\nabla\times\boldsymbol{H}^{(1)*} - \frac{\partial\boldsymbol{D}^{(1)*}}{\partial t} &= -ig(cA_0^{(2)}\boldsymbol{D}^{(3)} - cA_0^{(3)}\boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)}\times\boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)}\times\boldsymbol{H}^{(2)})\\
\nabla\times\boldsymbol{H}^{(2)*} - \frac{\partial\boldsymbol{D}^{(2)*}}{\partial t} &= -ig(cA_0^{(3)}\boldsymbol{D}^{(1)} - cA_0^{(1)}\boldsymbol{D}^{(3)} + \boldsymbol{A}^{(3)}\times\boldsymbol{H}^{(1)} - \boldsymbol{A}^{(1)}\times\boldsymbol{H}^{(3)})\\
\nabla\times\boldsymbol{H}^{(3)*} - \frac{\partial\boldsymbol{D}^{(3)*}}{\partial t} &= -ig(cA_0^{(1)}\boldsymbol{D}^{(2)} - cA_0^{(2)}\boldsymbol{D}^{(1)} + \boldsymbol{A}^{(1)}\times\boldsymbol{H}^{(2)} - \boldsymbol{A}^{(2)}\times\boldsymbol{H}^{(1)})
\end{aligned} \tag{83}$$

It can be seen that, in general, there are extra Noether charges and currents that define the photon mass gauge invariantly. The magnetic field strength and electric displacement is used in Eq. (83) because, in general, there may be vacuum polarization and magnetization, defined respectively as

$$\begin{aligned}\boldsymbol{D} &= \varepsilon_0\boldsymbol{E} + \boldsymbol{P}\\ \boldsymbol{B} &= \mu_0(\boldsymbol{H} + \boldsymbol{M})\end{aligned} \tag{84}$$

There may be a vacuum charge on the O(3) level provided that the term

$$\nabla \cdot \boldsymbol{D}^{(3)*} = ig(\boldsymbol{A}^{(1)} \cdot \boldsymbol{D}^{(2)} - \boldsymbol{D}^{(1)} \cdot \boldsymbol{A}^{(2)}) \tag{85}$$

is not zero. For this to be the case, the vacuum polarization must be such that the displacement $\boldsymbol{D}^{(1)}$ is not the complex conjugate of the displacement $\boldsymbol{D}^{(2)}$. It can be seen as follows that for this to be the case, polarization must develop asymmetrically as follows:

$$\begin{aligned} \boldsymbol{D}^{(1)} &= \varepsilon_0 \boldsymbol{E}^{(1)} + a\boldsymbol{P}^{(1)} \\ \boldsymbol{D}^{(2)} &= \varepsilon_0 \boldsymbol{E}^{(2)} + b\boldsymbol{P}^{(2)} \end{aligned} \tag{86}$$

If there is no vacuum polarization, then the photon mass resides entirely in the vacuum current.

In the preceding analysis, commutators of covariant derivatives always act on an eigenfunction, so, for example:

$$\begin{aligned} [D_\mu, D_\nu]\psi &= \left[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu\right]\psi \\ &= (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\psi - igA_\mu\partial_\nu\psi + ig\partial_\nu(A_\mu\psi) \\ &\quad - ig\partial_\mu(A_\nu\psi) + igA_\nu\partial_\mu\psi - g^2[A_\mu, A_\nu]\psi \\ &= -igA_\mu\partial_\nu\psi + ig\partial_\nu A_\mu\psi + igA_\mu\partial_\nu\psi - ig(\partial_\mu A_\nu)\psi \\ &\quad - igA_\nu\partial_\mu\psi + igA_\nu\partial_\mu\psi - g^2[A_\mu, A_\nu]\psi \\ &= -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])\psi \end{aligned} \tag{87}$$

giving the field tensor for all gauge groups:

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu] \tag{88}$$

In the literature, the operation $[D_\mu, D_\nu]\psi$ is often written simply as $[D_\mu, D_\nu]$ but this shorthand notation always implies that the operators act on the unwritten ψ.

On the O(3) level, the clearest insight into the meaning of the Jacobi identity (37) is obtained by writing the covariant derivative in terms of translation (P) and rotation (J) generators of the Poincaré group:

$$\begin{aligned} D_\sigma &= \partial_\sigma - igA_\sigma = \partial_\sigma - ig(A_\sigma^X J_X + A_\sigma^Y J_Y + A_\sigma^Z J_Z) \\ \partial_\sigma &= -iP_\sigma \end{aligned} \tag{89}$$

where J_X, J_Y, and J_Z are the rotation generators. The translation generator is defined [42,46] as

$$P_\sigma = i\partial_\sigma \tag{90}$$

The Jacobi identity of operators (37) therefore becomes, after index mat

$$\begin{aligned} &[P_\sigma + gA_\sigma^X J_X, [P_\kappa + gA_\kappa^Y J_Y, P_\lambda + gA_\lambda^Z J_Z]] \\ &\quad = [P_X + gA_X^X J_X, [P_Y + gA_Y^Y J_Y, P_Z + gA_Z^Z J_Z]] \end{aligned}$$

Now consider the component

$$\begin{aligned} &[P_X, [P_Y + gA_Y^Y, P_Z + gA_Z^Z J_Z]] \\ &\quad = [P_X, [P_Y, P_Z] + gA_Y^Y[J_Y, P_Z] + gA_Z^Z[P_Y, J_Z] + g^2 A_Y^Y A_Z^Z [J_Y, J_Z]] \end{aligned}$$

and use the Lie algebra [46]

$$\begin{aligned} [J_Y, P_X] &= -iP_X \qquad & [P_X, P_X] &= 0 \\ [P_Y, J_Z] &= iP_X & [P_X, J_X] &= 0 \\ [J_Y, J_Z] &= iJ_X \end{aligned}$$

to find that it vanishes. In vector notation, this result implies Eq. (52)

$$\partial_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \qquad [\mathrm{O}(3) \text{ level}]$$

and the result

$$[A_\sigma, G_{\kappa\lambda}] + [A_\kappa, G_{\lambda\sigma}] + [A_\lambda, G_{\sigma\kappa}] \equiv 0$$

which can be developed as

$$[\partial_\mu, \tilde{G}^{\mu\nu}]\psi \equiv 0$$

giving Eq. (94) again self-consistently. Similarly

$$[A_\mu, \tilde{G}^{\mu\nu}]\psi \equiv 0$$

giving Eq. (40). In operator form, this is

$$[gA_\sigma^X J_X, [P_\kappa + gA_\kappa^Y J_Y, P_\lambda + gA_\lambda^Z J_Z]] \equiv 0$$

and the factor $[A_\mu, \tilde{G}^{\mu\nu}]$ is a simple multiplication operation on ψ.

The overall result is that the homogeneous field equation for all group sy metries is the result of the Lie algebra of the Poincaré group, the group of s cial relativity. The Jacobi identity can be derived in turn from a round trip holonomy in Minkowski spacetime, as first shown by Feynman [46] for gauge groups. The Jacobi identity is Lorentz- and gauge-invariant.

III. CLASSICAL LEHNERT AND PROCA VACUUM CHARGE CURRENT DENSITY

In this section, gauge theory is used to show that there exist classical charge current densities in the vacuum for all gauge group symmetries, provided that the scalar field of gauge theory is identified with the electromagnetic field [O(3) level] or a component of the electromagnetic field [U(1) level]. The Lehnert vacuum charge current density exists for all gauge group symmetries without the Higgs mechanism. The latter introduces classical Proca currents and other terms that represent energy inherent in the vacuum. Some considerable mathematical detail is given as an aid to comprehension of the Lagrangian methods on which these results depend.

The starting point is the Lagrangian that leads to the vacuum d'Alembert equation for an electromagnetic field component, such as a scalar magnetic flux density component, denoted B, of the electromagnetic field. The identification of the scalar field, usually denoted ϕ [46], of gauge theory with a scalar electromagnetic field component was first made in the derivation [62,63} of the 't Hooft–Polyakov monopole. In principle, ϕ can be identified with a scalar component of the vacuum magnetic flux density (B), or electric field strength (E), or the Whittaker scalar magnetic fluxes G and F [64,65] from which all potentials and fields can be derived in the vacuum. The treatment is classical, and the field is regarded as a function of the spacetime coordinate x^μ, and not as an eigenfunction of quantum mechanics. The general mathematical method used is a functional variation on a given Lagrangian, and so it is helpful to illustrate this method in detail as an aid to understanding. The basic concept is that there exists, in the vacuum, an electromagnetic field whose scalar components are B and E, or G and F, scalar components that obey the d'Alembert, or relativistic wave, equation in the vacuum. The Lagrangian leading to this equation by functional variation is set up, and this Lagrangian is subjected to a local gauge transformation, or gauge transformation of the second kind [46]. Local gauge invariance leads directly to the inference, from the first principles of gauge field theory, of a vacuum charge current density first introduced phenomenologically by Lehnert [45]. Inclusion of spontaneous symmetry breaking with the Higgs mechanism leads to several more vacuum charge current densities on the U(1) and O(3) levels, and in general for any gauge group symmetry. Each of these charge current densities in vacuo provides energy inherent in the vacuum.

The method of functional variation in Minkowski spacetime is illustrated first through the Lagrangian (in the usual reduced units [46])

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{99}$$

where $F^{\mu\nu}$ is the field tensor on the U(1) level [46–61]. The relevant Euler Lagrange equation is

$$\partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu A_\mu)}\right) = \frac{\partial\mathscr{L}}{\partial A_\mu} \tag{100}$$

Consider the component

$$\partial_0\left(\frac{\partial\mathscr{L}}{\partial(\partial_0 A_\mu)}\right) = 0 \tag{101}$$

For indices $\nu = 0$ and $\mu = 1$, summation over repeated indices gives

$$F^{\mu\nu}F_{\mu\nu} = F^{10}F_{10} + F^{01}F_{01} \tag{102}$$

Therefore

$$\begin{aligned} F^{10}F_{10} &= (\partial^1 A^0 - \partial^0 A^1)(\partial_1 A_0 - \partial_0 A_1) \\ &= (\partial^1 A^0)(\partial_1 A_0) - (\partial^0 A^1)(\partial_1 A_0) - (\partial^1 A^0)(\partial_0 A_1) + (\partial^0 A^1)(\partial_0 A_1) \\ &= -\partial_X A_0 \partial_X A_0 + \partial_0 A_X \partial_X A_0 + \partial_X A_0 \partial_0 A_X - \partial_0 A_X \partial_0 A_X \end{aligned} \tag{103}$$

using contravariant–covariant notation. In the same notation, we have

$$\frac{\partial}{\partial(\partial_0 A_1)} = -\frac{\partial}{\partial(\partial_0 A_X)} \tag{104}$$

so

$$\frac{\partial(F^{10}F_{10})}{\partial(\partial_0 A_1)} = -\partial_X A_0 - \partial_X A_0 + \partial_0 A_X + \partial_0 A_X \tag{105}$$

Using the additional minus sign in the Lagrangian (99), we obtain

$$\frac{\partial(-F^{10}F_{10}/2)}{\partial(\partial_0 A_1)} = F^{10} \tag{106}$$

and repeating with the term

$$\begin{aligned} F^{01}F_{01} &= (\partial^0 A^1 - \partial^1 A^0)(\partial_0 A_1 - \partial_1 A_0) \\ &= -\partial_0 A_X \partial_0 A_X + \partial_X A_0 \partial_0 A_X + \partial_0 A_X \partial_X A_0 - \partial_X A_0 \partial_X A_0 \end{aligned} \tag{107}$$

gives the same as Eq. (103). So the final result of the functional variation is

$$\partial_\nu F^{\mu\nu} = 0 \tag{108}$$

which is the vacuum inhomogeneous field equation in the Maxwell–Heaviside theory. This equation is widely accepted, but it violates causality, because there is a field (effect) without a source (cause). This flaw is usually overlooked by stating that the field is in a source-free region, or that the field is infinitely distant from its source. Both explanations are unsatisfactory.

Another example of functional variation is the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{109}$$

which leads to the Proca equation in the received view [46]. The obvious problem with this Lagrangian is that for identically nonzero m, the product $A_\mu A^\mu$ is not gauge-invariant on the U(1) level. Setting that problem aside for the sake of argument, contravariant–covariant notation gives

$$A_\mu A^\mu = A_0^2 - A_X^2 - A_Y^2 - A_Z^2 \tag{110}$$

so that functional variation proceeds as follows:

$$\frac{\partial\mathscr{L}}{\partial A_0} = \frac{2m^2 A_0}{2}; \quad -\frac{\partial\mathscr{L}}{\partial A_X} = -\frac{2m^2 A_X}{2}; \quad -\frac{\partial\mathscr{L}}{\partial A_Y} = -\frac{2m^2 A_Y}{2}; \quad -\frac{\partial\mathscr{L}}{\partial A_Z} = -\frac{2m^2 A_Z}{2} \tag{111}$$

The overall result is

$$\frac{\partial\mathscr{L}}{\partial A_\mu} = m^2 A^\mu \tag{112}$$

giving the received Proca equation [46]:

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{113}$$

The Lagrangian (109) is not gauge-invariant, so Eq. (113) is not gauge-invariant. However, the foregoing illustrates the method of functional variation that will be used throughout this section.

In order to derive field equations in the vacuum that are self-consistent, cause must precede effect and the classical current of the Proca current must be gauge-invariant. The starting point for the development is the concept of scalar field

[46], which is usually denoted ϕ. The basic idea [46] behind the existence of tl scalar field ϕ is a transition from a point particle at coordinate $x(t)$ to a field

$$\phi(x^\mu) = \phi(X, Y, Z, t) \qquad (11$$

which is a function of X, Y, Z and t in Minkowski spacetime. The scalar field ϕ a classical concept and is governed by the Euler–Lagrange equation:

$$\frac{\partial\mathscr{L}}{\partial\phi} = \partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi)}\right) \qquad (11$$

The source of electric charge in this view is a symmetry of the action in Noether theorem, a symmetry that means that ϕ must be complex, that is, that there mu be two fields:

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad (11$$

$$\phi* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \qquad (11$$

These fields are regarded as independent functions in the method of function variation. In developing their concept of a magnetic monopole, 't Hooft an Polyakov identified ϕ with a scalar component of the electromagnetic field, component that they denoted F [46]. It is convenient for our purposes to identi ϕ with a scalar component B of the electromagnetic field in the vacuun Therefore, there are two independent magnetic flux density components:

$$B = \frac{1}{\sqrt{2}}(B_1 + iB_2) \qquad (11$$

$$B* = \frac{1}{\sqrt{2}}(B_1 - iB_2) \qquad (11$$

The Lagrangian governing these scalar components is

$$\mathscr{L} = (\partial_\mu B)(\partial^\mu B^*) \qquad (12$$

and is invariant under global gauge transformation, also known as "gaug transformation of the first kind"

$$B \to e^{-i\Lambda}B; \qquad B^* \to e^{i\Lambda}B^* \qquad (121$$

where Λ is any real number. The Euler–Lagrange equation

$$\frac{\partial\mathscr{L}}{\partial B} = \partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu B)}\right) \qquad (122$$

with the Lagrangian (120) gives the d'Alembert equations:

$$\Box B = 0 \tag{123}$$

$$\Box B^* = 0 \tag{124}$$

which are the relativistic wave equations in the vacuum satisfied by B and B^*. For example, if B and B^* are components of a plane wave, they satisfy the d'Alembert equations (123) and (124).

However, in special relativity, the number Λ is a function of the spacetime coordinate x^μ. This property defines the local gauge transformation

$$B \rightarrow e^{-i\Lambda(x^\mu)}B; \qquad B^* \rightarrow e^{i\Lambda(x^\mu)}B^* \tag{125}$$

$$\begin{aligned}\mathscr{L} &= (\partial_\mu B)(\partial^\mu B^*) - ig(B^*\partial^\mu B - B\partial^\mu B^*)A_\mu + g^2 A_\mu A^\mu B^* B - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= (\partial_\mu B + igA_\mu B)(\partial^\mu B^* - igA^\mu B^*) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\end{aligned} \tag{126}$$

or gauge transformation of the second kind. The Lagrangian (120) is invariant under the local gauge transformation (125) if it becomes [46]: The 4-potential becomes

$$A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu \Lambda \tag{127}$$

where Λ is any number and the derivative ∂_μ becomes the covariant derivatives:

$$D_\mu B = (\partial_\mu + igA_\mu)B \tag{128}$$

$$D_\mu B^* = (\partial_\mu - igA_\mu)B^* \tag{129}$$

acting respectively on B and B^*. The Lagrangian (126) is gauge invariant under a U(1) gauge transformation that introduces the electromagnetic field tensor $F^{\mu\nu}$. Using the Euler–Lagrange equation (100) gives the vacuum field equation:

$$\begin{aligned}\partial_\nu F^{\mu\nu} &= -ig(B^*\partial^\mu B - B\partial^\mu B^*) + 2g^2 A^\mu |B|^2 \\ &= ig(B^* D^\mu B - BD^\mu B^*) \\ &\equiv -gJ^\mu(\text{vac})\end{aligned} \tag{130}$$

where

$$J^\mu(\text{vac}) = i(B^* D^\mu B - BD^\mu B^*) \tag{131}$$

Therefore $J^\mu(\text{vac})$ is a covariant conserved charge current density in the vacuum The coefficient g of the covariant derivative has the units [47–61] of $\kappa/A^{(0)}$ in th vacuum. Using

$$g = \frac{\kappa}{A^{(0)}} \tag{132}$$

has been shown recently [47–61] to explain the Sagnac effect and interferometr in general using an O(3) invariant electrodynamics. The coefficient g is the sam on the U(1) and O(3) levels.

In SI units, Eq. (130) is

$$\partial_\nu F^{\mu\nu} = -igc(B^* D^\mu B - BD^\mu B^*)Ar \tag{133}$$

and shows that the electromagnetic field in the vacuum has its source in th conserved $J^\mu(\text{vac})$, which is divergentless.

In Eq. (133), Ar is the area of the electromagnetic beam, c the vacuum spee of light and μ_0 is the vacuum permeability in SI units.

The analysis can be repeated by identifying the scalar field ϕ with a scala component A of the vacuum four potential A^μ. Thus Eqs. (118) and (119 become

$$A = \frac{1}{\sqrt{2}}(A_1 + iA_2) \tag{134}$$

$$A^* = \frac{1}{\sqrt{2}}(A_1 - iA_2) \tag{135}$$

and the Lagrangian (120) becomes

$$\mathscr{L} = (\partial_\mu A)(\partial^\mu A^*) \tag{136}$$

Local gauge transformation is defined as

$$\begin{aligned} A &\rightarrow \exp(-i\Lambda(x^\mu))A \\ A^* &\rightarrow \exp(i\Lambda(x^\mu))A^* \end{aligned} \tag{137}$$

and the gauge-invariant Lagrangian (126) becomes

$$\mathscr{L} = (\partial_\mu A + igA_\mu A)(\partial^\mu A - igA^\mu A^*) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{138}$$

Finally, the inhomogeneous field equation in the vacuum becomes

$$\partial_\nu F^{\mu\nu} = -igc(A^* D^\mu A - AD^\mu A^*) \tag{139}$$

in SI units. This form has the advantage of eliminating any geometric variable such as Ar from the vacuum charge current density. The covariant derivatives (128) and (129) become

$$D_\mu A = (\partial_\mu + igA_\mu)A \tag{140}$$

$$D_\mu A^* = (\partial_\mu - igA_\mu)A^* \tag{141}$$

indicating the presence of self-interaction in the terms $A_\mu A$ and $A_\mu A^*$. This self-interaction is observed empirically [47–61] in a number of ways, including the inverse Faraday effect and the third Stokes parameter defining the circular polarization of electromagnetic radiation.

So it is also possible to use the form (139) for the vacuum charge current density, a form that eliminates any geometric unit such as Ar that is not fully relativistic. However, A is, strictly speaking, a potential energy difference and not a field.

Using the Euler–Lagrange equation (122) with the Lagrangian (126) produces the two complex conjugate equations (reduced units):

$$\Box B = -ig(B\partial^\mu A_\mu + A_\mu \partial^\mu B) + g^2 A_\mu A^\mu B \tag{142a}$$

$$\Box B^* = ig(B^*\partial^\mu A_\mu + A_\mu \partial^\mu B^*) + g^2 A_\mu A^\mu B^* \tag{142b}$$

or their representation in terms of the scalar A:

$$\Box A = -ig(A\partial^\mu A_\mu + A_\mu \partial^\mu A) + g^2 A_\mu A^\mu A \tag{143a}$$

$$\Box A^* = ig(A^*\partial^\mu A_\mu + A_\mu \partial^\mu A^*) + g^2 A_\mu A^\mu A^* \tag{143b}$$

Equations (133) and (142) or (139) and (143) can be solved simultaneously, because they are each two equations in two unknowns (B and A^μ) or (A and A^μ).

It can be shown on this U(1) level that the introduction of a Higgs mechanism [46], namely, spontaneous symmetry breaking, produces three more vacuum charge current densities in addition to the Lehnert-type charge current density (133) or (139). One of these is a Proca vacuum charge current density that is gauge-invariant on the classical level. The Higgs mechanism is introduced by considering the usual Lagrangian [46]

$$\mathscr{L} = T - V = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi^* \phi - \lambda(\phi^* \phi)^2 \tag{144}$$

and adapting it for the electromagnetic field in the vacuum by writing it as

$$\mathscr{L} = T - V = (\partial_\mu B)(\partial^\mu B^*) - m^2 B^* B - \lambda(B^* B)^2 \tag{145}$$

or

$$\mathscr{L} = T - V = (\partial_\mu A)(\partial^\mu A^*) - m^2 A^* A - \lambda(A^* A)^2 \qquad (14$$

depending on whether ϕ is chosen to be B or A. The appearance of three ne currents occurs for both choices and of course B is related to A through the vect equation:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (147$$

In Eq. (144), it is well known that the mass m is regarded as a parameter that ca become negative and that λ premultiplies the self-interaction term. The adapta tion of the Higgs mechanism for the vacuum electromagnetic field therefor automatically implies that scalar components of that field self-interact. The sel interaction of electromagnetic fields on the received U(1) level is observable i the Stokes parameters, energy and Poynting vector for example, and in nonlinea optical phenomena of various kinds [47–61].

Considering Eq. (145), we obtain

$$\frac{\partial V}{\partial B} = m^2 B^* + 2\lambda B^*(B^* B) \qquad (148$$

and if $m^2 < 0$, there is a local maximum at $B = 0$ and a minimum at

$$a^2 \equiv |B|^2 = -\frac{m^2}{2\lambda}; \qquad \text{i.e.,} \quad a = |B| \qquad (149$$

The scalar fields B and B^* therefore become

$$B(x^\mu) = a + \frac{1}{\sqrt{2}}(B_1 + iB_2) \qquad (150$$

$$B^*(x^\mu) = a + \frac{1}{\sqrt{2}}(B_1 - iB_2) \qquad (151$$

so the Lagrangian becomes

$$\mathscr{L} = \partial_\mu(a + B)\partial^\mu(a + B^*) - m^2(a + B^*)(a + B) - \lambda((a + B^*)(a + B))^2 \qquad (152$$

It is interesting to develop this expression as

$$\begin{aligned} \mathscr{L} &= BB^*(m^2 - \lambda BB^*) + \cdots \\ &= -\lambda BB^*(2a^2 + BB^*) + \cdots \end{aligned} \qquad (153$$

which can be expressed algebraically as

$$\mathscr{L} = -\lambda\left(a^2 + \frac{2a}{\sqrt{2}}B_1 + \frac{1}{2}(B_1^2 + B_2^2)\right)\left(3a^2 + \frac{2a}{\sqrt{2}}B_1 + \frac{1}{2}(B_1^2 + B_2^2)\right) + \cdots$$
$$= \partial_\mu B \partial^\mu B^* - 2\lambda a^2 B_1^2 - \sqrt{2}\lambda B_1(B_1^2 + B_2^2) - \frac{\lambda}{4}(B_1^2 + B_2^2)^2 - 3\lambda a^4 \quad (154)$$

In contemporary thought, the Higgs mechanism has acted in such a way as to produce a field component B_1 with mass, specifically, a scalar field with mass that is gauge-invariant. Therefore, spontaneous symmetry breaking of the vacuum introduces fields with effective mass.

Considering a local gauge transformation of the Lagrangian (145) produces the gauge-invariant Lagrangian:

$$\mathscr{L} = (\partial_\mu + igA_\mu)(a + B)(\partial^\mu - igA^\mu)(a + B^*) - m^2(a + B)(a + B^*)$$
$$- \lambda(a + B)^2(a + B^*)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (155)$$

Using this Lagrangian in Eq. (100) produces the following result (reduced units) by functional variation:

$$\partial_\nu F^{\mu\nu} = -ig(B^* D^\mu B - BD^\mu B^*) - \frac{g^2 m^2}{\lambda}A^\mu + 2\sqrt{2}g^2 aB_1 A^\mu + \sqrt{2}ag\partial^\mu B_2 \quad (156)$$

The term $-g^2m^2A^\mu/\lambda$ implies that the electromagnetic 4-potential A^μ has acquired mass. Simultaneously there appear two other terms. All four vacuum charge current densities produce vacuum energy through the equation

$$En(\text{vac}) = \int J^\mu(\text{vac})A_\mu \, dV \quad (157)$$

Alternatively, Eq. (156) can be written from Eq. (146) in terms of the scalar A:

$$\partial_\nu F^{\mu\nu} = -ig(A^* D^\mu A - AD^\mu A^*) - g^2 m^2 \frac{A^\mu}{\lambda} + 2\sqrt{2}g^2 aA_1 A^\mu + \sqrt{2}ag\partial^\mu A_2 \quad (158)$$

Therefore, spontaneous symmetry breaking of the vacuum on the U(1) level produces new vacuum charge current densities that act as sources for the electromagnetic field and produce energy inherent in the topology of the vacuum. The topology is described by gauge theory and group theory.

In an O(3) electromagnetic sector [47–61], the Lagrangian (120) become

$$\mathscr{L} = \frac{1}{2}\partial_\mu B_i \partial^\mu B^i \quad (15$$

where there are internal indices i to indicate the existence of an internal gau group of O(3) symmetry. In the complex basis ((1),(2),(3)), the Lagrangian c be expressed in terms of the physical magnetic field:

$$\boldsymbol{B} = B^{(2)}\boldsymbol{e}^{(1)} + B^{(1)}\boldsymbol{e}^{(2)} + B^{(3)}\boldsymbol{e}^{(3)} \quad (16$$

In vector notation, the Lagrangian (159) can be written as

$$\mathscr{L} = \frac{1}{2}\partial_\mu \boldsymbol{B} \cdot \partial^\mu \boldsymbol{B} \quad (16$$

and using the Euler–Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{B}} = \partial_\nu\left(\frac{\partial \mathscr{L}}{\partial_\nu \boldsymbol{B}}\right) \quad (16$$

produces the vacuum d'Alembert equation

$$\Box \boldsymbol{B} = \boldsymbol{0} \quad (163$$

which in component form becomes

$$\Box B^{(i)} = 0; \qquad i = 1, 2, 3 \quad (164$$

The Lagrangian (161) is invariant under a global O(3) transformation

$$\boldsymbol{B}' = e^{iJ_i\Lambda_i}\boldsymbol{B} \quad (165$$

where J_i are rotation generators of the O(3) group, and where Λ_i are angles in th physical internal space ((1),(2),(3)).

The local O(3) transformation corresponding to Eq. (165) is

$$\boldsymbol{B}' = e^{iJ_i\Lambda_i(x^\mu)}\boldsymbol{B} \quad (166$$

and the Lagrangian (161) is invariant under this if it becomes

$$\mathscr{L} = D_\mu \boldsymbol{B} \cdot D^\mu \boldsymbol{B} - \frac{1}{4}\boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}^{\mu\nu} \quad (167)$$

where the field $\boldsymbol{B}$ and the electromagnetic field $\boldsymbol{G}_{\mu\nu}$ are vectors of the internal gauge space and where $\boldsymbol{G}_{\mu\nu}$ is a tensor of Minkowski spacetime. Field equations are obtained from the Lagrangian (167) by functional variation using Euler–Lagrange equations such as

$$\partial^{\nu}\left(\frac{\partial\mathscr{L}}{\partial(\partial^{\nu}\boldsymbol{A}^{\mu})}\right)=\frac{\partial\mathscr{L}}{\partial\boldsymbol{A}^{\mu}} \tag{168}$$

where $\boldsymbol{A}^{\mu}$ is a vector in the internal gauge space and a 4-vector in Minkowski spacetime. The field tensor in O(3) is defined [46–61] as

$$\boldsymbol{G}^{\mu\nu}=\partial^{\mu}\boldsymbol{A}^{\nu}-\partial^{\nu}\boldsymbol{A}^{\mu}+g\boldsymbol{A}^{\mu}\times\boldsymbol{A}^{\nu} \tag{169}$$

In analogy with the Lagrangian (99), the factor $-\frac{1}{4}$ is needed because of double summation over repeated indices. So functional variation of the term $-\frac{1}{4}\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}^{\mu\nu}$ gives $\partial^{\nu}\boldsymbol{G}_{\mu\nu}$. However, on the O(3) level, we must consider the additional terms

$$\begin{aligned}\mathscr{L}_1&=-\frac{1}{4}g(\boldsymbol{G}^{\mu\nu}\cdot\boldsymbol{A}_{\mu}\times\boldsymbol{A}_{\nu}+\boldsymbol{A}^{\mu}\times\boldsymbol{A}^{\nu}\cdot\boldsymbol{G}_{\mu\nu})\\&=-\frac{1}{4}g(\boldsymbol{A}_{\mu}\cdot(\boldsymbol{G}^{\mu\nu}\times\boldsymbol{A}_{\nu})+\boldsymbol{A}^{\mu}\cdot(\boldsymbol{G}_{\mu\nu}\times\boldsymbol{A}^{\nu}))\end{aligned} \tag{170}$$

which have the same premultiplier $-\frac{1}{4}$ due to double summation over repeated indices. From the terms (170)

$$\frac{\partial\mathscr{L}}{\partial\boldsymbol{A}^{\mu}}=g\boldsymbol{G}_{\mu\nu}\times\boldsymbol{A}^{\nu}=-g\boldsymbol{A}^{\nu}\times\boldsymbol{G}_{\mu\nu} \tag{171}$$

So the sum of terms (which appear on the left-hand side of the field equation) from variation in the term $-\frac{1}{4}\boldsymbol{G}_{\mu\nu}\cdot\boldsymbol{G}^{\mu\nu}$in the Lagrangian (167) is

$$D^{\nu}\boldsymbol{G}_{\mu\nu}\equiv\partial^{\nu}\boldsymbol{G}_{\mu\nu}+g\boldsymbol{A}^{\nu}\times\boldsymbol{G}_{\mu\nu} \tag{172}$$

which is a covariant derivative in electrodynamics invariant under a local O(3) transformation. We must also consider functional variation of the term

$$\mathscr{L}_3=D_{\mu}\boldsymbol{B}\cdot D^{\mu}\boldsymbol{B}=(\partial_{\mu}+g\boldsymbol{A}_{\mu}\times)\boldsymbol{B}\cdot(\partial^{\mu}+g\boldsymbol{A}^{\mu}\times)\boldsymbol{B} \tag{173}$$

which can be expressed as

$$\begin{aligned}\mathscr{L}_3=\;&\partial_{\mu}\boldsymbol{B}\cdot\partial^{\mu}\boldsymbol{B}+g\boldsymbol{A}_{\mu}\cdot(\boldsymbol{B}\times\partial^{\mu}\boldsymbol{B})+g\boldsymbol{A}^{\mu}\cdot(\boldsymbol{B}\times\partial_{\mu}\boldsymbol{B})\\&+g^2((\boldsymbol{A}_{\mu}\cdot\boldsymbol{A}^{\mu})(\boldsymbol{B}\cdot\boldsymbol{B})-(\boldsymbol{A}_{\mu}\cdot\boldsymbol{B})(\boldsymbol{B}\cdot\boldsymbol{A}^{\mu}))\end{aligned} \tag{174}$$

We obtain

$$\begin{aligned}\frac{\partial\mathscr{L}_3}{\partial\boldsymbol{A}^{\mu}}&=g(\boldsymbol{B}\times\partial_{\mu}\boldsymbol{B})+g^2(\boldsymbol{A}_{\mu}(\boldsymbol{B}\cdot\boldsymbol{B})-(\boldsymbol{A}_{\mu}\cdot\boldsymbol{B})\boldsymbol{B})\\&=g(\boldsymbol{B}\times\partial_{\mu}\boldsymbol{B})+g^2\boldsymbol{B}\times(\boldsymbol{A}_{\mu}\times\boldsymbol{B})\end{aligned} \quad (17$$

So the complete field equation obtained from the Lagrangian (167) by functio variation is

$$D^{\nu}\boldsymbol{G}_{\mu\nu}=-g(D_{\mu}\boldsymbol{B})\times\boldsymbol{B}\equiv-g\boldsymbol{J}_{\mu}(\text{vac}) \quad (17$$

This equation in vector notation for the internal gauge space can be developed three equations in reduced units

$$\partial_{\mu}G^{\mu\nu(1)}=ig(A_{\mu}^{(2)}G^{\mu\nu(3)}-A_{\mu}^{(3)}G^{\mu\nu(2)}-B^{(2)}D^{\nu}B^{(3)}+B^{(3)}D^{\nu}B^{(2)}) \quad (17$$

$$\partial_{\mu}G^{\mu\nu(2)}=ig(A_{\mu}^{(3)}G^{\mu\nu(1)}-A_{\mu}^{(1)}G^{\mu\nu(3)}-B^{(3)}D^{\nu}B^{(1)}+B^{(1)}D^{\nu}B^{(3)}) \quad (17$$

$$\partial_{\mu}G^{\mu\nu(3)}=ig(A_{\mu}^{(1)}G^{\mu\nu(2)}-A_{\mu}^{(2)}G^{\mu\nu(1)}-B^{(1)}D^{\nu}B^{(2)}+B^{(2)}D^{\nu}B^{(1)}) \quad (17$$

where a covariant derivative acting on a component such as $B^{(1)}$ is

$$D^{\nu}B^{(1)}=\partial^{\nu}B^{(1)}-ig(A^{\nu(2)}B^{(3)}-A^{\nu(3)}B^{(2)}) \quad (18$$

Therefore there are several more vacuum current terms on the O(3) than on t U(1) level. The factor g is, however, the same on both levels. In SI units, the Ec (177)–(179) become

$$\begin{aligned}\partial_{\mu}G^{\mu\nu(1)}&=ig(A_{\mu}^{(2)}G^{\mu\nu(3)}-A_{\mu}^{(3)}G^{\mu\nu(2)})\\&\quad-igc(B^{(2)}D^{\nu}B^{(3)}-B^{(3)}D^{\nu}B^{(2)})Ar\\&\quad\cdots\end{aligned} \quad (18$$

If the field ϕ is identified with the space components of A in the bas ((1),(2),(3)), the following three vacuum equations are obtained

$$\begin{aligned}\partial_{\mu}G^{\mu\nu(1)}&=ig(A_{\mu}^{(2)}G^{\mu\nu(3)}-A_{\mu}^{(3)}G^{\mu\nu(2)})\\&\quad-igc(A^{(2)}D^{\nu}A^{(3)}-A^{(3)}D^{\nu}A^{(2)})\\&\quad\cdots\end{aligned} \quad (18$$

in which the vacuum currents have no geometric factor.

The structure of these vacuum charge current densities can be developed follows in terms of time-like, longitudinal and transverse components. In th development, we take the real parts of $\boldsymbol{A}$ and $\boldsymbol{A}_\mu$. The complete inhomogeneous field equation in the vacuum is

$$\partial^\nu \boldsymbol{G}_{\mu\nu} + g\boldsymbol{A}^\nu \times \boldsymbol{G}_{\mu\nu} = -g(D_\mu \boldsymbol{A}) \times \boldsymbol{A} \tag{183}$$

where the right-hand side can be expanded as

$$\boldsymbol{J}_\mu(\text{vac}) \equiv g\partial_\mu \boldsymbol{A} \times \boldsymbol{A} + g^2 \boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_\mu) \tag{184}$$

The longitudinal current density in vacuo is investigated, first in the plane-wave first approximation, by taking the real part of the potential

$$\boldsymbol{A} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{185}$$

which is

$$\text{Re}\,\boldsymbol{A} = \frac{A^{(0)}}{\sqrt{2}}(-\boldsymbol{i}\sin\phi + \boldsymbol{J}\cos\phi) \tag{186}$$

where

$$\phi \equiv \omega t - \kappa Z \tag{187}$$

The longitudinal current density is (in **SI** units)

$$\boldsymbol{J}_3 = \frac{g}{\mu_0 c}\partial_3 \boldsymbol{A} \times \boldsymbol{A} + \frac{g^2}{\mu_0 c}\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_3) \tag{189}$$

and the vector magnitude is

$$A^{(0)} = |\boldsymbol{A}| = (A_1^2 + A_2^2)^{1/2} \tag{189}$$

In general, the vacuum current density has a definite structure in the vacuum that is much richer than in the first plane-wave approximation: a structure that has to be computed because analytical solutions to Eq. (183) are not available.

In the plane-wave first approximation, the current density is therefore

$$\boldsymbol{J}^{(3)}(\text{vac}) = \frac{2\kappa}{\mu_0 c}\boldsymbol{B}^{(3)} \tag{190}$$

in SI units and is directly proportional to the vacuum $\boldsymbol{B}^{(3)}$ field. The structure of Eq. (190) was first derived by considering the inverse Faraday effect as Eq. (243) of Ref. 42. Equation (190) (above) was first derived phenomenologically on the O(3) level in Ref. 51 and first developed phenomenologically in Ref. 59. Equation (190) is its rigorous first-principles description in the vacuum. The first principles of gauge field theory therefore produce vacuum charge current densities in the vacuum for all gauge group symmetries. There are several experimental reasons [42,47–61] for preferring O(3) over U(1) for electrodynamics.

The vacuum charge density is also structured in general, but in the plane wave, first approximation is given by

$$J_0 = \frac{\kappa^2 A_0}{\mu_0 c} \tag{191}$$

because by definition, the time component of the vector $\boldsymbol{A}$ is zero. This is how it differs from the 4-vector A^μ, and why it is an independent variable in the method of functional variation used to derive Eq. (183) from an O(3) invariant Lagrangian.

The vacuum transverse current densities are also structured, and in general they are

$$\boldsymbol{J}_1 = \frac{g}{\mu_0 c}\partial_1 \boldsymbol{A} \times \boldsymbol{A} + \frac{g^2}{\mu_0 c}\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_1) \tag{192}$$

$$\boldsymbol{J}_2 = \frac{g}{\mu_0 c}\partial_2 \boldsymbol{A} \times \boldsymbol{A} + \frac{g^2}{\mu_0 c}\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_2) \tag{193}$$

In the plane-wave first approximation, they reduce to

$$\boldsymbol{J}_1 = -g^2 A_1 A_2^2 \boldsymbol{i} \tag{194}$$

$$\boldsymbol{J}_2 = g^2 A_1^2 A_2 \boldsymbol{j} \tag{195}$$

using the vector triple products:

$$\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_1) = -A_1 A_2^2 \boldsymbol{i} \tag{196}$$

$$\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_2) = -A_1^2 A_2 \boldsymbol{j} \tag{197}$$

In SI units, the transverse vacuum current densities are given in the plane-wave first approximation by

$$\boldsymbol{J}_1 = -g^2 \frac{A_1 A_2^2}{\mu_0 c}\boldsymbol{i} \tag{198}$$

$$\boldsymbol{J}_2 = g^2 \frac{A_1^2 A_2}{\mu_0 c}\boldsymbol{j} \tag{199}$$

It is emphasized, however, that there is no reason to assume plane waves. These are used as an illustration only, and in general the vacuum charge current densities of O(3) electrodynamics are richly structured, far more so than in U(1) electrodynamics, where vacuum charge current densities also exist from the first principles of gauge theory as discussed already.

The complete vacuum inhomogeneous equation is

$$\partial^{\nu}\boldsymbol{G}_{\mu\nu} = -g\boldsymbol{A}^{\nu} \times \boldsymbol{G}_{\mu\nu} - g(D_{\mu}\boldsymbol{A}) \times \boldsymbol{A} \tag{200}$$

If $\mu = 2$ and $\nu = 1$, the left-hand side vanishes because G_{21} contains only B_3, which is phaseless. The right-hand side gives the equation

$$B_3 = gA_1A_2 \tag{201}$$

which reduces in the notation that we have been using to

$$\boldsymbol{B}^{(3)} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{202}$$

In the usual complex circular basis used for O(3) electrodynamics [42], this is the definition of the field $\boldsymbol{B}^{(3)}$.

Therefore, a check for self-consistency has been carried out for indices $\mu = 2$ and $\nu = 1$. It has been shown, therefore, that in pure gauge theory applied to electrodynamics without a Higgs mechanism, a richly structured vacuum charge current density emerges that serves as the source of energy latent in the vacuum through the following equation:

$$En = \int \boldsymbol{J}^{\mu} \cdot \boldsymbol{A}_{\mu}\, dV \tag{203}$$

Therefore, on the O(3) level, there are several sources of energy latent in the vacuum. This conclusion is gauge-invariant because the Lagrangian is O(3) invariant. It is concluded that potentials can give rise to physical effects in the vacuum on both the U(1) and O(3) levels. These effects are reviewed experimentally by Barrett [50]. The best known is the Aharonov–Bohm effect, which Barrett has shown [50] to be supported self-consistently only by O(3) electrodynamics and not by U(1) electrodynamics. Both the O(3) and the U(1) group are non-singly connected, the O(3) group being doubly connected in topology [50]. The latter dictates the structure of the field equations in gauge theory applied to classical electrodynamics.

The wave equation in the vacuum for O(3) electrodynamics can be obtained by functional variation in the Euler–Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{A}} = \partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\boldsymbol{A})}\right) = \partial^{\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial^{\mu}\boldsymbol{A})}\right) \tag{204}$$

with the gauge-invariant Lagrangian

$$\mathscr{L} = D_{\mu}\boldsymbol{A} \cdot D^{\mu}\boldsymbol{A} - \frac{1}{4}\boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}^{\mu\nu} \tag{205}$$

obtained by a local gauge transformation on the Lagrangian:

$$\mathscr{L} = \frac{1}{2}\partial_{\mu}\boldsymbol{A} \cdot \partial^{\mu}\boldsymbol{A} \tag{206}$$

The only assumption therefore is that the Maxwell vector potential $\boldsymbol{A}$ exists in the physical internal space of O(3) symmetry. The gauge-invariant Lagrangian (205) can be developed as

$$\mathscr{L} = \partial_{\mu}\boldsymbol{A} \cdot \partial^{\mu}\boldsymbol{A} + g(\boldsymbol{A}_{\mu} \times \boldsymbol{A} \cdot \partial^{\mu}\boldsymbol{A} + \partial_{\mu}\boldsymbol{A} \cdot \boldsymbol{A}^{\mu} \times \boldsymbol{A}) + g^2(\boldsymbol{A}_{\mu} \times \boldsymbol{A}) \cdot (\boldsymbol{A}^{\mu} \times \boldsymbol{A}) \tag{207}$$

$$\mathscr{L} = \partial_{\mu}\boldsymbol{A} \cdot \partial^{\mu}\boldsymbol{A} + g(\boldsymbol{A} \cdot (\partial^{\mu}\boldsymbol{A} \times \boldsymbol{A}_{\mu}) + \boldsymbol{A} \cdot (\partial_{\mu}\boldsymbol{A} \cdot \boldsymbol{A}^{\mu})) + g^2(\boldsymbol{A}_{\mu} \times \boldsymbol{A}) \cdot (\boldsymbol{A}^{\mu} \times \boldsymbol{A}) \tag{208}$$

Using the vector identity

$$\boldsymbol{A}_{\mu} \times \boldsymbol{A} \cdot \boldsymbol{A}^{\mu} \times \boldsymbol{A} = (\boldsymbol{A}_{\mu} \cdot \boldsymbol{A}^{\mu})(\boldsymbol{A} \cdot \boldsymbol{A}) - (\boldsymbol{A}_{\mu} \cdot \boldsymbol{A})(\boldsymbol{A} \cdot \boldsymbol{A}^{\mu}) \tag{209}$$

gives the results

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{A}} = g\partial^{\mu}\boldsymbol{A} \times \boldsymbol{A}_{\mu} + g\partial_{\mu}\boldsymbol{A} \times \boldsymbol{A}^{\mu} + 2g^2\boldsymbol{A}(\boldsymbol{A}_{\mu} \cdot \boldsymbol{A}^{\mu}) - g^2(\boldsymbol{A}_{\mu}(\boldsymbol{A} \cdot \boldsymbol{A}^{\mu}) - (\boldsymbol{A} \cdot \boldsymbol{A}_{\mu})\boldsymbol{A}^{\mu}) \tag{210}$$

and

$$\partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\boldsymbol{A})}\right) = 2\partial_{\mu}\partial^{\mu}\boldsymbol{A} + 2g\partial_{\mu}(\boldsymbol{A}^{\mu} \times \boldsymbol{A}) \tag{211}$$

The vacuum wave equation in O(3) electrodynamics is therefore

$$\Box\boldsymbol{A} = -g\partial_{\mu}(\boldsymbol{A}^{\mu} \times \boldsymbol{A}) + g(\partial_{\mu}\boldsymbol{A}) \times \boldsymbol{A}^{\mu} + g^2(\boldsymbol{A}(\boldsymbol{A}_{\mu} \cdot \boldsymbol{A}^{\mu}) - \boldsymbol{A}_{\mu}(\boldsymbol{A} \cdot \boldsymbol{A}^{\mu})) \tag{212}$$

Using

$$\boldsymbol{A}^{\mu} \times (\boldsymbol{A} \times \boldsymbol{A}_{\mu}) = \boldsymbol{A}(\boldsymbol{A}^{\mu} \cdot \boldsymbol{A}_{\mu}) - \boldsymbol{A}_{\mu}(\boldsymbol{A}^{\mu} \cdot \boldsymbol{A}) \tag{213}$$

Eq. (212) simplifies to

$$\Box \boldsymbol{A} + g\partial_\mu(\boldsymbol{A}^\mu \times \boldsymbol{A}) = g(\partial_\mu \boldsymbol{A}) \times \boldsymbol{A}^\mu - g^2(\boldsymbol{A} \times \boldsymbol{A}_\mu) \times \boldsymbol{A}^\mu \tag{214}$$

which can be written as

$$\partial_\mu((\partial^\mu + g\boldsymbol{A}^\mu \times)\boldsymbol{A}) = g(\partial_\mu \boldsymbol{A}) \times \boldsymbol{A}^\mu + g^2 \boldsymbol{A}^\mu \times (\boldsymbol{A} \times \boldsymbol{A}_\mu) \tag{215}$$

This form further simplifies to

$$\partial_\mu(D^\mu \boldsymbol{A}) = g((\partial_\mu + g\boldsymbol{A}_\mu \times)\boldsymbol{A}) \times \boldsymbol{A}^\mu \tag{216}$$

which becomes

$$\partial_\mu(D^\mu \boldsymbol{A}) = g(D^\mu \boldsymbol{A}) \times \boldsymbol{A}_\mu \tag{217}$$

Therefore, we finally obtain the wave equation of O(3) electrodynamics in the form

$$D_\mu(D^\mu \boldsymbol{A}) = \boldsymbol{0} \tag{218}$$

which is a d'Alembert equation for $\boldsymbol{A}$ with O(3) covariant derivatives.

The derivation of Eq. (218) from Eq. (206) follows from local gauge invariance, and it is always possible to apply a local gauge transform to the vector $\boldsymbol{A}$, the Maxwell vector potential. The ordinary derivative of the d'Alembert wave equation is replaced by an O(3) covariant derivative. The U(1) equivalent of Eq. (218) in quantum-mechanical (operator) form is Eq. (13), and Eq. (212) is the rigorously correct form of the phenomenological Eq. (25). It can be seen that Eq. (212) is richly structured in the vacuum and must be solved numerically. The vacuum currents present in Eq. (218) can be computed from the right-hand side of the wave equation (212), and these vacuum currents follow from local gauge invariance.

On the U(1) level, the starting Lagrangian is

$$\mathscr{L} = \partial_\mu A \partial^\mu A^* \tag{219}$$

which on local gauge transformation becomes

$$\mathscr{L} = (\partial_\mu A + igA_\mu A)(\partial^\mu A^* - igA^\mu A^*) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{220}$$

Using the Euler–Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial A} = \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A)}\right); \qquad \frac{\partial \mathscr{L}}{\partial A^*} = \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A^*)}\right) \tag{221}$$

we obtain

$$\Box A^* = ig(\partial_\mu A^\mu)A^* + igA_\mu(\partial^\mu A^*) + g^2 A_\mu A^\mu A^* \tag{222}$$

which is Eq. (143), showing a richly structured vacuum charge current density. Equation (222) can be developed as

$$\partial_\mu(\partial^\mu A^* - igA^\mu A^*) = igA_\mu(\partial^\mu A^*) + g^2 A_\mu A^\mu A^* \tag{223}$$

that is

$$\partial_\mu(D^\mu A^*) = igA_\mu(\partial^\mu A^* - igA^\mu A^*) \tag{224}$$

$$D_\mu(D^\mu A^*) = 0 \tag{225}$$

which is a vacuum d'Alembert equation with U(1) covariant derivatives. To obtain Eq. (225) from Eq. (219), the only assumption is that the Lagrangian is invariant under the local U(1) gauge transform:

$$A \rightarrow \exp(-i\Lambda(x^\mu))A \tag{226}$$

Similarly, we obtain

$$\partial^\mu D_\mu A = -igA^\mu(\partial_\mu A + igA_\mu A) \tag{227}$$

and the d'Alembert equation

$$D^\mu(D_\mu A) = 0 \tag{228}$$

with covariant derivatives.

A possible solution of Eq. (228) is:

$$D_\mu A = 0 \tag{229}$$

specifically

$$\partial^\mu = -igA_\mu \tag{230}$$

Define

$$\kappa_\mu \equiv gA_\mu = \frac{\kappa}{A^{(0)}} A_\mu \tag{231}$$

and Eq. (230) becomes the following quantum ansatz:

$$\partial_\mu = -i\kappa_\mu = -\frac{i}{\hbar}p_\mu \tag{232}$$

On the quantum level, Eq. (229) becomes an operator equation, and, using the quantum ansatz, we obtain

$$D^{\mu*}D_{\mu}A = 0; \qquad \text{i.e.,} \quad \Box A = -\kappa^{\mu}\kappa_{\mu}A \tag{233}$$

which is Eq. (12) (above). In fully covariant form, Eq. (233) becomes the gauge invariant Proca equation:

$$\Box A^{\mu} = -\kappa^{\mu}\kappa_{\mu}A^{\mu} = -\kappa^2 A^{\mu} = -\left(\frac{m_0 c}{\hbar}\right)^2 A^{\mu} \tag{234}$$

Note that the Proca equation requires

$$\kappa^{\mu}\kappa_{\mu} \neq 0 \tag{235}$$

and has been obtained without the use of the Lorenz condition.

The equivalent procedure on the O(3) level is to choose a particular solution

$$D^{\mu}\boldsymbol{A} = (\partial^{\mu} + g\boldsymbol{A}^{\mu}\times)\boldsymbol{A} = \boldsymbol{0} \tag{236}$$

which, in the general notation of gauge field theory, is

$$\partial^{\mu}\psi = igA^{\mu(3)}\psi \tag{237}$$

giving again the quantum ansatz on the O(3) level. In the complex circular basis

$$\begin{aligned} \boldsymbol{A}^{\mu} &= \boldsymbol{A}^{(2)}\boldsymbol{e}^{(1)} + \boldsymbol{A}^{(1)}\boldsymbol{e}^{(2)} + \boldsymbol{A}^{(3)}\boldsymbol{e}^{(3)} \\ \boldsymbol{A} &= \boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)} \end{aligned} \tag{238}$$

and Eq. (236) becomes

$$(\partial^{\mu} + g\boldsymbol{a}^{\mu}\times)(\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) = \boldsymbol{0} \tag{239}$$

This equation can be developed as

$$\frac{\partial}{\partial Z}\boldsymbol{A}^{(1)} = -g\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \ldots \tag{240}$$

in other words, as

$$i\kappa\boldsymbol{A}^{(2)*} = \boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \tag{241}$$

which gives self-consistently the definition

$$\boldsymbol{B}^{(2)*} = -ig\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \tag{242}$$

Similarly, we obtain

$$\frac{\partial}{\partial Z}\boldsymbol{A}^{(2)} = g\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(3)} \tag{243}$$

which gives the following definition:

$$\boldsymbol{B}^{(1)*} = -ig\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(3)} \tag{244}$$

Using the relation $g = \kappa/A^{(0)}$ in Eqs. (242) and (244) gives two equations of the B cyclic theorem [42,47–61]:

$$\begin{aligned} \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*} \\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*} \end{aligned} \tag{245}$$

It follows from the quantum ansatz (237) that

$$\begin{aligned} -\frac{\partial}{\partial X}(\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) &= g\boldsymbol{A}_X^{(3)} \times (\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) = \boldsymbol{0} \\ -\frac{\partial}{\partial Y}(\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) &= g\boldsymbol{A}_Y^{(3)} \times (\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) = \boldsymbol{0} \end{aligned} \tag{246}$$

which is self-consistent because

$$\boldsymbol{A}_X^{(3)} = \boldsymbol{A}_Y^{(3)} = \boldsymbol{0} \tag{247}$$

Finally, the time-like component of Eq. (236) is

$$\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) = g\boldsymbol{A}_0^{(3)} \times (\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) \tag{248}$$

which gives again Eqs. (242) and (244).

Therefore the Proca equation can be recovered on the O(3) level from the special solution (236) as the operator equation:

$$\partial_{\mu}\partial^{\mu}\psi = -g^2 A^{\mu(3)}A_{\mu}^{(3)}\psi \tag{249}$$

This result is given in Eq. (22) of the preceding section.

A Lagrangian such as Eq. (219) is made up purely of a kinetic energy term:

$$\mathcal{L} = T = \partial_\mu A \partial^\mu A^* \tag{250}$$

and a local gauge transformation on the Lagrangian produces

$$\mathcal{L} = T - V = \partial_\mu A \partial^\mu A^* + ig(A_\mu A \partial^\mu A^* - A^\mu A^* \partial_\mu A) + g^2 A_\mu A A^\mu A^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{251}$$

where V is a potential energy term. In field theory [46], the ground state is the vacuum, and the ground state is obtained by minimizing the potential energy V with respect to a variable such as A or A^μ. The minimum of V in Eq. (251) with respect to A_μ is the vacuum charge current density, which is a ground state of the field theory and that is obviously a property of the vacuum itself. The ground state defined by the minimum

$$\frac{\partial V}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu} \tag{252}$$

is the electromagnetic field, which is also a vacuum property. So the inhomogeneous field equation

$$\partial_\nu F^{\mu\nu} = \frac{J^\mu(\text{vac})}{\varepsilon_0} \tag{253}$$

is a relation between ground states of the field theory, or a relation between vacuum states. Similarly, a ground state such as

$$\frac{\partial \mathcal{L}}{\partial A} = ig A_\mu D^\mu A^* \neq 0 \tag{254}$$

is a vacuum property. It can be seen that Eq. (254) is a minimum because

$$\frac{\partial^2 \mathcal{L}}{\partial A^2} = 2g^2 |A|^2 \tag{255}$$

is always greater than zero.

The source of the potential energy V in Eq. (251) is local gauge transformation, and so the source of V is the vacuum itself, as described by special relativity and gauge theory. The kinetic energy T appearing in Eq. (250) has no role in defining the ground state of the field theory, because the ground state is defined by the minimum of V with respect to a given variable, as just argued. In these equations, the physical A and A^* are excitations above the ground state o[f] vacuum, and the vacuum gives no contribution to the global Lagrangian (250). The potential energy V is part of the locally gauge-invariant Lagrangian tha[t] gives the field equation (253), a relation between vacuum properties. The vacuum charge current density gives energy latent in the vacuum, and rate o[f] doing work by the vacuum. These are given respectively by

$$En = \int J^\nu(\text{vac}) A_\mu \, dV \tag{256}$$

and by

$$\frac{\partial W}{\partial t} = \int \boldsymbol{J}(\text{vac}) \cdot \boldsymbol{E} \, dV \tag{257}$$

The volume V is arbitrary and, from Eq. (257) standard methods [66], give the vacuum Poynting theorem

$$\frac{\partial U}{\partial t}(\text{vac}) + \nabla \cdot \boldsymbol{S}(\text{vac}) = -\boldsymbol{J}(\text{vac}) \cdot \boldsymbol{E} \tag{258}$$

or law of conservation of energy and momentum for various vacuum properties. The vacuum energy flow is represented by the Poynting vector $\boldsymbol{S}$(vac):

$$\nabla \cdot \boldsymbol{S}(\text{vac}) = -\boldsymbol{J}(\text{vac}) \cdot \boldsymbol{E} \tag{259}$$

Integrating this equation gives

$$\boldsymbol{S}(\text{vac}) = -\int \boldsymbol{J}(\text{vac}) \cdot \boldsymbol{E} \, d\boldsymbol{r} + \text{ constant of integration} \tag{260}$$

where the constant of integration represents a physical component of energy flow whose magnitude is not limited by any concept in gauge field theory. The physical object $\boldsymbol{J}$(vac) also emanates from the vacuum, and its magnitude is not limited because the magnitude of A^μ is not limited by vacuum topology. The energy flow represented by S(vac) is electromagnetic energy flow generated by vacuum topology, and can be converted, in principle, to other forms of energy with suitable laboratory devices.

The physical meaning of the vacuum Poynting theorem [46] in Eq. (258) is that the time rate of change of electromagnetic energy within an arbitrary volume V, combined with the energy flowing out through the boundary surfaces of the volume per unit time, is equal to the negative of the total work done by the field (a vacuum property) on the source, interpreted as vacuum charge current density. This is a statement of conservation of energy applied within the vacuum and in the absence of matter (electrons). In the received view

$$J^{\mu}(\text{vac}) = 0 \tag{261}$$

and there is no vacuum Poynting theorem, but as argued already, the received view violates gauge invariance, special relativity, and causality. In the correctly gauge-invariant Eq. (253), work is done by the source (a vacuum property) on the field (another vacuum property), work that can be transmitted to rate of change of mechanical energy as follows [46]:

$$\frac{dEn}{dt}(\text{mech}) = \int \boldsymbol{J}(\text{vac}) \cdot \boldsymbol{E}\, dV \tag{262}$$

In general relativity, gravity is curvature of spacetime, and so the ordinary potential energy mgh emanates ultimately from the vacuum topology itself. Here m is mass, g is the acceleration due to gravity, and h is a difference in height. The electromagnetic field is orders of magnitude stronger than the gravitational field. Special relativity is a special case of general relativity, and sometimes A^{μ} is known [46] as a connection, in analogy with the affine connection of general relativity. The gravitational field is the vacuum, and the electromagnetic field is the vacuum. Mass and gravitational field, and charge and electromagnetic field, are therefore all consequences of relativity and vacuum topology.

In this view, the structures of the vacuum and matter currents are identical:

$$\begin{aligned} J^{\mu}(\text{vac}) &= -\frac{ig}{\mu_0 c}(A^* D^{\mu} A - A D^{\mu} A^*); \qquad g = \frac{\kappa}{A^{(0)}} \\ J^{\mu}(\text{matter}) &= -\frac{ig}{\mu_0 c}(A^* D^{\mu} A - A D^{\mu} A^*); \qquad g = \frac{e}{\hbar} \end{aligned} \tag{263}$$

and one is transformed into the other for one electron and one photon by the relation

$$\frac{\kappa}{A^{(0)}} = \frac{e}{\hbar} \tag{264}$$

Therefore, the momentum of one photon is transformed to the electron momentum

$$\hbar\kappa = eA^{(0)} \tag{265}$$

and the photon momentum and energy emanate from the vacuum itself, as just argued. In this way, the elementary charge e on the proton also becomes a topological property, arguing in analogy with the way in which mass in general

relativity is a property of the vacuum. Again, in analogy with general relativi photons are formed out of the vacuum as gravitons are formed out of the vacuu The relation (265) is true for all internal gauge group symmetries. In t foregoing, we happen to have been arguing on the U(1) level, but the concepts a the same on the O(3) level. Therefore, charge e is the result of the field, which is vacuum property.

The above is a pure gauge field theory. The Higgs mechanism on the U(level provides further sources of vacuum energy as discussed already. On t O(3) level, the Higgs mechanism can also be applied, resulting in yet mo sources of energy.

Gauge theory of any symmetry must have two mathematical space Minkowski spacetime and the internal gauge space. If electromagnetic theo in the vacuum is a U(1) symmetry gauge field symmetry, there is a scalar inte nal space of U(1) symmetry in the vacuum. This internal space is the space the scalar A and A^* used in the foregoing arguments. In geometric form

$$\boldsymbol{A} = A_1 \boldsymbol{i} + A_2 \boldsymbol{j} \tag{266}$$

is a vector in a two-dimensional space with orthonormal basis vectors $\boldsymbol{i}$ and This space is the internal gauge space of the U(1) gauge field theory applied t vacuum electromagnetism. A global gauge transform is a rotation of A throug an arbitrary angle Λ. Such a process is described [46] by the O(2) group c rotations in a plane, homomorphic with U(1). The invariance of action under th same global gauge transformation results in a conserved charge Q and divergentless current:

$$\frac{dQ}{dt} = 0; \qquad Q = \int J^0\, dV; \qquad \partial_{\mu} J^{\mu} = 0 \tag{267}$$

These concepts stem from a variational principle applied to the action

$$S = \int \mathscr{L}(A \partial_{\mu} A)\, d^4 x \tag{268}$$

which is stationary [46] under the condition

$$\frac{\partial \mathscr{L}}{\partial A} = \partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\mu} A)}\right) = 0 \tag{269}$$

which is the Euler–Lagrange equation for A in the internal U(1) gauge space of electromagnetic theory in the vacuum. The action is considered [46] in Noether's theorem to be unchanged by re-parameterization of x^{μ} and A, that is, is invariant under some group of transformations on x^μ and A. It follows [46] that there exist conserved quantities that are combinations of fields and derivatives, which are invariant under these transformations: energy, momentum, angular momentum, and charge.

For example, it can be shown that the energy momentum tensor due to A is [46]

$$\theta^\mu_\nu = \partial^\mu A \partial_\nu A - \frac{1}{2}\delta^\mu_\nu \partial_\sigma A \partial^\sigma A \tag{270}$$

For translation of the origin of space and time [46], Noether's theorem gives

$$J^\mu_\nu = -\theta^\mu_\nu = -\partial^\mu A \partial_\nu A + \frac{1}{2}\delta^\mu_\nu \partial_\sigma A \partial^\sigma A \tag{271}$$

The conserved quantity in this case is the energy momentum

$$\frac{d}{dt}\int \theta^0_\nu d^3x = 0 \tag{272}$$

in the internal gauge space. The energy and momentum of the field in the internal gauge space are given by

$$En = \int \theta^0_0 d^3x; \quad p = \int \theta^0_1 d^3x \tag{273}$$

Under the local gauge transformation (226) of the Lagrangian (219), the action is no longer invariant [46], and invariance must be restored by adding terms to the Lagrangian. One such term is

$$\mathscr{L}_1 = -gJ^\mu A_\mu \tag{274}$$

where g is a parameter such that gA_μ has the units of ∂_μ. It is important to realize that this is true under all conditions, including the vacuum, so if electromagnetic theory in the vacuum is a U(1) gauge theory, then both g and A_μ must be introduced in the vacuum. It is clear that

$$g = \frac{\kappa}{A^{(0)}} \tag{275}$$

satisfies the requirement that gA_μ have the same units as ∂_μ. The 4-potential A_μ is introduced from Minkowski spacetime and, under local U(1) gauge transformation

$$A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu \Lambda \tag{276}$$

where Λ is arbitrary. Local gauge transformation therefore results in the total Lagrangian (251) that is needed to render the action invariant.

Therefore the Lehnert equation (253) correctly conserves action under a local U(1) gauge transformation in the vacuum. Such a transformation leads to a vacuum charge current density as the result of gauge theory itself, because U(1) gauge theory has a scalar internal space that supports A and A^*. These must be complex in order to define the globally conserved charge:

$$Q = \int J^0 dV \tag{277}$$

from the globally invariant current:

$$J^\mu = i(A^*\partial^\mu A - A\partial^\mu A^*) \tag{278}$$

in the internal U(1) space of the gauge theory.

The existence of a vacuum charge current density in the vacuum was first introduced phenomenologically by Lehnert [45,49], and it has been shown that the Lehnert equations can describe phenomena that the Maxwell–Heaviside equations are unable to describe. The reason for this is now clear. The vacuum Maxwell–Heaviside equations do not conserve action under a local gauge transformation in the internal scalar space of a U(1) gauge field theory. In order to conserve action, a locally gauge-invariant charge current density of the type appearing in Eq. (253) is needed in the vacuum, and it has just been argued that such a conclusion has a solid basis in gauge theory. If the charge current density were absent, there would be no scalar internal space for U(1) gauge theory applied in the vacuum to electromagnetism. It follows, as argued already, that the vector potential A_μ and the electromagnetic field tensor $F^{\mu\nu}$ are the result of local gauge transformation and originate in the vacuum topology.

There is empirical evidence that electrons and positrons annihilate to give photons, and this process is represented symbolically by

$$e^- + e^+ = 2\gamma \tag{279}$$

This process cannot be described classically, because positrons are the result of the Dirac equation, but it illustrates the fact that a vacuum current (of photons) is made up of the interaction of two Dirac currents, one for the electron, one for the positron, and these are both matter currents. Therefore, there is a transmutation of matter current to vacuum current. On the classical level, this can be described in the scalar internal gauge space as

$$\phi \rightleftharpoons A \tag{280}$$

where ϕ is a matter field and A is the scalar component of an electromagnetic potential. As shown in Eqs. (263), the matter and vacuum fields have the same structure. The coefficient g in the vacuum field is $\kappa/A^{(0)}$ and is $e/\hbar$ in the matter field. The process

$$\hbar\kappa \to eA^{(0)} \tag{281}$$

is therefore a transfer of photon linear momentum to an electron, as in the Compton effect. As soon as $\hbar$ is introduced, Planck quantization is also introduced. Since e is a property of neither the electromagnetic field nor the Dirac electron, the equation

$$\hbar\kappa = eA^{(0)} \tag{282}$$

can be regarded [47–61] as a Planck quantization of the factor g in the vacuum:

$$g = \frac{\kappa}{A^{(0)}} = \frac{e}{\hbar} \tag{283}$$

The Lehnert equations are a great improvement over the Maxwell–Heaviside equations [45,49] but are unable to describe phenomena such as the Sagnac effect and interferometry [42], for which an O(3) internal gauge space symmetry is needed.

IV. DEVELOPMENT OF GAUGE THEORY IN THE VACUUM

Gauge theory can be developed systematically for the vacuum on the basis of material presented in Section II. Before doing so, recall that, on the U(1) level, A^μ exists in Minkowski spacetime and there is a scalar internal gauge space that can be denoted

$$\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} = A_X\boldsymbol{i} + A_Y\boldsymbol{i} \tag{284}$$

The internal gauge space has local symmetry, and is a physical space. In complex circular notation, the vector in the internal gauge space can be written as

$$\boldsymbol{A} = A^{(2)}\boldsymbol{e}^{(1)} + A^{(1)}\boldsymbol{e}^{(2)} \tag{285}$$

indicating two states of circular polarization. Therefore, we have $A^{\mu(1)}$ and $A^{\mu(2)}$ in the vacuum. Circular polarization becomes a prerequisite for the conserved Q of Eq. (277). In the notation of Eq. (285)

$$A^{(1)} = \frac{1}{\sqrt{2}}(A_X - iA_Y); \qquad A^{(2)} = \frac{1}{\sqrt{2}}(A_X + iA_Y) \tag{286}$$

Circular polarization appears in general if

$$A_X = A^{(0)}\exp(-i(\omega t - \kappa Z)) \quad (28$$

$$A_Y = A^{(0)}\exp(-i(\omega t - \kappa Z)) \quad (28$$

where we have included the electromagnetic phase on the U(1) level. The scal internal space in the vacuum is therefore described by the following two vecto

$$A = \frac{1}{\sqrt{2}}(A_X + iA_Y); \qquad A^* = \frac{1}{\sqrt{2}}(A_X - iA_Y) \quad (28$$

Global gauge transformation on these vectors produces a shift in the electr magnetic phase

$$A_X \to A^{(0)}\exp(-i(\omega t - \kappa Z + \Lambda)) \quad (29$$

$$A_Y \to A^{(0)}\exp(-i(\omega t - \kappa Z + \Lambda)) \quad (29$$

where Λ is an arbitrary number. So under global gauge transformation, tl electromagnetic phase in the vacuum is defined only up to an arbitrary Λ. Und local gauge transformation

$$A_X \to A^{(0)}\exp(-i(\omega t - \kappa Z + \Lambda(x^\mu))) \quad (29$$

$$A_Y \to A^{(0)}\exp(-i(\omega t - \kappa Z + \Lambda(x^\mu))) \quad (29$$

and the U(1) electromagnetic phase is defined up to an arbitrary number Λ, whic is a function of the spacetime coordinate x^μ. In consequence, it has been show elsewhere [42,47–61] that U(1) gauge theory applied to electromagnetism do not describe interferometry or physical optics in general.

There is an interrelation between the A and A^μ vectors of the scalar intern gauge space and components of $A^{\mu(1)}$ and $A^{\mu(2)}$ in the vacuum

$$\boldsymbol{A}^{(1)} = iA_X\boldsymbol{e}^{(1)} \quad (29$$

$$\boldsymbol{A}^{(2)} = -iA_Y\boldsymbol{e}^{(2)} \quad (29$$

so that $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is a vacuum plane wave. It can be seen that, on the U(1) leve local and global gauge transformation introduce arbitrariness into the electro magnetic phase factor:

$$\gamma = \exp(-i(\omega t - \kappa Z)) \quad (29$$

Dirac attempted to remedy this flaw on the U(1) level by defining the electromagnetic phase factor by [42]

$$\gamma = \exp\left(ig \oint A_\mu(x^\mu)\, dx^\mu \right) \tag{297}$$

On the O(3) level, vacuum gauge theory is defined by a Clifford algebra

$$\boldsymbol{A}_\mu = A_\mu^{(2)} \boldsymbol{e}^{(1)} + A_\mu^{(1)} \boldsymbol{e}^{(2)} + A_\mu^{(3)} \boldsymbol{e}^{(3)} \tag{298}$$

$$\boldsymbol{A} = A^{(2)} \boldsymbol{e}^{(1)} + A^{(1)} \boldsymbol{e}^{(2)} + A^{(3)} \boldsymbol{e}^{(3)} \tag{299}$$

where $\boldsymbol{A}_\mu$ is a vector in the internal gauge space of O(3) symmetry and a 4-vector in Minkowski spacetime. In the internal gauge space, the Maxwell vector potential is defined as

$$\boldsymbol{A} = A_X \boldsymbol{i} + A_Y \boldsymbol{j} + A_Z \boldsymbol{k} = A^{(2)} \boldsymbol{e}^{(1)} + A^{(1)} \boldsymbol{e}^{(2)} + A^{(3)} \boldsymbol{e}^{(3)} \tag{300}$$

indicating by ansatz the existence of a nonzero $\boldsymbol{A}^{(3)}$ in the vacuum. The latter describes the Sagnac effect with precision as demonstrated elsewhere [42] using a non-Abelian Stokes theorem. On the O(3) level, the electromagnetic phase factor is a Wu–Yang phase factor denoted

$$\gamma = P\exp\left(ig \oint A_\mu(x^\mu)\, dx^\mu \right) \tag{301}$$

where parallel transport is implied [42] with O(3) covariant derivatives. In the vacuum, the factor g is given by Eq. (275) for all gauge group symmetries. There is again a relation between the internal vector $\boldsymbol{A}$ and components in the vacuum of the four vector $\boldsymbol{A}^\mu$. For example

$$\boldsymbol{A}^{(1)} = iA_X \boldsymbol{e}^{(1)}; \qquad \boldsymbol{A}^{(2)} = -iA_Y \boldsymbol{e}^{(2)}; \qquad \boldsymbol{A}^{(3)} = A_Z \boldsymbol{k} \tag{302}$$

So it becomes clear that the description of the vacuum in gauge theory can be developed systematically by recognizing that, in general, A is an n-dimensional vector. On the U(1) level, it is one-dimensional; on the O(3) level, it is three-dimensional; and so on. The internal gauge space in this development is a physical space that can be subjected to a local gauge transform to produce physical vacuum charge current densities.

So in the general case where A is an n-dimensional vector [46], a local gauge transform on this vector is represented in the vacuum by

$$\begin{aligned} A(x^\mu) \rightarrow A'(x^\mu) &= \exp\left(i\mathrm{M}^a \Lambda^a(x^\mu)\right) A(x^\mu) \\ &\equiv S(x^\mu) A(x^\mu) \end{aligned} \tag{303}$$

where M^a are the generators of the group that describes the symmetr internal gauge space, and where the index a is summed from 1 to 3 w internal gauge group is O(3). It follows that

$$\partial_\mu A' = S(\partial_\mu A) + (\partial_\mu S) A$$

so $\partial_\mu A$ does not transform covariantly. This is the basis of the gauge princi the principle of parallel transport in the vacuum for any gauge group sy Parallel transport in the vacuum produces the vector δA, where

$$\delta A = ig \mathrm{M}^a A_\mu^a dx^\mu A$$

So the product $g\mathrm{M}^a A_\mu^a$ is the result of special relativity in the vacuum, adjusted for correct units. Ryder [46] simply describes A_μ^a as "an additio or potential;" Feynman describes it as "the universal influence." There argued in the foregoing section, both the potential and the electromagne in the vacuum originate in local gauge transformation, which, in turn, ori in special relativity itself.

The covariant derivative in the vacuum for any internal gauge group s try is therefore defined by

$$D_\mu A = (\partial_\mu - ig\mathrm{M}^a A_\mu^a) A$$

and is valid for an n-dimensional A and for any internal gauge group generators are represented by matrices M^a [46]. The U(1) covariant deriv the vacuum is given by $M = -1$, resulting in

$$D_\mu A = (\partial_\mu + igA_\mu) A$$

On the O(3) level, the covariant derivative in the vacuum is given by

$$D_\mu \boldsymbol{A} = \partial_\mu \boldsymbol{A} + g\boldsymbol{A}_\mu \times \boldsymbol{A}$$

Considering a rotation $A = SA$ in the vacuum, the covariant derivative tran as

$$D_\mu A \rightarrow D'_\mu A' = SD_\mu A$$

that is

$$(\partial_\mu - igA'_\mu) A' = S(\partial_\mu - igA_\mu) A$$

which [42] leads to the law governing A_μ under gauge transformation in any gauge group:

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \tag{311}$$

It is also possible to consider the holonomy of the generic A in the vacuum. This is a round trip or closed loop in Minkowski spacetime. The general vector A is transported from point A, where it is denoted $A_{A,0}$ around a closed loop with covariant derivatives back to the point $A_{A,0}$ in the vacuum. The result [46] is the field tensor for any gauge group

$$G_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{312}$$

and the field tensor is the result of rotating the vector A in the internal space of the gauge theory in the vacuum. It is seen that the field tensor is a commutator of covariant derivatives, and therefore originates in local gauge transformation. On the U(1) level, the field tensor in the vacuum is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{313}$$

and on the O(3) level is

$$\boldsymbol{G}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + g\boldsymbol{A}_\mu \times \boldsymbol{A}_\nu \tag{314}$$

The field tensor transforms covariantly [46] because

$$\begin{aligned} A_{A,0} &\rightarrow A'_{A,0} = SA_{A,0} \\ A_{A,1} &\rightarrow A'_{A,1} = SA_{A,1} \end{aligned} \tag{315}$$

in the vacuum.

Similarly, transport of the generic A around a three-dimensional closed loop [46] produces the Jacobi identity

$$\sum_{\text{cyclic}} [D_\sigma, [D_\mu, D_\nu]] = 0 \tag{316}$$

for any gauge group symmetry in the vacuum. On the U(1) level, it is the homogeneous field equation

$$\partial_\mu \tilde{F}^{\mu\nu} \equiv 0 \tag{317}$$

and on the O(3) level, the homogeneous field equation:

$$D_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{318}$$

The complete set of vacuum field and wave equations on the U(1) level therefore

$$\partial_\mu \tilde{F}^{\mu\nu} \equiv 0 \tag{319}$$

$$\partial_\mu F^{\mu\nu} = \frac{J^\mu(\text{vac})}{\varepsilon_0} \tag{320}$$

$$D^\mu D_\mu A = 0 \tag{321}$$

and the complete set on the O(3) level is

$$\partial_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv 0 \tag{322}$$

$$D^\nu \boldsymbol{G}_{\mu\nu} = -gc(D_\mu \boldsymbol{A}) \times \boldsymbol{A} \tag{323}$$

$$D_\mu(D^\mu \boldsymbol{A}) = \boldsymbol{0} \tag{324}$$

All these results are derived essentially by considering a rotation of the genera vector A in the internal space of the gauge theory in the vacuum.

In order to demonstrate that spontaneous symmetry breaking can affect th energy inherent in the vacuum, consider the globally invariant Higgs Lagran gian:

$$\mathscr{L} = \partial_\mu(a + A)\partial^\mu(a + A^*) - m^2(a + A^*)(a + A) - \lambda((a + A^*)(a + A))^2 \tag{325}$$

It has been demonstrated already that local gauge transformation on thi Lagrangian leads to Eq. (153), which contains new charge current density term due to the Higgs mechanism. For our present purposes, however, it is clearer t use the locally invariant Lagrangian obtained from Eq. (325), specifically

$$\begin{aligned} \mathscr{L} = {} & (\partial_\mu + igA_\mu)(a + A)(\partial^\mu - igA^\mu)(a + A^*) \\ & - m^2(a + A)(a + A^*) - \lambda(a + A)^2(a + A^*)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \end{aligned} \tag{326}$$

with the Euler–Lagrange equations:

$$\frac{\partial \mathscr{L}}{\partial A} = \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A)}\right); \qquad \frac{\partial \mathscr{L}}{\partial A^*} = \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A^*)}\right) \tag{327}$$

Such a procedure produces the equations:

$$\begin{aligned} D_\mu D^\mu A^* &= -m^2 A^* - 2\lambda A^*(AA^*) \\ D_\mu D^\mu A &= -m^2 A - 2\lambda A(A^* A) \end{aligned} \tag{328}$$

where we have used $a = a^*$. So the effect of the Higgs mechanism is to generate the inhomogeneous wave equations (328) from the homogeneous wave equations (225) and (228) by spontaneous symmetry breaking [46] of the vacuum. The charge current densities on the right-hand side of Eq. (328) can be used to generate the equivalent matter charge current densities as discussed later in this section.

Without the Higgs mechanism, the Lagrangian (325) is

$$\mathscr{L} = \partial_\mu A \partial^\mu A^* - m^2 A^* A - \lambda A^* A A^* A \tag{329}$$

and using Eqs. (327) produces the wave equations:

$$\begin{aligned} \Box A^* &= -(m^2 + 2\lambda A^* A)A^* \\ \Box A &= -(m^2 + 2\lambda AA^*)A \end{aligned} \tag{330}$$

At the Higgs minimum

$$a^2 = |A|^2 = -\frac{m^2}{2\lambda} \tag{331}$$

Eqs. (330) become

$$\begin{aligned} \Box A^* &= 0 \\ \Box A &= 0 \end{aligned} \tag{332}$$

At the local Higgs maximum [46] for $m^2 < 0$, that is, at $m = 0$, Eqs. (330) become

$$\begin{aligned} \Box A^* &= -2\lambda(A^* A)A^* \\ \Box A &= -2\lambda(AA^*)A \end{aligned} \tag{333}$$

and Eqs. (328) become

$$\begin{aligned} D_\mu D^\mu A^* &= -2\lambda A^*(AA^*) \\ D_\mu D^\mu A &= -2\lambda A(A^* A) \end{aligned} \tag{334}$$

So both the globally and locally invariant equations of motion of the internal gauge space [the Euler–Lagrange equations (327)] are different at the Higgs maximum and minimum. The minimum and local maximum are different ground states of the field, and are different vacuum states. The difference between the Higgs maximum and minimum represents potential energy difference within the vacuum itself. The Higgs mechanism is well known to lead to electroweak theory and to the existence of the Higgs boson, so it is well established that in the vacuum, there is a usable difference of potential energy, the different minima of which lead to different ground states of the field theory and to different vacua. In nineteenth-century classical electromagnetism, on which a text such as that by Jackson [66] is based, such concepts do not exist. There is no vacuum charge current density, and there are no potential energy maxima or minima in the vacuum itself.

It is well known that there is an interesting analogy between spontaneous symmetry breaking of the vacuum and the Landau–Ginzburg free energy in superconductors. The latter is obtained from the locally invariant Lagrangian (325) in the static limit [46]

$$\partial_0 A = 0 \tag{335}$$

where the mass term is defined as $m^2 = a(T - T_c)$ near the critical temperature T_c . At $T > T_c$, $m^2 > 0$ and the minimum free energy is at $|A| = 0$. When $T < T_c, m^2 < 0$ and the minimum free energy is at $|A|^2 = -(m^2/2\lambda) > 0$. This is an analogy with the case of spontaneous symmetry breaking in the vacuum, where there is a difference of free energy (or latent free energy) on the classical level that can be used for practical devices.

The effect of the Higgs mechanism can be seen most clearly by minimizing the Lagrangian (251) with respect to A:

$$\frac{\partial \mathscr{L}}{\partial A} = ig A_\mu D^\mu A^* = 0 \tag{336}$$

This minimum value defines the ground state and the true vacuum through the equation

$$\begin{aligned} D^\mu A^* &= 0 \\ D^\mu A &= 0 \end{aligned} \tag{337}$$

This means, however, that the vacuum charge current density disappears:

$$J^\mu(\text{vac}) = -\frac{ig}{\mu_0 c}(A^* D^\mu A - A D^\mu A^*) = 0 \tag{338}$$

It thus becomes clear that the vacuum charge current density introduced by Lehnert is an excitation above the true vacuum in classical electrodynamics. The true vacuum is defined by Eq. (337). It follows that in the true classical vacuum, the electromagnetic field also disappears.

Using the Higgs Lagrangian (326) however, the true vacuum is defined by

$$\frac{\partial \mathscr{L}}{\partial t} = igA_\mu D^\mu A^* - m^2 A^* - 2\lambda A^*(AA^*) = 0 \tag{339}$$

and the true vacuum itself carries a charge current density. The charge current density in the true vacuum is described by Eq. (339), which is consistent with the fact that the Lehnert charge current density implies photon mass, as does the Higgs mechanism.

The transfer of the energy associated with this true vacuum charge current density to a matter current is achieved by adjusting the value of the coupling constant g such that the vacuum value $g = \kappa / A^{(0)}$ becomes $e/\hbar$ in matter. The resulting equation is

$$g = \frac{\kappa}{A^{(0)}} = \frac{e}{\hbar} \tag{340}$$

specifically

$$\hbar\kappa = eA^{(0)} \tag{341}$$

which classically gives the minimal prescription:

$$\boldsymbol{p} = e\boldsymbol{A} \tag{342}$$

The momentum $\boldsymbol{p}$ is derived from a limit of general relativity, and so is derived from the structure of spacetime. Therefore $e\boldsymbol{A}$ is also derived from the structure of spacetime, or from the vacuum itself. The meaning of e is reinterpreted as the minimum value of

$$e = \hbar\frac{\kappa}{A^{(0)}} \tag{343}$$

and this minimum value is the charge on the proton.

At the Higgs minimum, the Lagrangian in the internal space of the U(1) gauge theory is

$$\mathscr{L} = \partial_\mu a \partial^\mu a^* - m^2 a^* a - \lambda(a^* a)^2 \tag{344}$$

which, on local gauge transformation, becomes

$$\mathscr{L} = (\partial_\mu + igA_\mu)a(\partial^\mu - igA^\mu)a^* - m^2 aa^* - \lambda(aa^*)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad (345$$

The equations of motion of the field at the Higgs minimum (the minimu potential energy of the vacuum) are the Euler–Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial a} = \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu a)}\right); \qquad \frac{\partial \mathscr{L}}{\partial a^*} = \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu a^*)}\right) \qquad (346$$

$$\frac{\partial \mathscr{L}}{\partial A_\mu} = \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu)}\right) \qquad (347$$

and using the globally invariant Lagrangian (344) in Eqs. (346) gives the resu

$$\Box a^* = -(m^2 + 2\lambda a^* a)a^* = 0$$
$$\Box a = -(m^2 + 2\lambda aa^*)a = 0 \qquad (348$$

and, using the locally invariant Lagrangian (345) in Eqs. (346) gives the resul

$$D_\mu(D^\mu a^*) = -(m^2 + 2\lambda a^* a)a^* = 0$$
$$D_\mu(D^\mu a) = -(m^2 + 2\lambda aa^*)a = 0 \qquad (349$$

Equation (348) is the globally invariant wave equation defining a, and Eq. (349 is its locally invariant equivalent. Using the locally invariant Lagrangian (345) i Eq. (347) gives the inhomogeneous field equation (SI units)

$$\partial_\nu F^{\mu\nu} = -igc(a^* D^\mu a - aD^\mu a^*) \qquad (350$$

where the charge current density on the right-hand side is obtained from the pur vacuum by local gauge transformation and local gauge invariance. Both the left and right-hand sides of Eq. (350) are defined by the minimum of potentia energy, and by the minimum value that A can attain. This minimum value is a and is the vacuum expectation value of A [46], associated with a nonzero potentia energy that gives rise to A^μ and $F^{\mu\nu}$ by local gauge invariance. Therefore th source of an electromagnetic field propagating in the vacuum is the Higg minimum value of A, which is denoted a. If we do not use a Higgs mechanism then the vacuum expectation value of A in the internal gauge space of the U(1 gauge theory is zero, and the globally invariant Lagrangian disappear

Therefore, in the presence of a Higgs mechanism

$$|\langle 0|A|0\rangle|^2 = a^2 \tag{351}$$

and in its absence:

$$|\langle 0|A|0\rangle|^2 = 0 \tag{352}$$

The Lagrangian (345) can be written as [see Eq. (158)]

$$\mathscr{L} = g^2 a^2 A_\mu A^\mu + \cdots \tag{353}$$

and if the photon mass is identified as

$$m_p^2 \equiv 2g^2 a^2 \tag{354}$$

the Lagrangian (353) gives a Proca equation that is locally gauge invariant on the U(1) level. Therefore, application of the Higgs mechanism in this way has produced one massive photon from one massless photon. The scalar field a remains unaffected, so degrees of freedom are conserved. Therefore, this theory identifies photon mass as the result of local gauge invariance applied at the Higgs minimum, that is, the minimum value that the potential energy of the globally invariant Lagrangian can take in the vacuum.

This minimum value provides the true vacuum energy

$$En(\text{vac}) = \int J^\mu(\text{vac}) A_\mu \, dV \tag{355}$$

and a rate of doing work:

$$\frac{dW}{dt}(\text{vac}) = \int \boldsymbol{J}(\text{vac}) \cdot \boldsymbol{E} \, dV \tag{356}$$

The Poynting theorem for the true vacuum can be developed as in Eqs. (258)–(262). The true vacuum energy (355) comes from the vacuum current in Eq. (350), which is transformed into a matter current by a minimal prescription as discussed already. This matter current in principle provides an electromotive force in a circuit. It is to be noted that the local Higgs maximum occurs at $\boldsymbol{A} = 0$ [46], so the local Higgs minimum occurs below the zero value of A.

The overall conclusion is that there is no objection in principle to extracting electromotive force from the true vacuum, defined by the minimum value, a, which can be attained by A in the internal scalar space of the gauge theory, which is the theory underlying electromagnetic theory.

On the O(3) level, the globally invariant Lagrangian corresponding to Eq. (344) is

$$\mathscr{L} = T - V = \frac{1}{2}\partial_\mu \boldsymbol{a} \cdot \partial^\mu \boldsymbol{a} - \frac{m^2}{2}\boldsymbol{a} \cdot \boldsymbol{a} - \lambda(\boldsymbol{a} \cdot \boldsymbol{a})^2 \tag{357}$$

with potential energy:

$$V = \frac{m^2}{2}\boldsymbol{a} \cdot \boldsymbol{a} + \lambda(\boldsymbol{a} \cdot \boldsymbol{a})^2 \tag{358}$$

Here, $\boldsymbol{a}$ is a vector in the internal space of O(3) symmetry. The equation of motion is

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{a}} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial_\mu \boldsymbol{a}} \right) = \partial^\mu \left(\frac{\partial \mathscr{L}}{\partial^\mu \boldsymbol{a}} \right) \tag{359}$$

and produces, from the Lagrangian (357), the result

$$\Box \boldsymbol{a} = m^2 \boldsymbol{a} + \lambda \boldsymbol{a}(\boldsymbol{a} \cdot \boldsymbol{a}) = \boldsymbol{0} \tag{360}$$

which is a globally invariant wave equation of d'Alembert type for the three components of $\boldsymbol{a}$. Local gauge transformation of the Lagrangian (357) produces [cf. Eq. (205)] the following equation:

$$\mathscr{L} = \frac{1}{2} D_\mu \boldsymbol{a} \cdot D^\mu \boldsymbol{a} - \frac{m^2}{2}\boldsymbol{a} \cdot \boldsymbol{a} - \lambda(\boldsymbol{a} \cdot \boldsymbol{a})^2 - \frac{1}{4}\boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}^{\mu\nu} \tag{361}$$

Use of Eq. (359) produces the wave equation

$$D^\mu(D_\mu \boldsymbol{a}) = m^2 \boldsymbol{a} + \lambda \boldsymbol{a}(\boldsymbol{a} \cdot \boldsymbol{a}) = \boldsymbol{0} \tag{362}$$

The Euler–Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{A}^\mu} = \partial^\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial^\nu \boldsymbol{A}^\mu)} \right) \tag{363}$$

produces the field equation

$$D^\nu \boldsymbol{G}_{\mu\nu} = -g(D_\mu \boldsymbol{a}) \times \boldsymbol{a} \tag{364}$$

where the current on the right-hand side is a current generated by the minimum value of $\boldsymbol{A}$ in the internal O(3) symmetry gauge space. This minimum value is the vacuum and is denoted by the vector $\boldsymbol{a}$

The Lagrangian (361) can be written as

$$\mathscr{L} = g^2 \boldsymbol{A}_\mu \times \boldsymbol{a} \cdot \boldsymbol{A}^\mu \times \boldsymbol{a} + \cdots \tag{365}$$

and produces three photons with mass from the vector identity

$$(\boldsymbol{A}_\mu \times \boldsymbol{a}) \cdot (\boldsymbol{A}^\mu \times \boldsymbol{a}) = (\boldsymbol{A}_\mu \cdot \boldsymbol{A}^\mu)(\boldsymbol{a} \cdot \boldsymbol{a}) - (\boldsymbol{A}_\mu \cdot \boldsymbol{a})(\boldsymbol{a} \cdot \boldsymbol{A}^\mu) \tag{366}$$

and the term

$$\mathscr{L} = g^2 (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{A}_\mu \cdot \boldsymbol{A}^\mu) + \cdots \tag{367}$$

One of these is the superheavy Crowell boson [42], associated with index (3) in the ((1),(2),(3)) basis, and the other two are massive photons associated with indices (1) and (2). The superheavy Crowell boson comes from electroweak theory with an SU(2) electromagnetic sector and may have been observed in a LEP collaboration at CERN [44,56].

On the O(3) level, the vacuum current (SI units)

$$\boldsymbol{J}^\mu(\text{vac}) = \frac{g}{\mu_0 c} (D^\mu \boldsymbol{a}) \times \boldsymbol{a} \tag{368}$$

gives the vacuum energy

$$En = \int \boldsymbol{J}^\mu(\text{vac}) \cdot \boldsymbol{A}_\mu \, dV \tag{369}$$

which can be transformed into a matter current by the minimal prescription (342). This matter current is effectively an electromotive force in a circuit. Gauge theory of any internal gauge symmetry applied to electromagnetism comes to the same result, that energy is available from the vacuum, defined as the Higgs minimum. This appears to be a substantial advance in understanding.

In order to check these results for self-consistency, the locally invariant Higgs Lagrangian, when written out in full, is

$$\begin{aligned}\mathscr{L} = {} & \partial_\mu (a_0 + A) \partial^\mu (a_0 + A)^* - ig((a_0 + A)^* \partial^\mu (a_0 + A) - (a_0 + A) \partial^\mu (a_0 + A)^*) A_\mu \\ & + g^2 A_\mu A^\mu (a_0 + A)^2 - m^2 (a_0 + A)(a_0 + A)^* \\ & - \lambda((a_0 + A)(a_0 + A)^*)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}\end{aligned} \tag{370}$$

where a_0 is the minimum value and where the complex scalar field is

$$A = \frac{1}{\sqrt{2}} (A_1 + iA_2) \tag{371}$$

in the internal space. In this Lagrangian, a_0 is a constant so the Lagrangian (3
can be written as

$$\begin{aligned}\mathscr{L} = {} & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g^2 a_0^2 A_\mu A^\mu + \frac{1}{2} (\partial_\mu A_1)^2 + \frac{1}{2} (\partial_\mu A_2)^2 \\ & - 2\lambda a_0^2 A_1^2 + \sqrt{2} g a_0 A^\mu \partial_\mu A_2 + \cdots\end{aligned} \quad (3$$

At its minimum value, this Lagrangian is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g^2 a_0^2 A_\mu A^\mu \quad (3$$

which gives the following locally gauge-invariant Proca equation:

$$\partial_\mu F^{\mu\nu} + m_p^2 A^\nu = 0 \quad (3$$

The photon mass is identified as argued already by

$$m_p^2 = 2g^2 |a_0^2| \quad (3$$

and if we further identify

$$g^2 \equiv \frac{\kappa}{2|a_0^2|} \quad (37$$

we obtain the de Broglie guidance theorem in SI units:

$$\hbar\omega = m_p c^2 \quad (37$$

So, as argued already, the photon mass is picked up from the vacuum, that
from the minimum value of the locally invariant Higgs Lagrangian (370). Tl
conclusion means that the Lehnert charge current density that leads to the Pro
equation [45,49] is also a property of the vacuum, as argued above. In order
show this result, the constant a_0 is expressed as the product of two complex fiel
$\boldsymbol{a}$ and $\boldsymbol{a}^*$. To illustrate this by analogy, one can show that the dot product of tv
conjugate plane waves gives a constant

$$A^{(0)2} = \frac{A^{(0)}}{\sqrt{2}} (\boldsymbol{i} - i\boldsymbol{j}) \cdot e^{i\phi} \frac{A^{(0)}}{\sqrt{2}} (\boldsymbol{i} + i\boldsymbol{j}) e^{-i\phi} = \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} \quad (37$$

but the individual plane waves are functions of coordinates and tim
Analogously, therefore, a and a^* are functions of x^μ. The vacuum Lagrangi

$$\mathscr{L} = \partial_\mu a \partial^\mu a^* - ig(a^* \partial^\mu a - a \partial^\mu a^*) A_\mu$$
$$+ g^2 A_\mu A^\mu a^2 - m^2 a^* a - \lambda (a^* a)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad (379)$$

From Eq. (373), it is known that this Lagrangian is

$$\mathscr{L} = g^2 A_\mu A^\mu a^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad (380)$$

There is therefore a balance between globally invariant Lagrangians:

$$\mathscr{L} = \partial_\mu a \partial^\mu a^* - m^2 a^* a - \lambda (a^* a)^2$$
$$= ig(a^* \partial^\mu a - a \partial^\mu a^*) A_\mu = g J^\mu A_\mu \qquad (381)$$

The globally invariant vacuum energy is therefore:

$$En = \int J^\mu A_\mu dV = \frac{1}{g} \int \partial_\mu a \partial^\mu a^* - m^2 a^* a - \lambda (a^* a)^2 dV \qquad (382)$$

and is defined in the internal space of the gauge theory being considered [in this case of U(1) symmetry]. It can be seen that the vacuum energy is essentially a volume integration over the original globally invariant Lagrangian

$$\mathscr{L} = \partial_\mu a \partial^\mu a^* - m^2 a^* a - \lambda (a^* a)^2 \qquad (383)$$

used in the Higgs mechanism. We have defined the mass of the photon by Eq. (375), and so the locally gauge-invariant Proca wave equation is

$$\Box A_\mu = -2 g^2 a_0^2 A_\mu \qquad (384)$$

Energy is usually written as the volume integral over the Hamiltonian, and not the Lagrangian, and Eq. (382) may be transformed into a volume integral over a Hamiltonian if we define the effective potential energy

$$V = -m^2 a^* a - \lambda (a^* a)^2 \qquad (385)$$

which is negative.

The locally gauge-invariant Lehnert field equation corresponding to Eq. (374) was derived as Eq. (350). The photon picks up mass from the vacuum itself, and having derived a locally gauge-invariant Proca equation, canonical quantization can be applied to produce a photon with mass with three space dimensions.

V. SCHRÖDINGER EQUATION WITH A HIGGS MECHANISM: EFFECT ON THE WAVE FUNCTIONS

In order to measure the effect of vacuum energy in atoms and molecules, in simplest case of the hydrogen atom, it is necessary to develop the nonrelativi Schrödinger equation with an inbuilt Higgs mechanism. The method used in t section is to start with the Lagrangian for the Higgs mechanism in matter fiel derive a Klein–Gordon equation, and from that, an Einstein equation, then to t the nonrelativistic limit of the Einstein equation, and finally quantize that to g the Schrödinger equation with a Higgs mechanism. It turns out that the Hi minimum is at an energy $\frac{1}{2}mc^2$ below the vacuum minimum with no Hi mechanism, meaning that this amount of energy is available in the vacuu Some examples of the effect of this negative potential energy on analyti solutions of the Schrödinger equation are given in this section.

The starting Lagrangian on the U(1) level for a free particle, such as an el tron, is the standard Lagrangian for the Higgs mechanism:

$$\mathscr{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \qquad (3$$

Using Eqs. (115) and (221), this Lagrangian gives the Klein–Gordon equati

$$(\Box + (m^2 + 2\lambda \phi^* \phi)) \phi^* = 0 \qquad (3$$
$$(\Box + (m^2 + 2\lambda \phi \phi^*)) \phi = 0 \qquad (3$$

in which ϕ and ϕ^* are considered to be complex-valued one-particle wa functions. It can be seen that the effect of the Higgs mechanism is to increase mass term m^2 to $m^2 + 2\lambda \phi^* \phi$.

This additional effective mass is introduced from spontaneous symme breaking of the vacuum. The two Klein–Gordon equations therefore take the form

$$(\Box + m^2) \phi^* = -2\lambda (\phi \phi^*) \phi^* \qquad (3$$
$$(\Box + m^2) \phi = -2\lambda (\phi \phi^*) \phi \qquad (3$$

The classical equivalent of these equations is the Einstein equation for particle

$$En^2 = p^2 c^2 + m_0^2 c^4 + 2\lambda (\phi \phi^*) c^4 \qquad (3$$

The Higgs mechanism has produced an additional rest energy:

$$En_0(\text{Higgs}) = 2\lambda (\phi \phi^*) c^4 \qquad (3$$

In Eq. (391), En is the total energy, and the equation can be written as follows:

$$p^2c^2 = En^2 - En_0^2 = m_0^2c^4\left(1-\frac{u^2}{c^2}\right)^{-1} - m_0^2c^4 - 2\lambda\langle\phi^2\rangle c^4 \qquad (393)$$

To reach the nonrelativistic limit of this equation, the right-hand side is expanded as

$$p^2c^2 = m^2c^4\frac{u^2}{c^2} - 2\lambda\langle\phi^2\rangle c^4 \qquad (u \ll c) \qquad (394)$$

which, for $u \ll c$, results in the nonrelativistic equation

$$p^2c^2 = m^2c^4\frac{u^2}{c^2} - 2\lambda\langle\phi^2\rangle c^4 = En^2 - En_0^2 \qquad (395)$$

which has the same form as the original, fully relativistic, equation (393). The nonrelativistic equation (395) can be written as

$$m^2u^2 = p^2 + 2\lambda\langle\phi^2\rangle c^2 \quad (u \ll c) \qquad (396)$$

that is

$$\frac{1}{2}mu^2 = \frac{p^2}{2m} + \frac{\lambda}{m}\langle\phi^2\rangle c^2 \qquad (397)$$

The left-hand side is the nonrelativistic kinetic energy of one particle. It can be seen that the Higgs mechanism changes the classical nonrelativistic expression

$$En = \frac{1}{2}mu^2 = \frac{p^2}{2m} = T \qquad (398)$$

to Eq. (397). The Schrödinger equation without the Higgs mechanism is obtained by applying the quantum ansatz

$$En \to i\hbar\frac{\partial}{\partial t}; \qquad \boldsymbol{p} \to -i\hbar\nabla \qquad (399)$$

to Eq. (398), giving

$$-i\hbar\frac{\partial\phi}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\phi \qquad (400)$$

The Schrö

$$-i\hbar\frac{\partial\phi}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\phi + \frac{\lambda}{m}\langle\phi^2\rangle c^2 \qquad (401$$

where $\langle\phi^2\rangle$ is the expectation value of the wave function. At the Higgs minimum this expectation value is [46]

$$\langle\phi^2\rangle = -\frac{m^2}{2\lambda} \qquad (402$$

and so the Schrödinger equation at the Higgs minimum is

$$-i\hbar\frac{\partial\phi}{\partial t} = \frac{\hbar^2}{2m}\phi - \frac{1}{2}mc^2\phi \qquad (403$$

which can be written in the familiar form

$$En\,\phi = H\phi = (T+V)\phi, \qquad T = -\frac{\hbar}{2m}\nabla^2 \qquad (404$$

where

$$V = -\frac{1}{2}mc^2 = \min\left(\frac{\lambda}{m}\langle\phi^2\rangle c^2\right) \qquad (405$$

is a negative potential energy produced by spontaneous symmetry breaking of the vacuum. The Schrödinger equation (404) shows that the Higgs minimum (the symmetry broken vacuum) is at an energy:

$$V(\text{Higgs}) = \frac{1}{2}mc^2 \qquad (406)$$

below the vacuum for the ordinary Schrödinger equation (400). The vacuum expectation value for the ordinary Schrödinger equation is

$$\langle\phi^2\rangle = 0 \qquad (407)$$

We have therefore derived a nonrelativistic Schrödinger equation for a free particle with an additional negative potential energy term $V = -\frac{1}{2}mc^2$. In order to apply this method to the hydrogen atom, the relevant Schrödinger equation is

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 - V_{\text{Coulomb}} + V\right)\phi = En\,\phi \tag{408}$$

$$V_{\text{Coulomb}} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} \tag{408a}$$

where V_{Coulomb} is the classical Coulomb interaction between one electron and one proton and μ is the reduced mass:

$$\mu = \frac{m_e m_p}{m_e + m_p} \tag{409}$$

The Higgs mechanism is the basis of electroweak theory and other elementary particle and gauge field theories, so it can be stated with confidence that to a good approximation the energy $\frac{1}{2}mc^2$ is released from the vacuum when a shift occurs between the Higgs minimum and the ground state of the hydrogen atom. The challenge is how to find a mechanism for releasing this energy. Mills [67] has found a working device based on the postulated collapse of the H atom below its ground state. The Schrödinger equation with a Higgs mechanism shows that there is an extra negative potential energy term that may account for the energy observed by Mills [67]. This possibility will be explored later by solving Eq. (408) analytically to find the effect of V on the states of the H atom. First, however, we illustrate the effect of V on analytical solutions of the Schrödinger equation, starting with the free-particle solution.

The wave function for Eq. (404) is well known [68] to be of the form

$$\phi = A'e^{i\kappa' Z} + B'e^{-i\kappa' Z}; \qquad \kappa' = \left(\frac{2m(E-V)}{\hbar^2}\right)^{1/2} \tag{410}$$

where the particle momentum is given by $\hbar\kappa'$. The scheme in the following equation group explains the role of the two parts of the wave function:

$$\begin{aligned} &\rightarrow p = \hbar\kappa'; \qquad \psi = A'e^{i\kappa' Z} \\ &\leftarrow p = \hbar\kappa'; \qquad \psi = B'e^{-i\kappa' Z} \end{aligned} \tag{411}$$

In the Schrödinger equation (404), the maximum value of the vacuum potential energy is the Newton vacuum

$$V = 0 \tag{412}$$

and its minimum value is the Higgs vacuum, or minimum of the symmetry-broken vacuum:

$$V = -\frac{1}{2}mc^2 \tag{413}$$

In Newtonian mechanics, the particle cannot be found below $V = 0$, therefo[re] Newtonian mechanics always corresponds to $V = 0$ [e.g., Eq. (398)], and th[is] represents, classically, an insurmountable barrier to a particle such as an electr[on] attempting to enter the Higgs region below $V = 0$. In quantum mechanic[s], however, an electron may enter the Higgs region by quantum tunneling, whic[h] occurs when $E < V = 0$. The wave function for this process is well known [to] be [68]

$$\phi = Ae^{-\kappa Z} \tag{414}$$

and has a nonzero amplitude. An electron of energy 1.6×10^{-19} J incident on [a] barrier of height 3.2×10^{-19} J has a wave function that decays with distanc[e] as $e^{-5.12\ (Z/\text{nm})}$, and decays to $1/e$ of its initial value after 0.2 nm, about th[e] diameter of an atom [68]. Therefore, quantum tunneling is important on atomi[c] scales. So quantum-mechanically, an electron can enter the Higgs region an[d] gain negative energy. This means that it radiates positive energy [46]. Th[e] maximum amount of energy that can be radiated is determined by the minimu[m] value of the Higgs region, which defines the ground state, namely, the Higg[s] vacuum. This is a result of Eq. (404) for a free electron. To see that negative energy states En are possible, write Eq. (395) as

$$En^2 = p^2c^2 + En_0^2 \tag{415}$$

and its solutions are

$$En = \pm(p^2c^2 + En_0^2)^{1/2} \tag{416}$$

The states of the hydrogen atom must be found from Eq. (408). When $V = 0$ the ground state of the H atom is well known [68] to be determined b[y] the expectation value

$$En = -\frac{\mu e^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n}; \qquad n = 1 \tag{417}$$

from the Schrödinger equation:

$$-\frac{\hbar^2}{2\mu}\nabla^2\phi - \frac{e^2}{4\pi\varepsilon_0 r}\phi = En\,\phi \tag{418}$$

When V is not zero, Eq. (418) becomes

$$-\frac{\hbar^2}{2\mu}\nabla^2\phi - \frac{e^2}{4\pi\varepsilon_0 r}\phi = (En - V)\phi \tag{419}$$

and the electronic orbital energy becomes

$$En = -\frac{\mu e^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n} + V \tag{420}$$

Here, n is the principal quantum number. So, for $V = 0$ the electronic orbital energy in the H atom becomes less negative as n increases. However, if we add $V < 0$ from the Higgs region to the ground state of H determined by $n = 1$, the electronic orbital energy falls below its ground state. This emits energy in the same way as an electron falling from a higher to a lower electronic atomic orbital emits energy. The energy emitted by driving the H orbital below its ground state has been observed experimentally by Mills et al. [67], repeatedly and reproducibly. The Higgs mechanism on the U(1) level accounts for this energy emission.

VI. VECTOR INTERNAL BASIS FOR SINGLE-PARTICLE QUANTIZATION

Conventional single particle quantization is based on the quantum ansatz (399) applied to the Einstein equation (415) to produce the Klein–Gordon equation

$$(\Box + m^2)\phi = 0 \tag{421}$$

$$(\Box + m^2)\phi^* = 0 \tag{422}$$

where ϕ is regarded as a single-particle wave function. In the nonrelativistic limit, the Schrödinger equation is obtained as demonstrated in Section IV. Formally, the Klein–Gordon equations (421) can be obtained from the U(1) Lagrangian [46]

$$\mathscr{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^* \tag{423}$$

which is globally invariant. Usually, the Lagrangian (423) is applied to complex fields, but formally, these can also be wave functions. On the U(1) level, they take the form

$$\phi = \phi^{(1)} = \frac{1}{\sqrt{2}}(\phi_X - i\phi_Y) \tag{424}$$

$$\phi^* = \phi^{(2)} = \frac{1}{\sqrt{2}}(\phi_X + i\phi_Y) \tag{425}$$

On the O(3) level, there are three wave functions:

$$\begin{aligned}\phi^{(1)} &= \frac{1}{\sqrt{2}}(\phi_X - i\phi_Y)\\ \phi^{(2)} &= \frac{1}{\sqrt{2}}(\phi_X + i\phi_Y)\\ \phi^{(3)} &= \phi_Z\end{aligned} \tag{42(}$$

and it is possible to collect these components in vector form through the relatic

$$\phi = \phi^{(2)}\boldsymbol{e}^{(1)} + \phi^{(1)}\boldsymbol{e}^{(2)} + \phi^{(3)}\boldsymbol{e}^{(3)} = \phi_X\boldsymbol{i} + \phi_Y\boldsymbol{j} + \phi_Z\boldsymbol{k} \tag{42'}$$

where ϕ_X, ϕ_Y, ϕ_Z are real-valued. The unit vectors of the circular basis a defined as

$$\begin{aligned}\boldsymbol{e}^{(1)} &= \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})\\ \boldsymbol{e}^{(2)} &= \frac{1}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})\\ \boldsymbol{e}^{(3)} &= \boldsymbol{k}\end{aligned} \tag{42ξ}$$

On the O(3) level, therefore, the probability density of the Schrödinger equatio is

$$\rho = \phi^{(1)}\phi^{(2)} = \phi^{(2)}\phi^{(1)} = \phi^{(3)}\phi^{(3)*} \tag{42ς}$$

and there are three Schrödinger equations:

$$\begin{aligned}\frac{\hbar^2}{2m}\nabla^2\phi^{(1)} &= -i\hbar\frac{\partial\phi^{(1)}}{\partial t}\\ \frac{\hbar^2}{2m}\nabla^2\phi^{(2)} &= -i\hbar\frac{\partial\phi^{(2)}}{\partial t}\\ \frac{\hbar^2}{2m}\nabla^2\phi^{(3)} &= -i\hbar\frac{\partial\phi^{(3)}}{\partial t}\end{aligned} \tag{429a}$$

which identify $\phi^{(1)}, \phi^{(2)}, \phi^{(3)}$ as angular momentum wave functions. Atkins [48 has shown that angular momentum commutator relations can be used to deriv the laws of nonrelativistic quantum mechanics. So the internal O(3) space, in thi instance, corresponds to ordinary three-dimensional space. In a U(1) interna space, the third component $\phi^{(3)}$ of angular momentum is missing and th

: functions are $\phi^{(1)}$ and $\phi^{(2)}$. In Newtonian and nonrelativistic quantum ıanics, the internal space is therefore O(3). The probability currents of the ödinger equation are

$$j = i\frac{\hbar}{2m}(\phi^{(2)}\nabla\phi^{(1)} - \phi^{(1)}\nabla\phi^{(2)}) \tag{430}$$

$$j = i\frac{\hbar}{2m}(\phi^{(3)}\nabla\phi^{(3)*} - \phi^{(3)*}\nabla\phi^{(3)}) = 0 \tag{431}$$

e complex circular basis. In a more general spherical harmonic [68] basis for :-dimensional space, the angular momentum wave functions are eigenfunc- ; such that

$$|n, m\rangle = Y(\theta, \phi) \tag{432}$$

re

$$Y_{lm_l}(\theta, \phi); \qquad l = 1; \quad m = 0, \pm 1 \tag{433}$$

he spherical harmonics. Therefore, it is also possible to describe the internal ı basis of electrodynamics in terms of spherical harmonics.

'he probability densities of the Klein–Gordon equation [46] in an O(3) inter- ɔasis contains terms such as

$$\rho = i\frac{\hbar}{2mc^2}\left(\phi^{(2)}\frac{\partial\phi^{(1)}}{\partial t} - \phi^{(1)}\frac{\partial\phi^{(2)}}{\partial t}\right) \tag{434}$$

; term is usually written as

$$\rho = i\frac{\hbar}{2mc^2}\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right) \tag{435}$$

in general can become negative. So the Klein–Gordon equation is abandoned eneral as an equation for single-particle quantum mechanics. However, for ɔhoton with mass, the probability density from the Klein–Gordon equation is tive definite, because it is possible to use the de Broglie wave functions:

$$\begin{aligned} \phi^* &= \phi^{(2)} = \exp(i(\omega t - \kappa Z)) \\ \phi &= \phi^{(1)} = \exp(-i(\omega t - \kappa Z)) \end{aligned} \tag{436}$$

to give

$$\rho = \frac{\hbar\omega}{mc^2} \tag{437}$$

When mass m is the rest mass, the de Broglie theorem states that

$$m_0c^2 = \hbar\omega_0 \tag{438}$$

and $\rho = 1$. For the free photon with mass, the Klein–Gordon equation gives positive definite probability density because the derivative $\partial\phi^{(1)}/\partial t$ is no independent of $\phi^{(2)}$. The equation shows that the free photon with mass can als take on negative energies. Therefore, the vector ϕ in this case can be interprete as a single-particle wave function. The probability 4-vector for the photon witl mass is given by [46]

$$j^\mu = i\frac{\hbar}{2mc}(\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi) \tag{439}$$

which for the de Broglie wave function gives

$$j_Z = -\frac{\hbar\kappa}{mc} \tag{440}$$

The 4-current j^μ is conserved:

$$\partial_\mu j^\mu = i\frac{\hbar}{2mc}(\phi^*\Box\phi - \phi\Box\phi^*) \tag{441}$$

If we define

$$\begin{aligned} \boldsymbol{A} &= A^{(2)}\boldsymbol{e}^{(1)} + A^{(1)}\boldsymbol{e}^{(2)} + A^{(3)}\boldsymbol{e}^{(3)} \\ &= A_X\boldsymbol{i} + A_Y\boldsymbol{j} + A_Z\boldsymbol{k} \end{aligned} \tag{442}$$

there emerge four Klein–Gordon equations that all give a positive probability density:

$$(\Box + m^2)A^{(i)} = 0; \qquad i = 0, 1, 2, 3 \tag{443}$$

for an O(3) invariant theory. In a U(1) invariant theory, there are only two equations:

$$(\Box + m^2)A^{(i)} = 0; \qquad i = 1, 2 \tag{444}$$

The four Klein–Gordon equations are for the photon regarded as a scalar particle without spin. If the scalar components $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ are regarded as fields and quantized, a many-particle interpretation of the photon emerges, and they are recognized as bosons, which have integral spin. Therefore, in an internal space that is globally invariant under a gauge transform, the four equations (443) give, after field quantization (second quantization), a globally gauge invariant Proca equation

$$(\Box + m^2)A^\mu = 0 \tag{445}$$

where the 4-vector is defined as

$$A^\mu = (A^{(0)}, A^{(1)}, A^{(2)}, A^{(3)}) \tag{446}$$

To an excellent approximation, the four Klein–Gordon equations (443) are d'Alembert equations, which are locally gauge-invariant.

However, there remains the problem of how to obtain a locally gauge-invariant Proca equation. To address this problem rigorously, it is necessary to use a non-Abelian Higgs mechanism applied within gauge theory.

The starting point of our derivation is the globally invariant O(3) Lagrangian of the Higgs mechanism

$$\mathscr{L} = \partial_\mu \boldsymbol{A} \cdot \partial^\mu \boldsymbol{A}^* - m^2 \boldsymbol{A} \cdot \boldsymbol{A}^* - \lambda(\boldsymbol{A} \cdot \boldsymbol{A}^*)^2 \tag{447}$$

where $\boldsymbol{A}$ and $\boldsymbol{A}^*$ are regarded as independent complex vectors in the O(3) internal space of the gauge theory. Application of the Euler–Lagrange equations (204) give the following results:

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial \boldsymbol{A}} &= -m^2 \boldsymbol{A}^* - 2\lambda \boldsymbol{A}^*(\boldsymbol{A} \cdot \boldsymbol{A}^*) \\ \frac{\partial \mathscr{L}}{\partial \boldsymbol{A}^*} &= -m^2 \boldsymbol{A} - 2\lambda \boldsymbol{A}(\boldsymbol{A} \cdot \boldsymbol{A}^*) \end{aligned} \tag{448}$$

Therefore, at the Higgs minimum

$$\boldsymbol{A} \cdot \boldsymbol{A}^* = -\frac{m^2}{2\lambda} \equiv a_0^2 \tag{449}$$

The wave equation obtained from Eqs. (204) and (448) with the Lagrangian (447) is

$$\partial^\mu \partial_\mu \boldsymbol{A} = -m^2 \boldsymbol{A} - 2\lambda \boldsymbol{A}(\boldsymbol{A} \cdot \boldsymbol{A}^*) \tag{450}$$

and, at the Higgs minimum, reduces to

$$\partial^\mu \partial_\mu \boldsymbol{A} = \boldsymbol{0} \tag{451}$$

If we define:

$$\boldsymbol{A} = A^{(2)} \boldsymbol{e}^{(1)} + A^{(1)} \boldsymbol{e}^{(2)} + A^{(3)} \boldsymbol{e}^{(3)} \tag{452}$$

then four globally invariant d'Alembert equations are obtained:

$$\begin{aligned} \Box \boldsymbol{A}^{(1)} &= \boldsymbol{0} \\ \Box \boldsymbol{A}^{(2)} &= \boldsymbol{0} \\ \Box \boldsymbol{A}^{(3)} &= \boldsymbol{0} \\ \Box A^{(0)} &= 0 \end{aligned} \tag{453}$$

The locally invariant Lagrangian obtained from the Lagrangian (447) is

$$\mathscr{L} = D_\mu \boldsymbol{A} \cdot D^\mu \boldsymbol{A}^* - \frac{1}{4} \boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}^{\mu\nu} - m^2 \boldsymbol{A} \cdot \boldsymbol{A}^* - \lambda(\boldsymbol{A} \cdot \boldsymbol{A}^*)^2 \tag{454}$$

where it is understood that

$$\begin{aligned} \boldsymbol{A} &\to \boldsymbol{a}_0 + \boldsymbol{A} \\ \boldsymbol{A}^* &\to \boldsymbol{a}_0^* + \boldsymbol{A}^* \end{aligned} \tag{455}$$

The following Euler–Lagrange equation is used next with the Lagrangian (454):

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{A}_\mu} = \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu \boldsymbol{A}_\mu)} \right) \tag{456}$$

The Lagrangian (454) contains terms such as

$$\begin{aligned} D_\mu \boldsymbol{A} \cdot D^\mu \boldsymbol{A}^* &= (\partial_\mu + \boldsymbol{A}_\mu \times)\boldsymbol{A} \cdot (\partial^\mu - \boldsymbol{A}^\mu \times)\boldsymbol{A}^* \\ &= \partial_\mu \boldsymbol{A} \cdot \partial^\mu \boldsymbol{A}^* + g\boldsymbol{A}_\mu \times \boldsymbol{A} \cdot \partial^\mu \boldsymbol{A}^* - g\partial_\mu \boldsymbol{A} \cdot \boldsymbol{A}^\mu \times \boldsymbol{A}^* \\ &\quad - g^2(\boldsymbol{A}_\mu \times \boldsymbol{A}) \cdot (\boldsymbol{A}^\mu \times \boldsymbol{A}^*) \end{aligned} \tag{457}$$

and a field equation emerges from the analysis by using

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial \boldsymbol{A}_\mu} &= g\boldsymbol{A} \times \partial^\mu \boldsymbol{A}^* - g^2(\boldsymbol{A}^\mu(\boldsymbol{A} \cdot \boldsymbol{A}^*) - \boldsymbol{A}^*(\boldsymbol{A} \cdot \boldsymbol{A}^\mu)) \\ &= -g\partial^\mu \boldsymbol{A}^* \times \boldsymbol{A} + g^2(\boldsymbol{A}^\mu \times \boldsymbol{A}^*) \times \boldsymbol{A} \\ &= -gD^\mu \boldsymbol{A}^* \times \boldsymbol{A} \end{aligned} \tag{458}$$

giving

$$D_\nu \boldsymbol{G}^{\mu\nu} = -gD^\mu \boldsymbol{A}^* \times \boldsymbol{A} \tag{459}$$

At the Higgs minimum, this field equation reduces to the locally gauge-invariant Proca equation

$$D_\nu \boldsymbol{G}^{\mu\nu} = -g^2 \boldsymbol{a}_0 \times (\boldsymbol{A}^\mu \times \boldsymbol{a}_0^*) \tag{460}$$

and the Lagrangian reduces to

$$\mathscr{L} = -\frac{1}{4}\boldsymbol{G}_{\mu\nu}\boldsymbol{\cdot}\boldsymbol{G}^{\mu\nu} - g^2(\boldsymbol{A}_\mu \times \boldsymbol{a}_0)\boldsymbol{\cdot}(\boldsymbol{A}^\mu \times \boldsymbol{a}_0^*) \tag{461}$$

Therefore, it can be seen that the mass of the photon in this analysis is derived from the Higgs vacuum, which is the minimum of the potential energy term in the Lagrangian (454). The field equation (460) is O(3) invariant and, therefore, the existence of photon mass is made compatible with the existence of the $\boldsymbol{B}^{(3)}$ field, as inferred originally by Evans and Vigier [42]. The Higgs mechanism is the basis of much of modern elementary particle theory; thus this derivation is based on rigorous gauge theory that is locally O(3) invariant.

VII. THE LEHNERT CHARGE CURRENT DENSITIES IN O(3) ELECTRODYNAMICS

We have established that, in O(3) electrodynamics, the vacuum charge current densities first proposed by Lehnert [42,45,49] take the form

$$\boldsymbol{J}_\mu(\text{vac}) = g\partial_\mu \boldsymbol{A} \times \boldsymbol{A} + g^2 \boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_\mu) \tag{462}$$

In this section, we illustrate the self-consistent calculation of these charge current densities in the plane-wave approximation, using plane waves in the X, Y, and Z directions. In general, the solution of the field equation (459) must be found numerically, and it is emphasized that the plane-wave approximation is a first approximation only. In the internal space, there is the real vector:

$$\boldsymbol{A} = A_X \boldsymbol{i} + A_Y \boldsymbol{j} + A_Z \boldsymbol{k} \tag{463}$$

and by definition

$$\boldsymbol{A}_\mu = A_{\mu,X}\boldsymbol{i} + A_{\mu,Y}\boldsymbol{j} + A_{\mu,Z}\boldsymbol{k} \tag{464}$$

First, we consider a plane-wave propagating in the Z direction, so that

$$\boldsymbol{A} = -\frac{A^{(0)}}{\sqrt{2}}\sin\phi\,\boldsymbol{i} + \frac{A^{(0)}}{\sqrt{2}}\cos\phi\,\boldsymbol{j} + A_Z\boldsymbol{k} \qquad (46$$

and adapt the following notation:

$$\boldsymbol{A}_3 = A_{3,Z}\boldsymbol{k} = -A_Z\boldsymbol{k} \qquad (46$$

Elementary vector algebra then gives

$$g^2\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{A}_3) = \kappa^2(-A_X\boldsymbol{i} - A_Y\boldsymbol{j} + A_Z\boldsymbol{k}) \qquad (46$$

and

$$g\partial_Z \boldsymbol{A} \times \boldsymbol{A} = \kappa^2 A^{(0)}\left(\boldsymbol{k} + \frac{1}{\sqrt{2}}\sin\phi\,\boldsymbol{i} - \frac{1}{\sqrt{2}}\cos\phi\,\boldsymbol{j}\right) \qquad (46$$

The $\boldsymbol{i}$ and $\boldsymbol{j}$ terms must cancel, so we obtain the following, self-consistently:

$$A_X = -\frac{A^{(0)}}{\sqrt{2}}\sin\phi; \qquad A_Y = \frac{A^{(0)}}{\sqrt{2}}\cos\phi \qquad (46$$

The Lehnert vacuum current density for a plane wave in the Z direction therefore

$$\boldsymbol{J}_Z = 2\frac{\kappa^2 A^{(0)}}{\mu_0}\boldsymbol{k} \qquad (47$$

If this is used in the third equation of Eq. (83), the B cyclic theorem [47–61] recovered self-consistently as follows. Without considering vacuum polarizatio and magnetization, the third equation of Eqs. (83) reduces to

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0} \qquad (47$$

because $\boldsymbol{B}$ is phaseless and $\boldsymbol{E}$ is zero by definition. This must mean that there is balance of terms on the right-hand side, giving

$$\begin{aligned} \kappa\boldsymbol{B}^{(3)*} &= -ig\boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)} \\ \kappa\boldsymbol{B}^{(3)*} &= -ig\boldsymbol{A}^{(2)} \times \boldsymbol{B}^{(1)} \end{aligned} \qquad (47$$

so that

$$\kappa A^{(0)}\boldsymbol{B}^{(3)*} = -i\kappa\boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)} \qquad (47$$

giving the B cyclic theorem self-consistently:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \tag{474}$$

The Lehnert charge density for a plane wave propagating in the Z direction is obtained similarly as

$$\rho = \frac{2\kappa^2 A^{(0)}}{\mu_0 c} \tag{475}$$

If a plane wave is now considered propagating in the X direction, the vector in the internal space is defined as

$$\boldsymbol{A} = -\frac{A^{(0)}}{\sqrt{2}} \sin\phi\, \boldsymbol{j} + \frac{A^{(0)}}{\sqrt{2}} \cos\phi\, \boldsymbol{k} + A_X \boldsymbol{i} \tag{476}$$

and it can be shown that the Lehnert vacuum current in the X direction is given self-consistently from Eq. (462) by

$$\boldsymbol{J}_X = 2\frac{\kappa^2 A^{(0)}}{\mu_0}\boldsymbol{i} \tag{477}$$

Finally, if we consider a plane wave propagating in the Y direction, the vector in the internal space is given by

$$\boldsymbol{A} = -\frac{A^{(0)}}{\sqrt{2}} \sin\phi\, \boldsymbol{k} + \frac{A^{(0)}}{\sqrt{2}} \cos\phi\, \boldsymbol{i} + A_Y \boldsymbol{j} \tag{478}$$

and the vacuum current density is given by

$$\boldsymbol{J}_Y = -2\frac{\kappa^2 A^{(0)}}{\mu_0}\boldsymbol{j} \tag{479}$$

Therefore, in order to obtain self-consistent results from Eq. (462), it is necessary to consider plane waves in all three directions. This is as far as an analytical approximation will go. In order to obtain solutions from the field equation (459), computational methods are required.

In summary, the Lehnert current densities in the Z, X, and Y directions, respectively, are

$$\boldsymbol{J}_Z = 2\frac{\kappa^2 A^{(0)}}{\mu_0}\boldsymbol{k}; \qquad \boldsymbol{J}_X = 2\frac{\kappa^2 A^{(0)}}{\mu_0}\boldsymbol{i}; \qquad \boldsymbol{J}_Y = -\frac{\kappa^2 A^{(0)}}{\mu_0}\boldsymbol{j} \tag{480}$$

and are accompanied by a vacuum charge density:

$$\rho = 2\frac{\kappa^2 A^{(0)}}{\mu_0} \qquad (4$$

These results are obtained self-consistently from using plane waves in the X and X directions.

VIII. EMPIRICAL TESTING OF O(3) ELECTRODYNAMICS: INTERFEROMETRY AND THE AHARONOV–BOHM EFFECT

In order to form a self-consistent description [44] of interferometry and Aharonov–Bohm effect, the non-Abelian Stokes theorem is required. It necessary, therefore, to provide a brief description of the non-Abelian Sto theorem because it generalizes the ordinary Stokes theorem, and is based on following relation between covariant derivatives for any internal gauge gr symmetry:

$$\oint D_\mu\, dx^\mu = -\frac{1}{2}\int [D_\mu, D_\nu]\, d\sigma^{\mu\nu} \qquad (48$$

This expression can be expanded in general notation [46] as

$$\oint (\partial_\mu - igA_\mu)\, dx^\mu = -\frac{1}{2}\int [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]\, d\sigma^{\mu\nu} \qquad (48$$

where g is a coupling constant, and A_μ is the potential for any gauge gro symmetry [44]. The coupling constant in the vacuum is

$$g = \frac{\kappa}{A^{(0)}} \qquad (48$$

as used throughout this review and the review by Evans in Part 1 of this thre volume compilation [44]. The terms

$$\oint \partial_\mu\, dx^\mu = [\partial_\mu, \partial_\nu] = 0 \qquad (48$$

are zero because by symmetry

$$\partial_\nu \partial_\mu = \partial_\mu \partial_\nu \qquad (48$$

so

$$\oint \partial_\mu dx^\mu = -\frac{1}{2}\int [\partial_\mu, \partial_\nu] d\sigma^{\mu\nu} = 0 \tag{487}$$

It can also be shown, as in the earlier part of this review, that

$$[A_\mu, \partial_\nu] = -\partial_\nu A_\mu; \qquad [\partial_\mu, A_\nu] = \partial_\mu A_\nu \tag{488}$$

Therefore a convenient and general form of the non-Abelian Stokes theorem is

$$\oint A_\mu dx^\mu = -\frac{1}{2}\int G_{\mu\nu} d\sigma^{\mu\nu} \tag{489}$$

where the field tensor for any gauge group is

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{490}$$

Equation (489) reduces to the ordinary Stokes theorem when U(1) covariant derivatives are used. First, define the units of the vector potential as

$$A_\mu \equiv (\phi, c\boldsymbol{A}) \tag{491}$$

and the units of the U(1) field tensor as

$$F_{\mu\nu} \equiv \begin{bmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & B_3 & -B_2 \\ -\frac{E_2}{c} & -B_3 & 0 & B_1 \\ -\frac{E_3}{c} & B_2 & -B_1 & 0 \end{bmatrix} \tag{492}$$

Summing over repeated indices gives the time-like part of the U(1) Stokes theorem:

$$\oint \phi\, dt = \frac{1}{2c^2}\left(\int E_X d\sigma^{01} + \int E_Y d\sigma^{02}\right) \tag{493}$$

where the SI units on either side are those of electric field strength multiplied by area. Summing over the space indices gives

$$\oint A_i dx^i = -\frac{1}{2}\int F_{ij} d\sigma^{ij} \tag{494}$$

which can be rewritten in Cartesian coordinates as

$$\oint A_X dX = \int B_X d\sigma^{YZ}$$
$$\oint A_Y dY = \int B_Y d\sigma^{ZX}$$
$$\oint A_Z dZ = \int B_Z d\sigma^{XY}$$

or as the vector relation

$$\oint \boldsymbol{A} \cdot d\boldsymbol{r} = \int \boldsymbol{B} \cdot d\boldsymbol{Ar}$$

which is the ordinary Stokes theorem in Maxwell–Heaviside ele
the vacuum, $\boldsymbol{A}$ is a plane wave and is perpendicular to the pro

$$\oint A_Z dZ = 0; \qquad \nabla \times A_Z \boldsymbol{k} = \boldsymbol{0}$$

which is self-consistent with $A_Z = 0$ for Maxwell–Heaviside el

If electrodynamics is a gauge theory with internal O(3) gaug
try, however, there are internal indices and the vector potentia

$$\boldsymbol{A}_\mu = A_\mu^{(2)}\boldsymbol{e}^{(1)} + A_\mu^{(1)}\boldsymbol{e}^{(2)} + A_\mu^{(3)}\boldsymbol{e}^{(3)}$$

The field tensor is similarly

$$\boldsymbol{G}_{\mu\nu} = G_{\mu\nu}^{(2)}\boldsymbol{e}^{(1)} + G_{\mu\nu}^{(1)}\boldsymbol{e}^{(2)} + G_{\mu\nu}^{(3)}\boldsymbol{e}^{(3)}$$

where

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}$$
$$\cdots$$

In O(3) electrodynamics therefore, Eq. (482) gives a term such

$$\oint A_3^{(3)} dx^3 = -i\frac{g}{2}\left(\int [A_1^{(1)}, A_2^{(2)}] d\sigma^{12} + \int [A_2^{(1)}, A_1^{(2)}] d\sigma^{21}\right.$$

which reduces to

$$\oint A_Z^{(3)} dZ = -ig\int [A_X^{(1)}, A_Y^{(2)}] dAr = \int B_Z^{(3)} dAr$$

Both $\boldsymbol{A}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are longitudinally directed and are nonzero in the vacuum. Both $\boldsymbol{A}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are phaseless, but propagate with the radiation [47–62] and with their (1) and (2) counterparts. The radiated vector potential $\boldsymbol{A}^{(3)}$ does not give rise to a photon on the low-energy scale, because it has no phase with which to construct annihilation and creation operators. On the high-energy scale, there is a superheavy photon [44] present from electroweak theory with an SU(2)× SU(2) symmetry. The existence of such a superheavy photon has been inferred empirically [44]. However, the radiated vector potential $\boldsymbol{A}^{(3)}$ is not zero in O(3) electrodynamics from first principles, which, as shown in this section, are supported empirically with precision.

On the O(3) level, there are time-like relations such as

$$\oint A_0\, dx^0 = -\frac{1}{2}\int \partial_0 A_\nu - \partial_\nu A_0 - ig[A_0, A_\nu]\, d\sigma^{(0\nu)} \tag{503}$$

which define the scalar potential on the O(3) level. The constant $\boldsymbol{A}^{(3)}$ can be expanded in a Fourier series:

$$A_Z = A_Z\left(\frac{\pi^2}{3} - 4\left(\cos\phi - \frac{1}{4}\cos 2\phi + \frac{1}{9}\cos 3\phi + \cdots\right)\right) \tag{504}$$

where α is chosen so that

$$\phi = \omega t - \kappa Z + \alpha \tag{505}$$

is always one radian. So both the scalar and vector potentials in O(3) have internal structure.

The non-Abelian Stokes theorem gives the homogeneous field equation of O(3) electrodynamics, a Jacobi identity in the following integral form:

$$\oint D_\mu\, dx^\mu + \frac{1}{2}\int [D_\mu, D_\nu]\, d\sigma^{\mu\nu} = 0 \tag{506}$$

To prove this, we again use

$$\oint D_\mu\, dx^\mu = -\frac{1}{2}\int [D_\mu, D_\nu]\, d\sigma^{\mu\nu} \tag{507}$$

to obtain the identity

$$\frac{1}{2}\int ([D_\mu, D_\nu] - [D_\mu, D_\nu])\, d\sigma^{\mu\nu} \equiv 0 \tag{508}$$

whose integrand is the identity

$$[D_\mu, D_\nu] - [D_\mu, D_\nu] \equiv 0 \qquad (50$$

From this, we obtain the Jacobi identity

$$\sum_{\sigma,\mu,\nu} [D_\sigma, [D_\mu, D_\nu]] \equiv 0 \qquad (51$$

straightforwardly for all group symmetries, including, of course, O(3). Tl homogeneous field equation in O(3) can be written in differential form as

$$D_\mu \tilde{G}^{\mu\nu} \equiv \mathbf{0}$$

$$\equiv D^\lambda \boldsymbol{G}^{\mu\nu} + D^\mu \boldsymbol{G}^{\nu\lambda} + D^\nu \boldsymbol{G}^{\lambda\mu} \qquad (51$$

and the equivalent in U(1) electrodynamics in the differential form is

$$\partial_\mu \tilde{F}^{\mu\nu} \equiv 0$$

$$\equiv \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} \qquad (51$$

As discussed in the earlier part of this review, Eq. (511) is an identity betwee generators of the Poincaré group, which differs from the Lorentz group becaus the former contains the generator of spacetime translations

$$p = i\partial_\mu \qquad (51$$

a group generator that also obeys the Jacobi identity. So we can write

$$\sum_{\sigma,\mu,\nu} [P_\sigma, [D_\mu, D_\nu]] \equiv 0 \qquad (51$$

which is:

$$D_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \mathbf{0} \qquad (51$$

and it follows that Eq. (515) can be written as

$$\partial_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \mathbf{0}$$

$$\boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \mathbf{0} \qquad (51$$

The homogeneous field equation (515) of O(3) electrodynamics therefore reduces to

$$\begin{aligned} \nabla \times \boldsymbol{E}^{(1)} + \frac{\partial \boldsymbol{B}^{(1)}}{\partial t} &= \mathbf{0} \\ \nabla \times \boldsymbol{E}^{(2)} + \frac{\partial \boldsymbol{B}^{(2)}}{\partial t} &= \mathbf{0} \\ \frac{\partial \boldsymbol{B}^{(3)}}{\partial t} &= \mathbf{0} \end{aligned} \tag{517}$$

Equation (515) can be expanded into the O(3) Gauss and Faraday laws

$$\nabla \cdot \boldsymbol{B}^{(1)*} = ig(\boldsymbol{A}^{(2)} \cdot \boldsymbol{B}^{(3)} - \boldsymbol{B}^{(2)} \cdot \boldsymbol{A}^{(3)}) = 0 \tag{518}$$

$$\cdots$$

$$\nabla \times \boldsymbol{E}^{(1)*} + \frac{\partial \boldsymbol{B}^{(1)*}}{\partial t} = -ig(cA_0^{(3)}\boldsymbol{B}^{(2)} - cA_0^{(2)}\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(2)})$$

$$\cdots \tag{519}$$

which are homomorphic with the SU(2) invariant Gauss and Faraday laws given by Barrett [50]:

$$\nabla \cdot \boldsymbol{B} = -iq(\boldsymbol{A} \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{A}) \tag{520}$$

$$\nabla \times \boldsymbol{B} + \frac{\partial \boldsymbol{B}}{\partial t} = -iq([A_0, \boldsymbol{B}] + \boldsymbol{A} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{A}) \tag{521}$$

The vacuum O(3) and SU(2) field equations, on the other hand, are more complicated in structure and highly nonlinear. The O(3) inhomogeneous field equation is given in Eq. (323) and must be solved numerically under all conditions.

These field equations are therefore the result of a non-Abelian Stokes theorem that can also be used to compute the electromagnetic phase in O(3) electrodynamics. It turns out that all interferometric and physical optical effects are described self-consistently on the O(3) level, but not on the U(1) level, a result of major importance. This result means that the O(3) (or SO(3) = SU(2)/Z2) field equations must be accepted as the fundamental equations of electrodynamics.

If we define

$$A^{(0)} \equiv |A_Z^{(3)}|; \qquad g = \frac{\kappa}{A^{(0)}} \tag{522}$$

then an equation is obtained for optics and interferometry:

$$\oint dZ = \kappa \int dAr \tag{523}$$

which relates the line integral on the left-hand side to the area integral. Multiplying both sides of Eq. (523) by $\boldsymbol{\kappa}$ gives a relation between the dynamical phase and topological phase on the right-hand side [44]:

$$\kappa \oint dZ = \kappa^2 \int dAr \tag{524}$$

Application of an O(3) gauge transform to Eq. (502) results in

$$\begin{aligned} A_Z^{(3)} &\rightarrow A_Z^{(3)} + \frac{1}{g}\partial_Z \Lambda^{(3)} \\ B_Z^{(3)} &\rightarrow SB_Z^{(3)}S^{-1} \end{aligned} \tag{525}$$

So after gauge transformation

$$\oint \left(A_Z^{(3)} + \frac{1}{g}\partial_Z \Lambda^{(3)}\right) dZ = \int SB_Z^{(3)}S^{-1}\, dAr \tag{526}$$

and if $A_Z^{(3)}$ is initially zero (vacuum without the Higgs mechanism), the gauge transform produces the nonzero result:

$$\oint \partial_Z \Lambda^{(3)} dZ = \Delta\Lambda^{(3)} = g \int SB_Z^{(3)}S^{-1} dAr \tag{527}$$

which is the Aharonov–Bohm effect, developed in more detail later.

The time-like part of the gauge transform gives the frequency shift [44]:

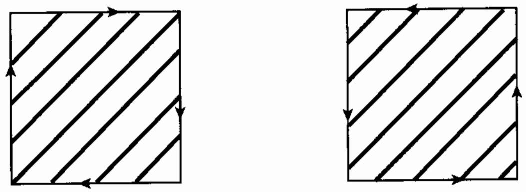
$$\omega \rightarrow \omega + \frac{\partial \Lambda^{(3)}}{\partial t} \equiv \omega + \Omega \tag{528}$$

The left-hand side of Eq. (523) denotes a round trip or closed loop in Minkowski spacetime [46]. On the U(1) level, this is zero in the vacuum because the line integral

$$\oint dZ = \kappa \int dAr \tag{529}$$

reduces in U(1) to a line integral of the ordinary Stokes theorem and is zero. In O(3) electrodynamics, Eq. (529) is a line integral over a closed path with O(3) covariant derivatives and is nonzero.

In the Sagnac effect, for example, the closed loop and area can be illustrated as follows:



There is no Sagnac effect in U(1) electrodynamics, as just argued, a result that is obviously contrary to observation [44]. In O(3) electrodynamics, the Sagnac effect with platform at rest is given by the phase factor [44]

$$\exp\left(i\oint_{A-C}\boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r}\right)=\exp(i\kappa^2 Ar) \tag{530}$$

because on the O(3) level, there is a component $\boldsymbol{\kappa}^{(3)}$ that is directed in the path $\boldsymbol{r}$. The phase factor (530) gives the interferogram

$$\gamma=\cos\left(2\frac{\omega^2}{c^2}Ar\pm 2\pi n\right) \tag{531}$$

as observed. The Sagnac effect with platform in motion is a rotation in the internal gauge space given by Eq. (528), which, when substituted into Eq. (530), gives the observed Sagnac effect to high accuracy:

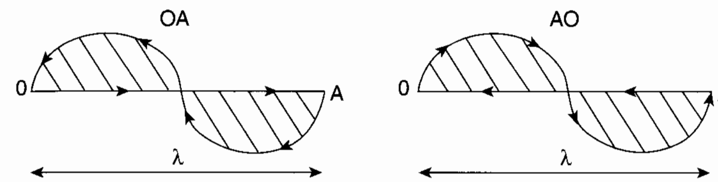
$$\Delta\gamma=\cos\left(4\frac{\omega\Omega Ar}{c^2}\pm 2\pi n\right) \tag{532}$$

The Sagnac effect is therefore due to a gauge transformation and a closed loop in Minkowski spacetime with O(3) covariant derivatives.

If we attempt the same exercise in U(1) electrodynamics, the closed loop gives the Maxwell–Heaviside equations in the vacuum, which are invariant under T and that therefore cannot describe the Sagnac effect [44] because one loop of the Sagnac interferometer is obtained from the other loop by T symmetry. The U(1) phase factor is $\omega t-\kappa Z+\alpha$, where α is arbitrary [44], and this phase factor is also T-invariant. The Maxwell–Heaviside equations in the vacuum are

also invariant under rotation, and are metric-invariant, so cannot describe t Sagnac effect with platform in motion.

Physical optics, and interferometry in general, are described by the pha equation of O(3) electrodynamics, Eq. (524). The round trip or closed loop Minkowski spacetime is illustrated as follows:

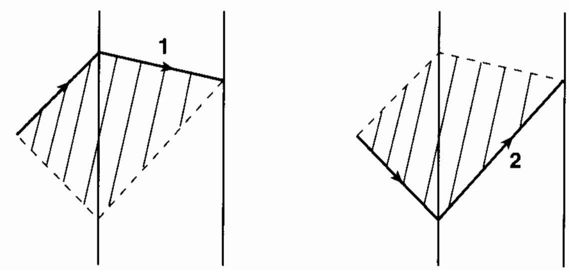


over one wavelength λ of radiation. If $k=\kappa/A^{(0)}$, the area is shown straig forwardly to be

$$Ar=\frac{\lambda^2}{\pi} \tag{53}$$

and if g is proportional to $\kappa/A^{(0)}$, the area is proportional to λ^2/π. Only the Z a contributes to the left hand side of Eq. (524), which correctly describes physical optical and interferometric effects. The closed loop is zero in U(electrodynamics because the line integral in Eq. (524) is zero from the ordina Stokes theorem. Therefore Maxwell–Heaviside electrodynamics cannot descri optics and interferometry. The root cause of this failure is that the phase random on the U(1) level.

The description of Young interferometry for electromagnetism is obtain immediately through the fact that the change in phase difference over traje tories 1 and 2 illustrated below



is given by

$$\Delta\delta=\frac{\kappa}{A^{(0)}}\oint_{2-1}\boldsymbol{A}^{(3)}\cdot d\boldsymbol{r}=\kappa\Delta r \tag{53}$$

where $A^{(0)} = |\boldsymbol{A}^{(3)}|$, and where $\boldsymbol{A}^{(3)}$ is directed along the path $\boldsymbol{r}$ in the vacuum. Equation (536) gives the correct result for Young interferometry for vacuum electromagnetism:

$$\Delta\delta = \kappa\Delta r = \frac{2\pi}{\lambda}\Delta r \tag{535}$$

The change in phase difference of the Young experiment is related through the non-Abelian Stokes theorem to the topological

$$\Delta\delta = g\int \boldsymbol{B}^{(3)} dAr \tag{536}$$

which is an integral over the $\boldsymbol{B}^{(3)}$ field of O(3) electrodynamics. The Young interferometer can therefore be regarded as a round trip in Minkowski spacetime with O(3) covariant derivatives, as can any type of interferometry or physical optical effect. If an attempt is made to describe the Young interferometer as a round trip with U(1) covariant derivatives, the change in phase difference (534) vanishes because the vector potential in U(1) electrodynamics is a transverse plane wave and is always perpendicular to the path. So on the U(1) level

$$\Delta\delta = 0 \tag{537}$$

and there is no Young interferometry, contrary to observation. The same result occurs in Michelson interferometry and therefore in ordinary reflection [44].

The O(3) description of the Aharonov–Bohm effect relies on developing the static magnetic field of a solenoid placed between the two apertures of the Young experiment as follows

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{538}$$

where

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(\boldsymbol{ii} + \boldsymbol{j})e^{i\omega t} \tag{539}$$

are nonpropagating and transverse. On the O(3) level, the following gauge transformations occur:

$$\begin{aligned} A_{\mu}^{(1)} &\to A_{\mu}^{(1)} + \frac{1}{g}\partial_{\mu}\Lambda^{(1)} \\ A_{\mu}^{(2)} &\to A_{\mu}^{(2)} + \frac{1}{g}\partial_{\mu}\Lambda^{(2)} \end{aligned} \tag{540}$$

This means that on O(3) gauge transformation

$$\boldsymbol{A}^{(1)} \to \boldsymbol{A}^{(1)} + \boldsymbol{A}^{(1)'}$$
$$\boldsymbol{A}^{(2)} \to \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(2)'}$$

In regions outside the solenoid, the static magnetic field is represented

$$S\boldsymbol{B}^{(3)}S^{-1} = -ig\boldsymbol{A}^{(1)'} \times \boldsymbol{A}^{(2)'}$$

and is not zero. The Aharonov–Bohm effect is therefore described by

$$\Delta\delta = \frac{e}{\hbar}\int S\boldsymbol{B}^{(3)}S^{-1}\cdot d\boldsymbol{S}$$

as observed [46]. On the U(1) level, the static magnetic field is represer

$$\boldsymbol{B} = \nabla \times A$$

but in regions outside the solenoid

$$\boldsymbol{B} = \nabla \times \left(\frac{1}{g}\nabla\Lambda\right) = \boldsymbol{0}$$

and the magnetic field is zero. So there is no Aharonov–Bohm effect on tl level because $\boldsymbol{B}^{(3)}$ is zero in the integral (543). This has also been pointed Barrett [50] with an O(3) invariant electrodynamics.

Therefore, in this section, several effects have been demonstrated to scribable accurately by O(3) electrodynamic and to have no explanation ; Maxwell–Heaviside electrodynamics. It is safe to infer, therefore, thε electrodynamics must replace U(1) electrodynamics if progress is to be

IX. THE DEBATE PAPERS

There has been an unusual amount of debate concerning the development (electrodynamics, over a period of 7 years. When the $\boldsymbol{B}^{(3)}$ field was first pr [48], it was not realized that it was part of an O(3) electrodynamics homom with Barrett's SU(2) invariant electrodynamics [50] and therefore had ; basis in gauge theory. The first debate published [70,79] was between Barr Evans. The former proposed that $\boldsymbol{B}^{(3)}$ violates C and CPT symmetry. T correct assertion was adequately answered by Evans at the time, but it : clear that if $\boldsymbol{B}^{(3)}$ violated C and CPT, so would classical gauge theory, a red to absurdity. For example, Barrett's SU(2) invariant theory [50] would vic and CPT. The CPT theorem applies only on the quantum level, something that Barron did not seem to realize.

In chronological order, the next critical papers to appear were by Lakhtakia [71] and Grimes [72]. Both papers are obscure, and were adequately answered by Evans [73]. Neither critical paper realized that the $\boldsymbol{B}^{(3)}$ field is part of a classical gauge theory homomorphic with the SU(2) invariant theory by Barrett, published earlier in a volume edited by Lakhtakia [50] himself. This fact reflects the depth of Lakhtakia's confusion. Critical papers were published next by Buckingham and Parlett [74] and by Buckingham [75], essentially duplicating Barron's argument. If these papers were correct, then classical gauge theory would violate CPT and T, a reduction to absurdity. This has been pointed out by Evans [42] and by Evans and Crowell [76]. The next critical paper to appear was by Lakhtakia [77], answered by Evans [78]. Lakhtakia had already published Barrett's SU(2) invariant theory [50] 2 years earlier, so his critical paper is invalidated by the fact that the SU(2) and O(3) invariant theories discussed, for example, in the preceding section, are homomorphic. Then appeared a paper by Rikken [79] answered by Evans [80]. The former claimed erroneously that $\boldsymbol{B}^{(3)}$ is a nonradiated static magnetic field and set about finding it experimentally on this basis. His estimate was orders of magnitude too big, as pointed out by Evans [42] and in the third volume of Ref. 42. The correct use of $\boldsymbol{B}^{(3)}$ gives the empirically observed inverse Faraday effect [42].

These papers were followed by a letter by van Enk [81], answered by Evans [82]. Although not denying the possibility of a $\boldsymbol{B}^{(3)}$, van Enk made the error of arguing on a U(1) level, because, again, he did not realize that $\boldsymbol{B}^{(3)}$ is part of an O(3) invariant electrodynamics and does not exist on the U(1) level. All critical papers cited to this point argued on the U(1) level and are automatically incorrect for this reason. This error was next repeated by Comay [83], who was answered by Evans and Jeffers [84]. Comay attempted to apply the ordinary Abelian Stokes theorem to $\boldsymbol{B}^{(3)}$ and is automatically incorrect because the non-Abelian Stokes theorem should have been applied. The Lorentz covariance of the B cyclic theorem was next challenged by Comay [85], and answered by Evans [86]. The B cyclic theorem is the basic definition of $\boldsymbol{B}^{(3)}$ in an O(3) invariant gauge theory, which is therefore automatically Lorentz covariant, as are all gauge theories for all gauge group symmetries. Comay [87] then challenged the ability of $\boldsymbol{B}^{(3)}$ theory to describe dipole radiation and was answered by Evans [42,88]. It is clear that an O(3) or SU(2) invariant electrodynamics can produce multipole radiation of many types. These comments by Comay are therefore trivially incorrect, not least because they argue again on the U(1) level.

Two papers by Raja et al. [89,90] erroneously claimed once more that $\boldsymbol{B}^{(3)}$ is a static magnetic field and should have produced Faraday induction vacuo. These papers were answered by Evans [91,92]. In the O(3) invariant electrodynamics defining $\boldsymbol{B}^{(3)}$, the latter is a radiated, phaseless, field, and does not produce Faraday induction.

Independent confirmation of the invariance of the B cyclic theorem was next produced by Dvoeglazov [93], but he did not argue on the O(3) level as required. His argument is therefore only partially valid, but produces the correct result.

Comay [94] then repeated the earlier arguments [69,74] on C and CPT violation and was answered by Evans and Crowell [76], who showed that all gauge theories trivially conserve CPT and C on the quantum level. Comay again made the error of arguing on the U(1) and classical levels, whereas $\boldsymbol{B}^{(3)}$ exists only on the O(3) level and the CPT theorem exists only on the quantum level. The argument by Comay using the Stokes theorem [83] was next duplicated by Hunter [95], who again argued erroneously on the U(1) level. The reply to Hunter [96] pointed this out. Next in chronological order, Hunter again duplicated Comay's argument [97] and was again replied to by Evans [98], on the correct O(3) level. Additionally, Comay and Dvoeglazov [99,100] have argued erroneously on the U(1) level concerning the Lorentz covariance of the B cyclic theorem, something that follows trivially from the O(3) gauge invariance of the gauge theory that defines $\boldsymbol{B}^{(3)}$.

The preceding section, and a review in Part 1 of this compilation, supply copious empirical evidences of the fact that the $\boldsymbol{B}^{(3)}$ field is part of the topological phase that describes interferometry through a non-Abelian Stokes theorem. Therefore, the early critical papers are erroneous because they argue on a U(1) level.

X. THE PHASE FACTOR FOR O(3) ELECTRODYNAMICS

The phase factor in classical electrodynamics is the starting point for quantization in terms of creation and annihilation operators, and so it is important to establish its properties on the classical O(3) level. In this context, Barrett [50] has provided a useful review of the development of the phase factor, and Simon [101] has shown that the phase factor is in general due to parallel transport in the presence of a gauge field. On the O(3) level, therefore, the phase factor must be due to parallel transport around a closed loop in Minkowski spacetime (a holonomy) with O(3) covariant derivatives and is governed by the non-Abelian Stokes theorem, Eq. (482). This inference means that all phases in O(3) electrodynamics have their origin in topology on the classical level. This inference is another step in the evolution of understanding of topological phase effects. As pointed out by Barrett [50], the origin of such effects was the development of the Dirac phase factor by Wu and Yang [102], who argued that the wave function of a system will be multiplied by a path-dependent phase factor after its transport around a closed curve in the presence of a potential in ordinary space. This process is now understood to be the origin of the non-Abelian Stokes theorem (482) and to explain the Aharonov–Bohm effect. The phases proposed by Berry [103], Aharonov and Anandan [104], and Pancharatnam [105] are due to a closed loop in parameter or momentum space. These effects occur both on the classical and quantum levels [50].

Originally, Berry [103] proposed a geometric phase for a nondegenerate quantum state that varied adiabatically over a closed loop in parameter space. This occurred in addition to the dynamical phase. It was shown later [50] that the effect is present without the need for an adiabatic approximation, and is also present for degenerate states. Aharonov and Anandan [104] showed that the effect is present for any cyclic evolution of a quantum system, and Bhandari and Samuel [106] showed that the effect is closely related to the geometrical phase discovered by Pancharatnam [105]. The topological phase, therefore, has its origin in topology, either on the classical or quantum level, and is equivalent to a gauge potential in the parameter space of the system on the classical or quantum level.

There are at least three variations of topological phases [50]:

1. A phase arising from cycling in the direction of a beam of light
2. The Pancharatnam phase from cycling of polarization states while keeping the direction of the beam of light constant, a phase change due to polarization change
3. The phase change due to a cycle of changes in squeezed states of light

If the topological phase is denoted Φ, then it obeys the conservation law

$$\Phi(C) = -g\oint \boldsymbol{A}\cdot d\boldsymbol{r} \tag{546}$$

and occurs on the classical level from polarization changes due to changes in the topological path of a light beam. The angle of rotation of linearly polarized light is a direct measure of the topological phase at the classical level. An example of this is the Sagnac effect, which can be explained using O(3) as discussed already. The Sagnac effect can be considered as one loop in the Tomita–Chiao effect [107], which is the rotation of the plane of polarization of a light beam when propagating through an optical fiber.

The next level in the evolution of understanding of the electromagnetic phase is to consider that all optical phases are derived from the non-Abelian Stokes theorem (482), so all optical phases originate in the phase factor

$$\gamma = \exp\left(ig\oint D_\mu\, dx^\mu\right) = \exp\left(-\frac{1}{2}ig\int [D_\mu, D_\nu]\, d\sigma^{\mu\nu}\right) \tag{547}$$

which originates directly in the non-Abelian Stokes theorem (482). Therefore, o the O(3) level, all optical phases are topological in origin. We have brief discussed how the phase factor reduces to a line integral over the dynamic phase and this property of Eq. (547) is also reviewed in Part 1 by Evans [44]. has been argued that the most general equation (547) reduces to

$$\gamma = \exp\left(ig\oint A_\mu\, dx^\mu\right) = \exp\left(-i\frac{g}{2}\int G_{\mu\nu}\, d\sigma^{\mu\nu}\right) \qquad (54$$

for a round trip in Minkowski spacetime for all internal gauge group symmetrie The notation used in Eq. (548) is the condensed notation used by Ryder [46], which the field tensor is in general defined by

$$G_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu - ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \qquad (54$$

In free space, as argued already, the factor g is $\kappa/A^{(0)}$.

If we attempt to apply Eq. (548) on the U(1) level, relations such as

$$\gamma = \exp\left(ig\oint \boldsymbol{A}\cdot d\boldsymbol{r}\right) = \exp\left(ig\int \boldsymbol{B}\cdot d\boldsymbol{Ar}\right) \qquad (55$$

are obtained. In free space, on the U(1) level, $\boldsymbol{A}$ is, however, a plane wave, and therefore always perpendicular to the path $\boldsymbol{r}$ of the radiation. Therefore, on th U(1) level in free space

$$\oint \boldsymbol{A}\cdot d\boldsymbol{r} = \int \boldsymbol{B}\cdot d\boldsymbol{Ar} = 0 \qquad (55$$

On the O(3) level in free space, however, relations such as

$$\gamma = \exp\left(ig\oint \boldsymbol{A}^{(3)}\cdot d\boldsymbol{r}\right) = \exp\left(ig\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{Ar}\right) \qquad (55$$

are obtained, where $\boldsymbol{A}^{(3)}$ is parallel to the path of the radiation. Using $g = \kappa/A^{($ in free space, Eq. (552) reduces to

$$\gamma = \exp\left(i\oint \kappa^{(3)}\cdot d\boldsymbol{r}\right) = \exp\left(ig\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{Ar}\right) \qquad (55$$

and the left-hand side can be recognized as a line integral over what is usuall termed the *dynamical phase*. By definition, the line integral changes sign o traversing a closed loop from O to A to A to O, and this fundament mathematical property is responsible for all optics and interferometry as argued in this review and in Ref. 44. This inference is an evolution in understanding of the phase in optics and electrodynamics.

The $\boldsymbol{B}^{(3)}$ field appearing on the right-hand side of the non-Abelian Stokes theorem (553) changes sign [47–62] between left- and right-handed circularly polarized states, and a linearly polarized state is a superposition of two circularly polarized states. This inference gives rise to Pancharatnam's phase, which is due to polarization changes and also to the phase caused by the cycling of the tip of the vector in a circularly polarized electromagnetic field. Therefore, we reach the important conclusion that the $\boldsymbol{B}^{(3)}$ field is an observable of the phase in all optics and electrodynamics. It has been argued briefly in this review and in Part 1 of this series [44] that the $\boldsymbol{B}^{(3)}$ field provides an explanation of the Sagnac effect.

The U(1) phase factor in the received view, on the other hand, is well known to be

$$\gamma = \exp(i(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} + \alpha)) \tag{554}$$

where α is an arbitrary number. So the phase factor (γ) is defined only up to an arbitrary α, an unphysical result. If $\alpha = 0$ for the sake of argument, the phase factor (γ) is invariant under motion reversal symmetry (T) and parity inversion symmetry (P) [44]. Since one loop of the Sagnac effect is generated from the other by T, it follows that the received phase factor (γ) is invariant in the Sagnac effect with platform at rest and there is no phase shift, contrary to observation [44]. The phase factor (553), on the other hand, changes sign under T and produces the observed Sagnac effect. The phase factor (554) is invariant under P and cannot explain Michelson interferometry or normal reflection [44]. The phase factor (553) changes sign under P and explains Michelson interferometry as observed [44]. We have argued earlier in this review that the phase factor (553) also explains Young interferometry straightforwardly.

Therefore, the distinction between the topological and dynamical phase has vanished, and the realization has been reached that the phase in optics and electrodynamics is a line integral, related to an area integral over $\boldsymbol{B}^{(3)}$ by a non-Abelian Stokes theorem, Eq. (553), applied with O(3) symmetry-covariant derivatives. It is essential to understand that a non-Abelian Stokes theorem must be applied, as in Eq. (553), and not the ordinary Stokes theorem. We have also argued, earlier, how the non-Abelian Stokes explains the Aharonov-Bohm effect without difficulty.

We also infer that, in the vacuum, there exists the topological charge

$$g_m = \frac{1}{V}\oint A_\mu \, dx^\mu \tag{555}$$

where V is a volume, and for one photon, the quantum of electromagnetic energy, the phase becomes

$$\phi = g\oint \boldsymbol{A}^{(3)}\cdot d\boldsymbol{r} = g\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{Ar} = \pm 1 \tag{556}$$

where $g = \kappa/A^{(0)}$. The flux due to one photon is classically

$$\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{Ar} = \frac{A^{(0)}}{\kappa} = \frac{\hbar}{e} \tag{557}$$

and so we have the quantum classical equivalence

$$eA^{(0)} = \hbar\kappa \tag{558}$$

which is a Planck quantization. In quantum theory, the magnetic flux of one photon is $\pm\hbar/e$, depending on the sense of circular polarization.

It can be shown that the Sagnac effect with platform at rest is the rotation of the plane of linearly polarized light as a result of radiation propagating around a circle in free space. Such an effect cannot exist in the received view where the phase factor in such a round trip is always the same and given by Eq. (554). However, it can be shown as follows that there develops a rotation in the plane of polarization when the phase is defined by Eq. (553). It is now known that the phase must always be defined by Eq. (553). Therefore, proceeding on this inference, we construct plane polarized light as the sum of left and right circularly polarized components:

$$\mathrm{Re}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi} = \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j} \tag{559}$$

$$\mathrm{Re}(\boldsymbol{i} - i\boldsymbol{j})e^{-i\phi} = \cos\phi\,\boldsymbol{i} - \sin\phi\,\boldsymbol{j} \tag{560}$$

where the phase factor $e^{i\phi}$ is given by Eq. (553). Plane-polarized light at the beginning of the 180° round trip of the Sagnac effect is therefore

$$(\boldsymbol{i} - i\boldsymbol{j})(e^{i\phi} + e^{-i\phi}) = 2\boldsymbol{i}\cos\phi \tag{561}$$

The round trip of the Sagnac effect in a given—say, clockwise—direction produces the effect

$$(\boldsymbol{i} - i\boldsymbol{j})e^{i(\phi+\phi_S)} + (\boldsymbol{i} - i\boldsymbol{j})e^{-i(\phi-\phi_S)} \tag{562}$$

where

$$\phi_S = g\oint \boldsymbol{A}^{(3)}\cdot d\boldsymbol{r} = g\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{Ar} \tag{563}$$

is generated by the round trip over 2π radians. The extra phase factor for the left circularly polarized component is ϕ_s, and the extra phase factor for the right circularly polarized component is $-\phi_S$ because $\boldsymbol{B}^{(3)}$ changes sign between senses of circular polarization. The effect of the round trip in the Sagnac effect on the plane of linearly polarized light is therefore

$$(\cos(\phi + \phi_S) + \cos(\phi - \phi_S))\boldsymbol{i} + (\sin(\phi + \phi_S) + \sin(\phi - \phi_S))\boldsymbol{j} \tag{564}$$

Using the angle formulas

$$\begin{aligned} \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \end{aligned} \tag{565}$$

the effect can be expressed as

$$2\cos\phi(\boldsymbol{i}\cos\phi_S - \boldsymbol{j}\sin\phi_S) \tag{566}$$

The original plane-polarized light at the beginning of the round trip is described by

$$2\cos\phi\boldsymbol{i} \tag{567}$$

so the overall effect is to rotate the plane of polarized light. Therefore, a linearly polarized laser beam sent around an optical fiber in a circle arrives back at the origin with its plane rotated as in Eq. (566). This is a description of the Sagnac effect with the platform at rest. Spinning the platform produces an extra phase effect with the platform at rest. Spinning the platform produces an extra phase shift that is described [44] by a gauge transformation of $\boldsymbol{A}^{(3)}$ [a rotation in the physical O(3) internal space]. This extra phase shift produces an extra rotation in the plane of polarization of linearly polarized light.

Therefore, it becomes clear that the Sagnac effect is one loop of the Tomita–Chiao effect [107], which is the rotation of the plane of a linearly polarized light beam sent through a helical optical fiber. In both the Sagnac and Tomita–Chiao effects, the angle of rotation (or phase shift) is a direct measure of the phase factor (γ), whose origin is in topology. A circle can always be drawn out into a helix of given pitch (p), length(s), and radius (r). This can be seen by straightening out the helix into a line, and bending the line into a circle. So the Tomita–Chiao effect must reduce to the Sagnac effect for this reason. The former effect can be expressed in general as

$$\phi = 2\pi\left(1 - \frac{p}{S}\right) g \int \boldsymbol{B}^{(3)} \cdot d\boldsymbol{Ar} \tag{568}$$

be(

$$g \int \boldsymbol{B}^{(3)} \cdot d\boldsymbol{Ar} = \pm 1$$

Therefore, the Tomita–Chiao effect reduces to the Sagnac effect u condition

$$2\pi\left(1 - \frac{p}{S}\right) = 1$$

that is

$$\frac{p}{S} = 1 - \frac{1}{2\pi}$$

or when the pitch : length ratio of the helix is this number, which consistently less than one (the length s is always greater than the pitch

The received view, in which the phase factor of optics and electrodyn given by Eq. (554), can describe neither the Sagnac nor the Tomita–C fects, which, as we have argued, are the same effects, differing only by try. Both are non-Abelian, and both depend on a round trip in Mi spacetime using O(3) covariant derivatives.

Having argued thus far, it becomes clear that the phase factor (553 generalized and put on a rigorous footing in topology [50]. It is precise ned from a set of angles associated with a group element, and only c angle can correspond to a holonomy transformation of a vector bundle a closed curve on a sphere. For example, in a SU(2) invariant electrody there is a single angle from the holonomy of the Riemannian connecti sphere. Thus, we infer that gauge structure appears at a very fundamen in all optical effects that depend on the electrodynamical phase. We infer new effects, for example, if the helix of the Tomita–Chiao exper spun, an effect equivalent to the Sagnac effect should be observable. T eral conclusion is that all electrodynamical phases are non-Abelian, ar tization proceeds naturally on this basis. For example, Berry's phase inferred in quantum mechanics. We can conclude that all phases are top

The properties of the phase factor (548) on O(3) gauge transformati been shown [47] to explain the Sagnac effect with platform in motion. densed notation, gauge transformation produces the results

$$\begin{aligned} A'_\mu &= S A_\mu S^{-1} - \frac{i}{g}(\partial_\mu S) S^{-1} \\ G'_{\mu\nu} &= S G_{\mu\nu} S^{-1} \end{aligned}$$

where S is defined by

$$S = \exp(i\mathrm{M}^a \Lambda^a(x^\mu)) \tag{573}$$

In the O(3) gauge group, M^a are rotation generators, and Λ^a are angles in three-dimensional space, which coincides with the internal gauge space. Rotation about the Z axis leaves the $\boldsymbol{B}^{(3)}$ field unaffected. In matrix notation, this can be demonstrated by

$$\begin{bmatrix} 0 & -B_Z & 0 \\ B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -B_Z & 0 \\ B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{574}$$

The gauge transformation of A_Z has been shown [44] to be given by

$$A_Z \to A_Z + \frac{1}{g}\partial_Z \alpha \tag{575}$$

Therefore, the phase factor on O(3) gauge transformation becomes

$$\exp\left(ig \oint (\boldsymbol{A}^{(3)} + \nabla\alpha) \cdot d\boldsymbol{r}\right) = \exp\left(ig \int \boldsymbol{B}^{(3)} \cdot d\boldsymbol{Ar}\right) \tag{576}$$

and using the property

$$\oint \nabla\alpha \cdot d\boldsymbol{r} = 0, \qquad \text{i.e.,} \quad \nabla \times (\nabla\alpha) = \boldsymbol{0} \tag{577}$$

it is seen that the phase factor is invariant under an O(3) gauge transformation. The phase factor, however, contains only the space part of the complete expression (548). Gauge transformation of the time part gives the result [44]

$$\omega \to \omega \pm \Omega; \qquad \Omega = \frac{\partial\alpha}{\partial t} \tag{578}$$

which explains the Sagnac effect with platform in motion.

On the U(1) level, the ordinary Stokes theorem applies, and this can be written as

$$\oint \boldsymbol{A} \cdot d\boldsymbol{r} = \int \nabla \times \boldsymbol{A} \cdot d\boldsymbol{Ar} \tag{579}$$

which is gauge-invariant because of the property

$$\oint \nabla\chi \cdot d\boldsymbol{r} = 0 \tag{580}$$

which is equivalent to the fundamental vector property:

$$\nabla \times (\nabla\chi) = \boldsymbol{0} \tag{581}$$

However, as argued, $\boldsymbol{A}$ is always perpendicular to the path $\boldsymbol{r}$ on the U(1) level, and so the phase factor (548) cannot be applied on this level.

Barrett [50] has interestingly reviewed and compared the properties of the Abelian and non-Abelian Stokes theorems, a review and comparison that makes it clear that the Abelian and non-Abelian Stokes theorems must not be confused [83,95]. The Abelian, or original, Stokes theorem states that if $\boldsymbol{A}(x)$ is a vector field, S is an open, orientable surface, C is the closed curve bounding S, dl is a line element of C, n is the normal to S, and C is traversed in a right-handed (positive direction) relative to n, then the line integral of A is equal to the surface integral over S of $\nabla \times \boldsymbol{A} \cdot \boldsymbol{n}$:

$$\oint \boldsymbol{A} \cdot d\boldsymbol{l} = \int_S (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}\, da \tag{582}$$

and, as pointed out by Barrett [50], the original Stokes theorem just described takes no account of boundary conditions.

In the non-Abelian Stokes theorem (482), on the other hand, the boundary conditions are defined because the phase factor is path-dependent, that is, depends on the covariant derivative [50]. On the U(1) level [50], the original Stokes theorem is a mathematical relation between a vector field and its curl. In O(3) or SU(2) invariant electromagnetism, the non-Abelian Stokes theorem gives the phase change due to a rotation in the internal space. This phase change appears as the integrals

$$\oint \boldsymbol{A}^{(3)} \cdot d\boldsymbol{r} = \int \boldsymbol{B}^{(3)} \cdot d\boldsymbol{Ar} \tag{583}$$

which do not exist in Maxwell–Heaviside electromagnetism. There is a profound ontological difference therefore between the original Stokes theorem, in which $\boldsymbol{B}^{(3)}$ is zero, and the non-Abelian Stokes theorem, in which $\boldsymbol{B}^{(3)}$ is nonzero and of key importance. Therefore progress from a U(1) to an O(3) or SU(2) invariant electromagnetism is a striking evolution in understanding, as argued throughout Ref. 44 and references cited therein and in several reviews of this volume.

Equation (482) is a simple form of the non-Abelian Stokes theorem, a form that is derived by a round trip in Minkowski spacetime [46]. It has been adapted directly for the O(3) invariant phase factor as in Eq. (547), which gives a simple and accurate description of the Sagnac effect [44]. A U(1) invariant electrodynamics has failed to describe the Sagnac effect for nearly 90 years, and kinematic explanations are also unsatisfactory [50]. In an O(3) or SU(2) invariant electrodynamics, the Sagnac effect is simply a round trip in Minkowski spacetime and an effect of special relativity and gauge theory, the most successful theory of the late twentieth century. There are open questions in special relativity [108], but no theory has yet evolved to replace it.

By using the O(3) invariant phase factor (547), we have also removed the distinction between the topological phase and the dynamical phase, reaching, as argued earlier, a new level of understanding in all optical effects that depend on electromagnetic phase.

For example, the description of the Aharonov–Bohm effect and other types of interferometry become closely similar. The Young interferometer, for example, is described by

$$\frac{\kappa}{A^{(0)}}\oint_{2-1} \boldsymbol{A}^{(3)}\cdot d\boldsymbol{r} = \frac{\kappa}{A^{(0)}}\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{S} \tag{584}$$

and the Aharonov–Bohm effect can be described by

$$\frac{e}{\hbar}\oint_{2-1} A^{(3)}\cdot d\boldsymbol{r} = \frac{e}{\hbar}\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{S} = \frac{e}{\hbar}\Phi^{(3)} \tag{585}$$

In both cases, the magnetic flux

$$\Phi^{(3)} = \int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{S} \tag{586}$$

is generated by the round trip in Minkowski space with O(3) covariant derivatives (holonomy) on the left-hand side of Eqs. (584) and (585). So the original magnetic field inside the solenoid does not contribute to the Aharonov–Bohm effect, as pointed out by Barrett [50], and the U(1) invariant description [46] of the effect is erroneous. The effect is due to the magnetic field $\boldsymbol{B}^{(3)}$ of O(3) electrodynamics. The Sagnac, Michelson, and Mach–Zehnder effects, and all interferometric effects are similarly described by Eq. (584), and all interferometry and optics originate in topology. The only difference between these effects and the Aharonov–Bohm effect is that in the latter, interaction with electrons takes place, so the factor $\kappa/A^{(0)}$ is replaced by $e/\hbar$ in a minimal prescription.

The interpretation of Eq. (584) is that the potential $\boldsymbol{A}^{(3)}$ is defined along the integration path of the line integral. The field $\boldsymbol{B}^{(3)}$ is defined as being perpendicular to the plane or surface enclosed by the line integral. Neither $\boldsymbol{A}^{(3)}$ nor $\boldsymbol{B}^{(3)}$ exists in a U(1) invariant electrodynamics. Effects attributed to the topological phase, such as those of Pancharatnam and Tomita and Chiao, reviewed already, do not exist in a U(1) invariant electrodynamics, but are described by Eq. (584) in an O(3) invariant theory. Equation (3) is for circularly polarized radiation propagating in a plane, and so allowance may have to be made for the geometry of a particular experiment. We have illustrated this with the Tomita–Chiao effect. The key to this evolution in understanding is that there exists in an O(3) invariant electrodynamics, an internal gauge space with index (3). The existence of this index gives rise to the non-Abelian Stokes theorem (584). The internal space on a ((1),(2),(3)) level is considered to be the physical space of three dimensions and not an isospace. Therefore, a rotation in the internal space ((1),(2),(3)) is a physical rotation in three-dimensional space. The spinning platform of the Sagnac effect is an example of one such rotation, about the axis perpendicular to the platform, and results in Eq. (578), which, as shown elsewhere [44], gives the observed Sagnac effect, again through Eq. (584). Such concepts are available in neither a U(1) invariant electrodynamics nor gauge theory, which considers the internal space as an isospace.

Therefore, it has been shown convincingly that electrodynamics is an O(3) invariant theory, and so the O(3) gauge invariance must also be found in experiments with matter waves, such as matter waves from electrons, in which there is no electromagnetic potential. One such experiment is the Sagnac effect with electrons, which was reviewed in Ref. 44, and another is Young interferometry with electron waves. For both experiments, Eq. (584) becomes

$$\oint \boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r} = \kappa^2 Ar \tag{587}$$

and for matter waves

$$\omega^2 = c^2\kappa^2 + \frac{m_0^2 c^4}{\hbar^2} \tag{588}$$

where m_0 is the mass of the particle. The Sagnac effect in electrons [44] is therefore the same as the Sagnac effect in photons, and is given [44] by

$$\begin{aligned}\Delta\phi &= \frac{Ar}{c^2}\left(((\omega+\Omega)^2 - (\omega-\Omega)^2) - \frac{m_0^2c^4}{\hbar^2} + \frac{m_0^2c^4}{\hbar^2}\right)\\ &= \frac{4\omega\Omega Ar}{c^2}\end{aligned} \tag{589}$$

from the gauge transform (578). This is the observed result [44]. The Young effect for electrons is similarly

$$\Delta\phi = \oint_{2-1} \boldsymbol{\kappa}^{(3)}\cdot d\boldsymbol{r} \tag{590}$$

and also more generally for particles such as atoms and molecules, the famous two-slit experiment.

On this empirical evidence, it is possible to reach a far-reaching conclusion that all wave functions in quantum mechanics are of the form (590). For example, the electron wave function from the Dirac equation is

$$\text{Positive energy: } \psi^{(\alpha)}(r) = u^{(\alpha)}(p)\exp\left(-i\oint \boldsymbol{p}\cdot d\boldsymbol{r}\right) \tag{591}$$

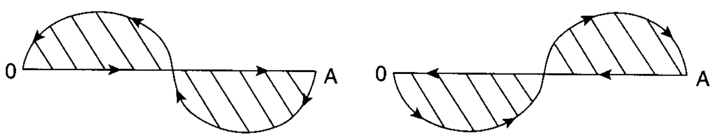
$$\text{Negative energy: } \psi^{(\alpha)}(r) = v^{(\alpha)}(p)\exp\left(i\oint \boldsymbol{p}\cdot d\boldsymbol{r}\right) \tag{592}$$

instead of the conventional [46]

$$\psi^{(\alpha)}(r) = u^{(\alpha)}(p)\exp(-i\boldsymbol{p}\cdot\boldsymbol{r}) \tag{593}$$

$$\psi^{(\alpha)}(r) = v^{(\alpha)}(p)\exp(i\boldsymbol{p}\cdot\boldsymbol{r}) \tag{594}$$

The path and area in Eq. (584) and in wave functions such as those of the photon and electron are given by the following sketch:



The shaded area in this sketch is not arbitrary, as it is determined by the right-hand side of Eq. (587). The line integrals *OA* and *AO* change sign, and this accounts for reflection of matter waves and for the Sagnac and Young effects in matter waves, such as electron waves. Therefore, the electron is an O(3) invariant entity, as shown by the Sagnac effect for electron waves [44]. It follows that the Dirac equation should be developed as an O(3) invariant equation.

The Fermat principle can now be reworked into an O(3) invariant form and the principles of quantum mechanics on a nonrelativistic level developed from it. In so doing, we modify the discussion by Atkins [68] for an O(3) invariant treatment. Fermat's principle of least time is the basic law governing light propagation in geometric optics. In the received view, light travels in a straight line in geometric optics, but the physical nature of light is a wave motion. These two fundamental aspects are unified in the sketch following Eq. (594), constructed in an O(3) invariant theory, in which the phase now correctly describes both the wave nature of light and the fact that it travels in a straight line in the vacuum or a uniform medium. The U(1) invariant phase shows only the latter property of light, and consequently is a number invariant under motion reversal symmetry

(*T*) and parity inversion symmetry (*P*). Similarly, particles travel in a straight line by Newton's first law, but de Broglie demonstrated that particles have a wave nature–wave particle duality. Therefore, the phase in classical electrodynamics becomes the wave function in quantum mechanics, and the general and important conclusion is reached that both the electromagnetic phase and the wave function of particles are O(3) invariant. We have already argued that this new general principle is supported by the Sagnac and Young effects in matter waves. In retrospect, it is not surprising that the wave function should reflect wave–particle duality, for both the photon and matter waves.

A simple example of the Fermat principle may be used to show the weakness inherent in a U(1) invariant phase. Fermat's principle states that the path taken by a light ray through a medium is such that its time of passage is a minimum. Following Atkins [68], consider the relation between angles of incidence and reflection. The least-time path is the one corresponding to the angle of incidence being equal to the angle of reflection, giving Snell's law. However, reflection is a parity inversion, under which the U(1) invariant phase

$$P(\omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}) = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r} \tag{595}$$

does not change [44]. This is seen at its clearest in normal reflection. Therefore, the U(1) invariant phase cannot describe normal reflection and Snell's law, and violates Fermat's principle. The O(3) invariant phase

$$\phi = \oint \omega\, dt - \oint \boldsymbol{\kappa}\cdot d\boldsymbol{r} \tag{596}$$

on the other hand, changes sign on reflection, because of the property of the path integral

$$P\left(\oint \boldsymbol{\kappa}\cdot d\boldsymbol{r}\right) = -\oint \boldsymbol{\kappa}\cdot d\boldsymbol{r} \tag{597}$$

and so is in accordance with the Fermat principle. This conclusion is a major evolution in understanding because Fermat's principle is at the root of quantum mechanics, in particular, the time-dependent Schrödinger equation.

Following Atkins [68], the propagation of particles follows a path dictated by Newton's laws, equivalent to Hamilton's principle, that particles select paths between two points such that the action associated with the path is a minimum. Therefore, Fermat's principle for light propagation is Hamilton's principle for particles. The formal definition of action is an integral identical in structure with the phase length in physical optics. Therefore, particles are associated with wave motion, the wave–particle dualism. Hamilton's principle of least action leads directly to quantum mechanics. The final touch to this development was made by de Broglie. Therefore, a particle is also described by an amplitude $\psi(r)$, and amplitudes at different points are related by an expression of the following form [68]:

$$\psi(P_2) = e^{i(r_2-r_1)\cdot\kappa}\psi(P_1) \tag{598}$$

If this is to be O(3) invariant, the phase in quantum mechanics must take the form (597). In the classical limit, the particle propagates along a path that makes the action S a minimum. Therefore, the O(3) invariant phase is proportional to S through the Planck constant. It is concluded that the O(3) invariant phase in quantum mechanics is given by

$$\phi = \oint \boldsymbol{\kappa}\cdot d\boldsymbol{r} \tag{599}$$

The amplitude describing a particle in O(3) invariant quantum mechanics is

$$\psi = \psi_0 \exp^{-i\phi} = \psi_0 \exp^{-i(S/\hbar)} \tag{600}$$

where S is the action associated with the path from P_1 (a point at x_1, t_1) to P_2 (a point at x_2, t_2). An equation of motion can be developed from this form by differentiating with respect to time t_2:

$$\frac{\partial}{\partial t}\psi(x,t) = -\frac{i}{\hbar}En\,\psi(x,t) \tag{601}$$

The rate of change of the action is equal to $-En$, where En is the total energy $T + V$:

$$\frac{\partial S}{\partial t} = -En \tag{602}$$

Therefore, the equation of motion is

$$\frac{\partial}{\partial t}\psi(x,t) = \frac{i}{\hbar}\frac{\partial S}{\partial t}\psi(x,t) \tag{603}$$

and if En is interpreted as the Hamiltonian operator H, the O(3) invariant time-dependent Schrödinger equation is obtained:

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{604}$$

So, if the O(3) invariant wave function is defined as

$$\psi = \psi_0 \exp\left(-i\oint \omega\,dt - \oint \boldsymbol{\kappa}\cdot d\boldsymbol{r}\right)$$

where

$$S = -\hbar\left(\oint \omega\,dt - \oint \boldsymbol{\kappa}\cdot d\boldsymbol{r}\right)$$

the energy is given by

$$En = \hbar\omega = -\frac{\partial S}{\partial t}$$

which is the energy for one photon. Equation (605) is the O(3) inva Broglie wave function.

XI. O(3) INVARIANCE: A LINK BETWEEN ELECTROMAGNETISM AND GENERAL RELATIVITY

In order to develop a Riemannian theory of classical electromagnetis necessary [109] to consider a curve corresponding to a plane wave:

$$f(Z) = (\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}$$

In terms of the retarded time $[t] = t - Z/c$, the U(1) phase ϕ is $\omega[t]$, retarded distance is $Z - Z_0 = c[t]$. The electromagnetic wave propagat the Z axis, and the trajectory of the real part is

$$f_R(Z) = \mathrm{Re}(f(Z)) = (\cos\phi, \sin\phi, \phi)$$

which is a circular helix. The curve (609) is a function of Z with Z_0 regar constant in partial differentiation of $f(Z)$ with respect to Z. More gen Z-dependent phase angle must be incorporated in ϕ, which becomes [4

$$f_R(Z) = (\cos(\kappa(Z - Z_0) + \Phi), \sin(\kappa(Z - Z_0) + \Phi), \kappa(Z - Z_0) + \Phi)$$

Frenet's tangent vector (T) is obtained by differentiation:

$$\frac{\partial f_R(Z)}{\partial Z} = \kappa T = (-\kappa\sin\phi, \kappa\cos\phi, \kappa)$$

In elementary differential geometry, therefore, the electromagnetic helix produces a nonzero T, and tangent vectors are characteristic of curved spacetime in general relativity. The scalar curvature in elementary differential geometry is

$$R = \left|\frac{\partial^2 f_R(Z)}{\partial Z^2}\right| = \left|\kappa^2(\cos\phi, -\sin\phi, 0)\right| = \kappa^2 \tag{612}$$

and this is also the scalar curvature of the electromagnetic wave in general relativity, specifically, the scalar curvature of Riemann's tensor, obtained by suitable antisymmetric index contraction. The electromagnetic field therefore becomes a property of spacetime, or the vacuum.

The metric coefficient in the theory of gravitation [110] is locally diagonal, but in order to develop a metric for vacuum electromagnetism, the antisymmetry of the field must be considered. The electromagnetic field tensor on the U(1) level is an angular momentum tensor in four dimensions, made up of rotation and boost generators of the Poincaré group. An ordinary axial vector in three-dimensional space can always be expressed as the sum of cross-products of unit vectors

$$\boldsymbol{I} = \boldsymbol{i} \times \boldsymbol{j} + \boldsymbol{j} \times \boldsymbol{k} + \boldsymbol{k} \times \boldsymbol{i} \tag{613}$$

a sum that can be expressed as the metric

$$g = g_{\mu\nu}^{(A)} i^\mu j^\nu \tag{614}$$

where the $g_{\mu\nu}^{(A)}$ coefficient in three dimensions is the fully antisymmetric 3×3 matrix. This becomes the right-hand side in four dimensions. In the language of differential geometry, the field tensor becomes the Faraday 2-form [110]

$$F = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \tag{615}$$

where the wedge product $dx^\alpha \wedge dx^\beta$ between differential forms is an exterior product. Equation (615) translates in tensor notation into

$$F = F_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \tag{616}$$

We have argued here and elsewhere [44] that the plane-wave representation of classical electromagnetism is far from complete. In tensor language, this incompleteness means that the antisymmetric electromagnetic field tensor on the O(3) level must be proportional to an antisymmetric frame tensor of spacetime, $R_{\mu\nu}^{(A)}$, derived from the Riemannian tensor by contraction on two indices:

$$R_{\mu\nu}^{(A)} = R^{\lambda}_{\lambda\mu\nu} \tag{617}$$

Therefore $R_{\mu\nu}^{(A)}$ is an antisymmetric Ricci tensor obtained from the index contraction from the Riemann curvature tensor. Further contraction of $R_{\mu\nu}^{(A)}$ leads to the scalar curvature R, which, for electromagnetism, is κ^2. The contraction must be

$$R = \frac{1}{12} g_{\mu\nu}^{(A)} R^{\mu\nu(A)} \tag{618}$$

The principle of equivalence between electromagnetism and the antisymmetric Ricci tensor is

$$R_{\mu\nu}^{(A)} = gG_{\mu\nu} = \frac{\kappa}{A^{(0)}} G_{\mu\nu} \tag{619}$$

whose scalar form is

$$R = gG^{(0)} \tag{620}$$

where $G^{(0)}$ is a scalar field amplitude and where $R = \kappa^2$ is the scalar curvature of vacuum electromagnetism, whose metric coefficient is antisymmetric. In this view, vacuum electromagnetism is the antisymmetric Ricci 2-form [110], and gravitation is the symmetric Ricci 2-form.

Geodesic equations can be developed for the vacuum plane wave from the starting point [110]

$$D\kappa^\mu = \frac{d\kappa^\mu}{d\lambda} + \Gamma^{\mu}_{\nu\sigma}\kappa^\nu\kappa^\sigma = 0 \tag{621}$$

where $\kappa^\mu = dx^\mu/d\lambda$ is the wave 4-vector and $\Gamma^{\mu}_{\nu\sigma}$ is the affine connection. The symbol D in Eq. (621) is therefore a covariant derivative. In the received view, on the U(1) level, Eq. (621) becomes

$$d\kappa^\mu = 0 \tag{622}$$

in which the wave-vector does not vary along its path. Equation (621), on the other hand, has a parameter that varies along the ray, and the world line is a helix. This is a conclusion reminiscent of the fact that the O(3) electromagnetic phase is described by a line integral, as developed in the previous section.

A relation is first established between κ^μ and the A^μ 4-vector:

$$\kappa^\mu = \frac{\kappa}{A^{(0)}} A^\mu \tag{523}$$

Using this equation in Eq. (621) gives

$$\frac{dA^\mu}{d\lambda} + \frac{\kappa}{A^{(0)}}\Gamma^{\mu}_{\nu\sigma}A^\nu A^\sigma = \frac{dA^\mu}{d\lambda} + \frac{\kappa}{A^{(0)}}A^2\Gamma^\mu = 0 \tag{624}$$

where A is a scalar. The contracted affine connection Γ^{μ} is proportional to A^{μ} in general gauge theory, and we adopt this rule to give

$$\Gamma^{\mu} = \frac{\kappa}{A^{(0)}} A^{\mu} \tag{625}$$

which is an equivalence principle between field and frame (or vacuum) properties. Such an equivalence does not appear on the U(1) level if the ordinary derivative replaces the covariant derivative.

Equation (625) can be written as

$$\frac{dA^{\mu}}{d\lambda} + \kappa^2 A^{\mu} = 0 \tag{626}$$

where the dimensionality of λ is κ^2, the inverse of the Thomson area of a photon [42], and if $\lambda = Z^2/2$, Eq. (626) become

$$\frac{d^2A^{\mu}}{dZ^2} + RA^{\mu} = 0 \tag{627}$$

This has the form of a geodesic equation [111], and is obeyed by a plane wave. Similarly, we obtain:

$$\frac{1}{c^2}\frac{d^2A^{\mu}}{dt^2} + RA^{\mu} = 0 \tag{628}$$

an equation that is also obeyed by a plane wave. Now, subtract Eq. (627) from Eq. (628) to give the d'Alembert wave equation:

$$\Box A^{\mu} = (R - R)A^{\mu} = 0 \tag{629}$$

which is the Proca equation

$$\Box A^{\mu} = -\frac{m_0^2 c^4}{\hbar^2} A^{\mu} = 0 \tag{630}$$

whose right-hand side happens to be zero because we have used a plane wave to derive it. The Proca equation (629) is an equation of a spacetime or vacuum whose curvature is $R = \kappa^2$, and not zero.

Equations (627) and (628) are special cases of the usual definition of the Riemann tensor in curvilinear geometry

$$A_{\mu;\nu;\kappa} - A_{\mu;\kappa;\nu} \equiv R^{\lambda}_{\mu\nu\kappa} A_{\lambda} \tag{631}$$

where A_{λ} is a general 4-vector field [111]. Equation (631) can

$$(D_{\nu}D_{\kappa} - D_{\kappa}D_{\nu})A_{\mu} + R^{\lambda}_{\mu\nu\kappa}A_{\lambda} = 0$$

and this is a geodesic equation. Multiply Eq. (632) by the antis coefficient $g^{(A)}_{\mu\nu}$ to obtain

$$g^{\nu\kappa}_{(A)}(D_{\nu}D_{\kappa} - D_{\kappa}D_{\nu}) + g^{\nu\kappa}_{(A)}R^{\lambda}_{\mu\nu\kappa}A_{\lambda} = 0$$

and identify

$$R \equiv g^{\nu\kappa}_{(A)}R^{\mu}_{\mu\nu\kappa}; \qquad \frac{d^2}{dZ^2} = g^{\nu\kappa}_{(A)}(D_{\nu}D_{\kappa} - D_{\kappa}D_{\nu})$$

This procedure reduces Eq. (631) to Eqs. (627) and (628), which obtained by tensor contraction.

Electromagnetism can therefore be defined geometrically i ordinates, and has vacuum properties such as scalar curvature, m affine connection, and Ricci tensor that manifest themselves f level:

$$G_{\mu\nu} = \frac{A^{(0)}}{\kappa} R^{(A)}_{\mu\nu}$$

This equation can be written in precise analogy with the Einste

$$T^{(A)}_{\mu\nu} = \hbar\omega\left(\frac{R^{(A)}_{\mu\nu}}{R}\right)$$

where $T^{(A)}_{\mu\nu}$ is an antisymmetric electromagnetic energy-mome $R = \kappa^2$ is the scalar curvature in O(3) electromagnetism. Eq therefore a rotational Einstein equation. The scalar curvature in ism is defined through the antisymmetric metric coefficient ($g^{(A}_{\mu\nu}$

$$R = \kappa^2 = g^{\mu\nu}_{(A)}R^{(A)}_{\mu\nu}$$

The analogous definition of scalar curvature in gravitation is gi metric $g_{\mu\nu}$:

$$R(\text{grav}) = g^{\mu\nu}R^{(S)}_{\mu\nu}$$

and the symmetric part of the Ricci tensor $R_{\mu\nu}^{(s)}$, that is, through the equation

$$R_{\lambda\nu}^{(S)} = R_{\lambda\kappa\nu}^{\kappa} \tag{639}$$

If O(3) electromagnetism [denoted e.m. in Eq. (640)] and gravitation are both to be seen as phenomena of curved spacetime, then both fields are derived ultimately from the same Riemann curvature tensor as follows:

$$T_{\mu\nu}^{(A)}(\text{e.m.}) = \hbar\omega \frac{R_{\mu\nu}^{(A)}}{R} \tag{640}$$

$$T_{\mu\nu}^{(S)}(\text{grav.}) = \frac{c^4}{8\pi k}\left(R_{\mu\nu}^{(S)} - \frac{1}{2}g_{\mu\nu}R\right) \tag{641}$$

$$R_{\mu\nu} = R_{\mu\nu}^{(S)} + R_{\mu\nu}^{(A)} \tag{642}$$

The unification of O(3) electromagnetism and gravitation using these concepts is summarized in Table I.

TABLE I
Some Concepts in the Unified Theory of Fields

Concept of Quantity	Gravitation	Electromagnetism
Riemann tensor	$R_{\lambda\mu\nu}^{\kappa}$	$R_{\lambda\mu\nu}^{\kappa}$
Ricci tensor	$R_{\mu\nu}^{(S)} = R_{\lambda\alpha\nu}^{\alpha}$	$R_{\mu\nu}^{(A)} = R_{\alpha\mu\nu}^{\alpha}$
Metric coefficient	$g_{\mu\nu}$(diagonal)	$g_{\mu\nu}^{(A)}$(off-diagonal)
Scalar curvature	$R = g^{\mu\nu}R_{\mu\nu}^{(S)}$	$R = g^{\mu\nu(A)}R_{\mu\nu}^{(A)} = \kappa^2$
Einstein tensor	$R_{\mu\nu}^{(S)} - \frac{1}{2}g_{\mu\nu}R \equiv G_{\mu\nu}^{(E)}$	$R_{\mu\nu}^{(A)}$
Field equation	$G_{\mu\nu}^{(E)} = \frac{8\pi k}{c^4}T_{\mu\nu}^{(S)}$	$R_{\mu\nu}^{(A)} = \frac{\kappa^2}{\hbar\omega}T_{\mu\nu}^{(A)}$
Connection	$\Gamma_{\mu\nu}^{\lambda}$	$\Gamma_{\mu\nu}^{\lambda} = \frac{\kappa}{A^{(0)}}M_\mu M_\nu A^\lambda$
Local group	Poincaré	Poincaré
Group generator	Bianchi identity	Feynman Jacobi
Identity	$D_\rho R_{\lambda\mu\nu}^{\kappa} + D_\mu R_{\lambda\nu\rho}^{\kappa} + D_\nu R_{\lambda\rho\mu}^{\kappa} = 0$	identity($\kappa = \lambda$)
Energy-momentum tensor	$T_{\mu\nu}^{(S)}$(translational)	$T_{\mu\nu}^{(A)} \equiv \omega J_{\mu\nu}$ $= \frac{\hbar\omega}{R}R_{\mu\nu}^{(A)}$(rotational)
Equivalence principle	Gravitation is a noninertial frame	Electromagnetism is a noninertial frame
Universal constant	k (Einstein's constant)	$\frac{\kappa}{A^{(0)}}$

The electromagnetic field equations on the O(3) level can be obtained fro this purely geometrical theory by using Eq. (631) in the Bianchi identity

$$\begin{aligned} D_\kappa G_{\mu\nu} + D_\nu G_{\kappa\mu} + D_\mu G_{\nu\kappa} &= R_{\mu\nu\kappa}^{\lambda}A_\lambda + R_{\nu\kappa\mu}^{\lambda}A_\lambda + R_{\kappa\mu\nu}^{\lambda}A_\lambda \\ &= D_\rho R_{\lambda\mu\nu}^{\kappa} + D_\mu R_{\lambda\nu\rho}^{\kappa} + D_\nu R_{\lambda\rho\mu}^{\kappa} = 0 \qquad (64 \end{aligned}$$

with appropriate index contraction. The end result is the Feynman Jacobi identit discussed in earlier sections of this review

$$D_\mu \tilde{G}^{\mu\nu} \equiv 0 \qquad (644$$

an identity that can be written as

$$D_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \qquad (645$$

The O(3) field equations can be obtained from the fundamental definition c the Riemann curvature tensor, Eq. (631), by defining the O(3) field tensor usin covariant derivatives of the Poincaré group.

Equation (643) is also a Bianchi identity in the theory of gravitation becaus $G_{\mu\nu}$ is derived from the antisymmetric part of the Riemann tensor, whose sym metric part can be contracted to the Einstein tensor.

Similarly, Eq. (643) can be developed into an inhomogeneous equation of th unified field. First, raise indices in the Riemann tensor and field tensor:

$$G^{\nu\kappa} = g^{\nu\rho}g^{\kappa\sigma}G_{\rho\sigma}; \qquad R_{\mu}^{\lambda\nu\kappa} = g^{\nu\rho}g^{\kappa\sigma}R_{\mu\rho\sigma}^{\lambda} \qquad (646$$

From the equivalence of $G_{\mu\nu}$ and $R_{\mu\nu}^{(A)}$ in Eq. (635), individual terms in th identity (643) can be equated:

$$D_\kappa G^{\mu\nu} = R_{\kappa}^{\lambda\mu\nu}A_\lambda \qquad (647a$$

$$D_\nu G^{\kappa\mu} = R_{\nu}^{\lambda\kappa\mu}A_\lambda \qquad (647b$$

$$D_\mu G^{\nu\kappa} = R_{\mu}^{\lambda\nu\kappa}A_\lambda \qquad (647c$$

Consider the antisymmetric part of the Riemann tensor in Eqs. (647) by suitabl contraction. In Eq. (647c), for example, the contraction is $\lambda = \mu$. The resul reduces to the O(3) inhomogeneous field equation of electromagnetism in th form

$$D_\mu G^{\nu\mu} = R_{\sigma}^{\sigma\nu\mu}A_\mu \equiv \frac{J^\nu(\text{vac})}{\varepsilon_0} \qquad (648$$

where the term

$$J^{\nu}(\text{vac}) = \varepsilon_0 R_{\sigma}^{\sigma\nu\mu} A_{\mu} \tag{649}$$

is the O(3) charge current density, which can be seen to exist in the vacuum as argued earlier.

There are well known similarities between the Riemann curvature tensor of general relativity and the field tensor in non-Abelian electrodynamics. The Riemann tensor is

$$R^{\kappa}_{\lambda\mu\nu} = \partial_{\nu}\Gamma^{\kappa}_{\lambda\mu} - \partial_{\mu}\Gamma^{\kappa}_{\lambda\nu} + \Gamma^{\rho}_{\lambda\mu}\Gamma^{\kappa}_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\kappa}_{\rho\mu} \tag{650}$$

and is made up of a Ricci tensor and a Weyl conformal tensor. The following contraction of indices

$$R^{\kappa}_{\kappa\mu\nu} = \partial_{\nu}\Gamma^{\kappa}_{\kappa\mu} - \partial_{\mu}\Gamma^{\kappa}_{\kappa\nu} + \Gamma^{\rho}_{\kappa\mu}\Gamma^{\kappa}_{\rho\nu} - \Gamma^{\rho}_{\kappa\nu}\Gamma^{\kappa}_{\rho\mu} \tag{651}$$

leads to an expression similar to the field tensor as argued. The holonomy [46] in general relativity is

$$\triangle V^{\mu} = \frac{1}{2} R^{\mu}_{\rho\sigma\lambda} V^{\rho} \triangle S^{\sigma\lambda} \tag{652}$$

which can be compared with the holonomy in gauge theory

$$\triangle\psi_A = -ig\triangle S^{\mu\nu} G_{\mu}\psi_A \tag{653}$$

In both cases, the $\triangle S^{\mu\nu}$ factor is a hypersurface. This suggests that the Ricci tensor is in general complex, and given by

$$R_{\mu\nu} = R^{(S)}_{\mu\nu} + iR^{(A)}_{\mu\nu} \tag{654}$$

where the real part is symmetric and the imaginary part is antisymmetric. Barrett [50] has pointed out that O(3) gauge theory is non-Minkowskian in general, and requires an extrapolation of twistor algebra to non-Minkowski spacetime, requiring the presence of a Weyl tensor, complex spacetime, and curved twistor space. In O(3) electrodynamics, therefore, Minkowski spacetime applies only locally, and Minkowski vector spaces are tangent spaces of spacetime events. The Weyl anti-self-dual spacetime is independent of the self-dual spacetime. There is conformally curved, complex spacetime, as reflected in the complex Ricci tensor discussed already. The Weyl tensor is not zero. A complex spacetime [50] is defined by a four-dimensional complex manifold, M, with a holomorphic

metric g_{ab}. A differential function defined on an open set of complex holomorphic [50] if it satisfies the Cauchy–Riemann equations. With ı holomorphic coordinate basis $x^{\mu} = (x^0, x^1, x^2, x^3)$, the metric is a 4 × · holomorphic functions of x^{μ}, and its determinant is nowhere vanishing tensor becomes complex-valued as argued already. Self-consistently checked that the determinant of the metric

$$g_{ab} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

is nonzero, (i.e., -1). So the use of an antisymmetric Ricci tensor from first principles.

XII. BASIC ALGEBRA OF O(3) ELECTRODYNAMI AND TESTS OF SELF-CONSISTENCY

In this section, some elementary details of the complex circular ba generated by ((1),(2),(3)) are given. The basis vectors are

$$\boldsymbol{e}^{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}); \qquad \boldsymbol{i} = \frac{1}{\sqrt{2}}(\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)})$$

$$\boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j}); \qquad \boldsymbol{j} = \frac{1}{\sqrt{2}}(\boldsymbol{e}^{(1)} - \boldsymbol{e}^{(2)})$$

$$\boldsymbol{e}^{(3)} = \boldsymbol{k}$$

Within a phase factor and amplitude, $\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*}$ is the vectorial complex description of right and left circularly polarized radiation. unit vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, and $\boldsymbol{e}^{(3)}$ form the O(3) cyclic permutation relati

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*}$$

$$\boldsymbol{e}^{(2)} \times \boldsymbol{e}^{(3)} = i\boldsymbol{e}^{(1)*}$$

$$\boldsymbol{e}^{(3)} \times \boldsymbol{e}^{(1)} = i\boldsymbol{e}^{(2)*}$$

A closely similar complex circular basis has been described by Silve three-dimensional space. This space forms the internal gauge spac electrodynamics, as argued already. In the complex circular basis, the ı dot product is

$$\boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} = \boldsymbol{e}^{(2)} \cdot \boldsymbol{e}^{(1)} = \boldsymbol{e}^{(3)} \cdot \boldsymbol{e}^{(3)} = 1$$

$$\boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)} \cdot \boldsymbol{e}^{(2)} = 0$$

as compared with the same concept in the Cartesian basis

$$\begin{aligned} i \cdot i &= j \cdot j = k \cdot k = 1 \\ i \cdot j &= i \cdot k = j \cdot k = 0 \end{aligned} \tag{659}$$

Vectors are defined as

$$\begin{aligned} A &\equiv A^{(1)} + A^{(2)} + A^{(3)} \\ &= A^{(2)} e^{(1)} + A^{(1)} e^{(2)} + A^{(3)} e^{(3)} \end{aligned} \tag{660}$$

where

$$\begin{aligned} A^{(1)} &= \frac{1}{\sqrt{2}}(A_X - iA_Y) = A^{(2)*} \\ A^{(3)} &= A_Z \end{aligned} \tag{661}$$

The dot product of two vectors is therefore

$$\begin{aligned} A \cdot B &= A^{(1)} B^{(2)} e^{(1)} \cdot e^{(2)} + A^{(2)} B^{(1)} e^{(2)} \cdot e^{(1)} + A^{(3)} B^{(3)} e^{(3)} \cdot e^{(3)} \\ &= A^{(1)} B^{(2)} + A^{(2)} B^{(1)} + A^{(3)} B^{(3)} \end{aligned} \tag{662}$$

The del operator in the circular basis is defined by

$$\begin{aligned} \nabla_X &= \frac{\partial}{\partial X} = \frac{1}{\sqrt{2}}(\nabla^{(1)} + \nabla^{(2)}); & \nabla^{(1)} &= \frac{1}{\sqrt{2}}(\nabla_X - i\nabla_Y) \\ \nabla_Y &= \frac{\partial}{\partial Y} = \frac{i}{\sqrt{2}}(\nabla^{(1)} - \nabla^{(2)}); & \nabla^{(2)} &= \frac{1}{\sqrt{2}}(\nabla_X + i\nabla_Y) \\ \nabla_Z &= \frac{\partial}{\partial Z} = \nabla^{(3)}; & \nabla^{(3)} &= \nabla_Z \end{aligned} \tag{663}$$

and the divergence of a vector is therefore

$$\nabla \cdot A = \nabla^{(1)} A^{(2)} + \nabla^{(2)} A^{(1)} + \nabla^{(3)} A^{(3)} \tag{664}$$

and the gradient of a scalar is

$$\nabla \phi = \nabla^{(1)} \phi e^{(2)} + \nabla^{(2)} \phi e^{(1)} + \nabla^{(3)} \phi e^{(3)} \tag{665}$$

The curl operator in the complex circular basis is

$$\nabla \times A = -i \begin{vmatrix} e^{(1)} & e^{(2)} & e^{(3)} \\ \nabla^{(1)} & \nabla^{(2)} & \nabla^{(3)} \\ A^{(1)} & A^{(2)} & A^{(3)} \end{vmatrix} = \begin{vmatrix} i & j & k \\ \nabla_X & \nabla_Y & \nabla_Z \\ A_X & A_Y & A_Z \end{vmatrix} \tag{666}$$

and the vector cross-product is

$$A \times B = -i \begin{vmatrix} e^{(1)} & e^{(2)} & e^{(3)} \\ A^{(1)} & A^{(2)} & A^{(3)} \\ B^{(1)} & B^{(2)} & B^{(3)} \end{vmatrix} = \begin{vmatrix} i & j & k \\ A_X & A_Y & A_Z \\ B_X & B_Y & B_Z \end{vmatrix} \tag{667}$$

It is helpful to exemplify the basis by calculating the vector cross-product in detail and comparing it with the Cartesian counterpart. This procedure shows that the ((1),(2),(3)) and Cartesian representations are equivalent when correctly worked out.

The $e^{(3)}$ component can be developed as

$$\begin{aligned} & -ie^{(3)}(A^{(1)}B^{(2)} - A^{(2)}B^{(1)}) \\ &= -i\left(\frac{1}{\sqrt{2}}(A_X + iA_Y)\frac{1}{\sqrt{2}}(B_X - iB_Y) - \frac{1}{\sqrt{2}}(A_X - iA_Y)\frac{1}{\sqrt{2}}(B_X + iB_Y)\right) \\ &= A_X B_Y - A_Y B_X \end{aligned} \tag{668}$$

and is equivalent to the Cartesian component obtained from the well-known expression

$$\begin{vmatrix} i & j & k \\ A_X & A_Y & A_Z \\ B_X & B_Y & B_Z \end{vmatrix} = (A_X B_Y - A_Y B_X)k + \cdots \tag{669}$$

The other two components are evaluated by developing the sum

$$\begin{aligned} A \times B &= -i[e^{(1)}(A^{(2)}B^{(3)} - A^{(3)}B^{(2)}) - e^{(2)}(A^{(1)}B^{(3)} - A^{(3)}B^{(1)})] + \cdots \\ &= -i\left[\frac{1}{\sqrt{2}}(i - ij)\left(\frac{1}{\sqrt{2}}(A_X + iA_Y)B_Z - \frac{A_Z}{\sqrt{2}}(B_X + iB_Y)\right)\right. \\ &\quad \left. - \frac{1}{\sqrt{2}}(i + ij)\left(\frac{1}{\sqrt{2}}(A_X - iA_Y)B_Z - \frac{A_Z}{\sqrt{2}}(B_X - iB_Y)\right)\right] + \cdots \\ &= i(A_Y B_Z - A_Z B_Y) - j(A_X B_Z - A_Z B_X) + \cdots \end{aligned} \tag{670}$$

and again we obtain a result equivalent to the Cartesian sum.

A conjugate product such as $A^{(1)} \times A^{(2)}$ is evaluated as

$$-i \begin{vmatrix} e^{(1)} & e^{(2)} & e^{(3)} \\ A^{(2)} & 0 & 0 \\ 0 & A^{(1)} & 0 \end{vmatrix} = -iA^{(0)2}k \tag{671}$$

and is the same as the Cartesian equivalent:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_X^{(2)} & A_Y^{(2)} & 0 \\ A_X^{(1)} & A_Y^{(1)} & 0 \end{vmatrix} = -iA^{(0)2}\boldsymbol{k} \tag{672}$$

In the logic of the complex circular basis, unity is expressed as the product of two complex conjugates, referred to hereinafter as *complex unity*

$$1^2 = 1^{(1)}1^{(2)} \tag{673}$$

where

$$1^{(1)} = \frac{1}{\sqrt{2}}(1-i); \qquad 1^{(2)} = \frac{1}{\sqrt{2}}(1+i) \tag{674}$$

Therefore, developments such as the following are possible:

$$\begin{aligned} \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} &= 1^{(2)}\boldsymbol{e}^{(1)} \cdot 1^{(1)}\boldsymbol{e}^{(2)} = 1^{(1)}1^{(2)} = 1^2 = 1 \\ \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} &= A^{(2)}\boldsymbol{e}^{(1)} \cdot A^{(1)}\boldsymbol{e}^{(2)} = A^{(1)}A^{(2)} = A^{(0)2} \end{aligned} \tag{675}$$

Since the product $1^{(1)}1^{(2)}$ is always unity, it makes no difference to the dot product of unit vectors or of conjugate vectors such as $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, but the dot product of a vector $\boldsymbol{A}^{(1)}$ and a unit vector $\boldsymbol{e}^{(2)}$ is

$$\begin{aligned} \boldsymbol{A}^{(1)} \cdot \boldsymbol{e}^{(2)} &= A^{(2)}1^{(1)}\boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} = \frac{1}{2}(A_X - iA_Y)(1+i) \\ &= \frac{1}{2}(A_X - iA_Y + iA_X + A_Y) \end{aligned} \tag{676}$$

Similarly [42], the dot product of a complex circular Pauli matrix $\boldsymbol{\sigma}^{(1)}$ and a unit vector $\boldsymbol{e}^{(2)}$ is

$$\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)} = \frac{1}{2}(\sigma_X - i\sigma_Y + i\sigma_X + \sigma_Y) \tag{677}$$

leading to

$$(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{e}^{(2)})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{e}^{(1)}) = \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} + i\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} \tag{678}$$

and the prediction of radiatively induced fermion resonance.

As we have argued, the basis ((1),(2),(3)) defines an internal space in ele trodynamics, and was first applied as such by Barrett [50] in an SU(2) invaria gauge theory. As a consequence of this hypothesis, we can write

$$A^{\mu} = A^{\mu(2)}\boldsymbol{e}^{(1)} + A^{\mu(1)}\boldsymbol{e}^{(2)} + A^{\mu(3)}\boldsymbol{e}^{(3)} \qquad (67$$

so $\boldsymbol{A}^{\mu}$ is developed as a vector in the internal space. The object $A^{\mu(1)}, A^{\mu(2)}$, ar $A^{\mu(3)}$ are scalar coefficients in the internal space. The boldface character $\boldsymbol{A}^{\mu}$ simultaneously a vector in the basis ((1),(2),(3)) and a 4-vector in spacetime. we consider to start with the received view of ordinary plane waves, the boldfa character in this case is a vector of three-dimensional space in the bas ((1),(2),(3)) and so is also a vector in the internal space of O(3) electrodynamic As we have argued, the phase factor $e^{i\phi}$ on the O(3) level is made up of a lir integral, related to an area integral by a non-Abelian Stokes theorem. In order expand the horizon of the gauge structure of electrodynamics to the O(3) leve an additional spacetime index must appear in the definition of the plane wav and the (1) and (2) indices must become indices of the internal space. This achieved by recognizing that

$$\begin{aligned} A^{1(1)} &= A_X^{(1)} = i\frac{A^{(0)}}{\sqrt{2}}e^{-i\phi} = A^{1(2)*} \\ A^{2(1)} &= A_Y^{(1)} = \frac{A^{(0)}}{\sqrt{2}}e^{-i\phi} = A^{2(2)*} \\ A^{0(1)} &= A^{3(1)} = A^{0(2)} = A^{3(2)} = 0 \end{aligned} \qquad (68$$

These equations define two of the scalar coefficients of the complete 4-vector A

$$\begin{aligned} A^{\mu(1)} &= (0, \boldsymbol{A}^{(1)}) \\ A^{\mu(2)} &= (0, \boldsymbol{A}^{(2)}) \end{aligned} \qquad (68$$

a deduction that follows from the fact that $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ are transverse and so ca have X and Y components only. The scalar coefficients $A^{\mu(1)}$ and $A^{\mu(2)}$ are ligh like invariants

$$A^{\mu(1)}A_{\mu}^{(1)} = A^{\mu(2)}A_{\mu}^{(2)} = 0 \qquad (68$$

of polar 4-vectors in spacetime. The third index (3) of the non-Abelian theor must therefore be in the direction of propagation of radiation and must also be light-like invariant

$$A^{\mu(3)}A_{\mu}^{(3)} = 0 \qquad (68$$

in the vacuum.

One possible solution of Eq. (683) is

$$A^{\mu(3)} = (cA^{(0)}, \boldsymbol{A}^{(3)}) \tag{684}$$

where

$$cA^{(0)} = |\boldsymbol{A}^{(3)}| \tag{685}$$

Such a solution is proportional directly to the wave 4-vector

$$\kappa^{\mu(3)} \equiv (c\kappa, \boldsymbol{\kappa} e^{(3)}) = gA^{\mu(3)} \tag{686}$$

and to the photon energy momentum:

$$p^{\mu(3)} = \hbar g A^{\mu(3)} = \hbar \kappa^{\mu(3)} \tag{687}$$

in the vacuum. Therefore, the complete vector in the internal ((1),(2),(3)) space is the light-like polar vector

$$\boldsymbol{A}^{\mu} = (0, \boldsymbol{A}^{(2)})\boldsymbol{e}^{(1)} + (0, \boldsymbol{A}^{(1)})\boldsymbol{e}^{(2)} + (cA^{(0)}, \boldsymbol{A}^{(3)})\boldsymbol{e}^{(3)} \tag{688}$$

and has time-like, longitudinal, and transverse components, which are all physical components in the vacuum. On the U(1) level, the time-like and longitudinal components are combined in an admixture [46].

Similarly, the field tensor on the O(3) level is a vector in the internal space:

$$\boldsymbol{G}^{\mu\nu} = G^{\mu\nu(2)}\boldsymbol{e}^{(1)} + G^{\mu\nu(1)}\boldsymbol{e}^{(2)} + G^{\mu\nu(3)}\boldsymbol{e}^{(3)} \tag{689}$$

and the coefficients $G^{\mu\nu(i)}$ are scalars in the internal space. They are also antisymmetric tensors in spacetime. General gauge field theory for O(3) symmetry then gives

$$\begin{aligned} \boldsymbol{G}^{\mu\nu(1)*} &= \partial^{\mu}\boldsymbol{A}^{\nu(1)*} - \partial^{\nu}\boldsymbol{A}^{\mu(1)*} - ig\boldsymbol{A}^{\mu(2)} \times \boldsymbol{A}^{\nu(3)} \\ \boldsymbol{G}^{\mu\nu(2)*} &= \partial^{\mu}\boldsymbol{A}^{\nu(2)*} - \partial^{\nu}\boldsymbol{A}^{\mu(2)*} - ig\boldsymbol{A}^{\mu(3)} \times \boldsymbol{A}^{\nu(1)} \\ \boldsymbol{G}^{\mu\nu(3)*} &= \partial^{\mu}\boldsymbol{A}^{\nu(3)*} - \partial^{\nu}\boldsymbol{A}^{\mu(3)*} - ig\boldsymbol{A}^{\mu(1)} \times \boldsymbol{A}^{\nu(2)} \end{aligned} \tag{690}$$

which is a relation between vectors in the internal space ((1),(2),(3)). The cross-product notation is also a vector notation; for example, $\boldsymbol{A}^{\mu(2)} \times \boldsymbol{A}^{\nu(3)}$ is a cross-product of a vector $\boldsymbol{A}^{\mu(2)}$ with the vector $A^{\nu(3)}$ in the internal space. In forming the cross-product, the Greek indices are not transmuted and the complex basis is used, so that the terms quadratic in $\boldsymbol{A}$ become natural descriptions of the

empirically observable conjugate product. As we have argued, th coefficient $g = \kappa/A^{(0)}$ is a scalar in both the internal gauge space and sp In field–matter interaction, g changes magnitude [44]. The field tensc O(3) level is therefore a vector in the internal space and is nonline: potential. It contains the longitudinal field $\boldsymbol{B}^{(3)}$ in the vacuum. The field t the U(1) level does not define $\boldsymbol{B}^{(3)}$, which exists only on the O(3) level

Equation (690) is a concise description that contains a considerable of information about the O(3) theory of electromagnetism in the vacuu mation that is available without assuming any form of field equation. portant to give details of the correct algebraic form of reduction of E Consider, for example, the equation

$$\boldsymbol{G}^{\mu\nu(1)*} = \partial^{\mu}\boldsymbol{A}^{\nu(1)*} - \partial^{\nu}\boldsymbol{A}^{\mu(1)*} - ig\boldsymbol{A}^{\mu(2)} \times \boldsymbol{A}^{\nu(3)}$$

which consists of components such as

$$\boldsymbol{G}^{12(1)*} = \partial^1 A^{2(1)*} - \partial^2 A^{1(1)*} - ig\varepsilon_{(1)(2)(3)}A^{1(2)}A^{2(3)}$$

where $\varepsilon_{(1)(2)(3)}$ is the Levi–Civita symbol defined by

$$\varepsilon_{(1)(2)(3)} \equiv 1 = -\varepsilon_{(1)(2)(3)} = \cdots$$

Now take the vector potential as defined already with

$$\partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$$

then we obtain

$$\begin{aligned} G^{12(1)*} &= \partial^1 A^{2(1)*} - \partial^2 A^{1(1)*} - ig(A^{1(2)}A^{2(3)} - A^{1(3)}A^{2(2)}) \\ &= 0 \end{aligned}$$

This is a self-consistent result because there is no Z component of $G^{\mu\nu(1)}$ is defined as transverse. Both the linear and nonlinear components are

Consider next the element:

$$\begin{aligned} G^{13(1)*} &= \partial^1 A^{3(1)*} - \partial^3 A^{1(1)*} - ig\varepsilon_{(1)(2)(3)}A^{1(2)}A^{3(3)} \\ &= \partial^1 A^{3(2)} - \partial^3 A^{1(2)} - ig(A^{1(2)}A^{3(3)} - A^{1(3)}A^{3(2)}) \\ &= -(\partial^3 + igA^{3(3)})A^{1(2)} = -(\partial^3 + i\kappa)A^{1(2)} \end{aligned}$$

where we have used

$$g = \frac{\kappa}{A^{(0)}}; \qquad A^{3(3)} = A_Z^{(3)} = A^{(0)} \tag{697}$$

There are two contributions to the field element $G^{13(2)}$, a magnetic component:

$$-\partial^3 A^{1(2)} \tag{697a}$$

and

$$-igA^{3(3)}A^{1(2)} \tag{697b}$$

In vector notation, Eq. (696) is a component of

$$\begin{aligned} 2\boldsymbol{B}^{(1)} &\equiv \nabla \times \boldsymbol{A}^{(1)} - ig\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \\ &= (\nabla - ig\boldsymbol{A}^{(3)}) \times \boldsymbol{A}^{(1)} \\ &= \nabla \times \boldsymbol{A}^{(1)} - \frac{i}{B^{(0)}}\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} \end{aligned} \tag{698}$$

Furthermore:

$$\partial^3 A^{1(2)} = i\kappa A^{1(2)} \tag{699}$$

and so it follows that

$$\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} \tag{700}$$

Similarly:

$$\boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} \tag{701}$$

Therefore, the definition of the field tensor in O(3) electrodynamics gives the first two components of the B cyclic theorem [47–62]

$$\begin{aligned} \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*} \\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*} \end{aligned} \tag{702}$$

together with the definition of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ in terms of the curl of vector potentials:

$$\begin{aligned} \boldsymbol{B}^{(1)} &= \nabla \times \boldsymbol{A}^{(1)} \\ \boldsymbol{B}^{(2)} &= \nabla \times \boldsymbol{A}^{(2)} \end{aligned} \tag{703}$$

It is convenient to write this result as

$$\boldsymbol{H}(\text{vac}) = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}(\text{vac})$$

where $\boldsymbol{H}$(vac) is the vacuum magnetic field strength and μ_0 is t permeability. The object $\boldsymbol{M}$(vac) does not exist on the U(1) level termed *vacuum magnetization*:

$$\boldsymbol{M}^{(1)}(\text{vac}) = -\frac{1}{i\mu_0 B^{(0)}}\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)}$$

The objects $\boldsymbol{M}^{(1)}$(vac) and $\boldsymbol{M}^{(2)}$(vac) depend on the phaseless vacuu field $\boldsymbol{B}^{(3)}$ and so do not exist as concepts in U(1) electrodynamics. Th itself is defined through

$$G^{\mu\nu(3)*} = \partial^\mu \boldsymbol{A}^{\nu(3)*} - \partial^\nu \boldsymbol{A}^{\mu(3)*} - ig\boldsymbol{A}^{\mu(1)} \times \boldsymbol{A}^{\nu(2)}$$

with (3) aligned in the Z axis. So, by definition, the only nonzero com

$$G^{12(3)*} = -G^{21(3)*} = B_Z^{(3)}$$

It follows that

$$B_Z^{(3)} = -ig(A^{1(1)}A^{2(2)} - A^{1(2)}A^{2(1)})$$

or

$$\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$$

giving the third component of the B cyclic theorem $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{($ the vacuum magnetization:

$$\boldsymbol{M}^{(3)*} = -\frac{1}{i\mu_0 B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$$

On the U(1) level, $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is considered to be an operator [44] o optics with no third axis, but on the O(3) level it defines $\boldsymbol{B}^{(3)}$ as argu

Therefore, on the O(3) level, the magnetic part of the complete f defined as a sum of a curl of a vector potential and a vacuum ma inherent in the structure of the B cyclic theorem. On the U(1) lev no $\boldsymbol{B}^{(3)}$ field by hypothesis.

The following field coefficients can be calculated:

$$\begin{aligned} G^{01(2)} &= (\partial^0 + igA^{0(3)})A^{1(2)} = -G^{10(2)} \\ G^{02(2)} &= (\partial^0 + igA^{0(3)})A^{2(2)} = -G^{20(2)} \\ G^{03(2)} &= 0 \\ G^{13(2)} &= -(\partial^3 + igA^{3(3)})A^{1(2)} = -G^{31(2)} \\ G^{23(2)} &= -(\partial^3 + igA^{3(3)})A^{2(2)} = -G^{32(2)} \\ G^{12(2)} &= 0 \end{aligned} \tag{711}$$

so that

$$\begin{aligned} G^{01(1)} &= G^{01(2)*} = (\partial^0 + igA^{0(3)})A^{1(1)} \\ G^{12(3)*} &= -G^{21(3)*} = -ig(A^{1(1)}A^{2(2)} - A^{1(2)}A^{2(1)}) \end{aligned} \tag{712}$$

The three field tensors are therefore the transverse

$$G^{\mu\nu(1)} = G^{\mu\nu(2)*} = \begin{bmatrix} 0 & -E^{1(1)} & -E^{2(1)} & 0 \\ E^{1(1)} & 0 & 0 & cB^{2(1)} \\ E^{2(1)} & 0 & 0 & -cB^{1(1)} \\ 0 & -cB^{2(1)} & cB^{1(1)} & 0 \end{bmatrix} \tag{713}$$

and the longitudinal:

$$G^{\mu\nu(3)*} = G^{\mu\nu(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -cB^{3(3)} & 0 \\ 0 & cB^{3(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{714}$$

On the O(3) level, there also exists a vacuum polarization, because the complete electric field strength is given in the vacuum by

$$\begin{aligned} 2\boldsymbol{E}^{(2)} &= -\frac{\partial \boldsymbol{A}^{(2)}}{\partial t} - igcA^{(0)}\boldsymbol{A}^{(2)} \\ &= -\left(\frac{\partial}{\partial t} + igcA^{(0)}\right)\boldsymbol{A}^{(2)} = 2\boldsymbol{E}^{(1)*} \end{aligned} \tag{715}$$

Using $g = \kappa/A^{(0)}$, then

$$\boldsymbol{E}^{(2)} = -\frac{\partial \boldsymbol{A}^{(2)}}{\partial t} = -ic\kappa\boldsymbol{A}^{(2)} = -i\omega\boldsymbol{A}^{(2)} \tag{716}$$

and it is convenient to express this result as

$$\boldsymbol{D}^{(2)}(\text{vac}) = \varepsilon_0\boldsymbol{E}^{(2)} + \boldsymbol{P}^{(2)}(\text{vac}) \qquad (717$$

where $\boldsymbol{D}^{(2)}$(vac) is the electric displacement in vacuo, and where the vacuur polarization is

$$\boldsymbol{P}^{(2)}(\text{vac}) = -i\varepsilon_0\omega\boldsymbol{A}^{(2)} \qquad (718$$

The vacuum polarization is well known to have an analog in quantur electrodynamics [46], the photon self-energy. The latter has no classical analo on the U(1) level, but one exists on the O(3) level, thus saving the correspon dence principle. The classical vacuum polarization on the O(3) level is trans verse and vanishes when $\omega = 0$. It is pure transverse because, as follows, th hypothetical $\boldsymbol{E}^{(3)}$ field is zero on the O(3) level

$$\begin{aligned} G^{03(3)*} &= \partial^0 A^{3(3)*} - \partial^3 A^{0(3)*} - ig(A^{0(1)}A^{3(2)} - A^{3(2)}A^{0(1)}) \\ &= 0 \end{aligned} \qquad (719$$

giving

$$G^{03(1)} = G^{03(2)} = G^{03(3)} = 0 \qquad (720$$

in the vacuum. In the presence of field–matter interaction, this result is no longe true because of the Coulomb field, indicating polarization of matter.

In the presence of field–matter interaction [44]

$$\boldsymbol{H}^{\mu\nu(i)*} = \varepsilon_0\boldsymbol{F}^{\mu\nu(i)*} - \boldsymbol{M}^{\mu\nu(i)*} \qquad (721$$

where $i = 1, 2, 3$. Here

$$\begin{aligned} \boldsymbol{F}^{\mu\nu(i)*} &\equiv \partial^\mu\boldsymbol{A}^{\nu(i)} - \partial^\nu\boldsymbol{A}^{\mu(i)} \\ \boldsymbol{M}^{\mu\nu(1)} &\equiv i\varepsilon_0 g'\boldsymbol{A}^{\mu(2)} \times \boldsymbol{A}^{\nu(3)} \end{aligned} \qquad (722$$

n cyclic permutation, with $g' \ll g$ empirically [44].

There are therefore obvious points of similarity between the O(3) theory o lectrodynamics and the Yang–Mills theory [44]. Both are based, as we have rgued, on an O(3) or SU(2) invariant Lagrangian. However, in O(3) electrody amics, the particle concomitant with the field has the topological charge $/A^{(0)}$. In O(3) electrodynamics, the internal space and spacetime are not inde endent spaces but form an extended Lie algebra [42]. In elementary particle theory, the internal space is usually an abstract isospin space [46]. The overall structures of O(3) electrodynamics and of Yang–Mills theory are the same.

XIII. QUANTIZATION FROM THE B CYCLIC THEOREM

The B cyclic theorem is a Lorentz invariant construct in the vacuum and is a relation between angular momentum generators [42]. As such, it can be used as the starting point for a new type of quantization of electromagnetic radiation, based on quantization of angular momentum operators. This method shares none of the drawbacks of canonical quantization [46], and gives photon creation and annihilation operators self-consistently. It is seen from the B cyclic theorem:

$$\begin{aligned} \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} &= iB^{(0)}\boldsymbol{B}^{(3)*} \\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*} \\ \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*} \end{aligned} \tag{723}$$

that if any one of the magnetic fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, or $\boldsymbol{B}^{(3)}$ is zero, this implies that the other two will also be zero. The B cyclic theorem can be put in commutator form by using the result that an axial vector is equivalent to a rank 2 antisymmetric polar tensor

$$B_k = \frac{1}{2}\varepsilon_{ijk}B_{ij} \tag{724}$$

where ε_{ijk} is the Levi–Civita symbol. The rank 2 tensor representation of the axial vector B_k is mathematically equivalent but has the advantage of being accessible to commutator (matrix) algebra, allowing $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$ to be expressed as infinitesimal rotation generators and as quantum-mechanical angular momentum operators. These methods show that the photon has an elementary longitudinal flux quantum, the photomagneton operator $B^{(3)}$, which is directly proportional to its intrinsic spin angular momentum [42].

The unit vector components of the classical magnetic fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$ in vacuo are all axial vectors by definition, and it follows that their unit vector components must also be axial in nature. In matrix form, they are, in the Cartesian basis

$$\boldsymbol{i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}; \quad \boldsymbol{j} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad \boldsymbol{k} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{725}$$

and in the circular basis

$$\boldsymbol{e}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{bmatrix}; \quad \boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{bmatrix}; \quad \boldsymbol{e}^{(3)} = \begin{bmatrix} 0 & 1 & \cdots \\ -1 & 0 & \cdots \\ 0 & 0 & \cdots \end{bmatrix} \tag{726}$$

The latter form a commutator Lie algebra, which is mathematically equivalent the vectorial Lie algebra:

$$[\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}] = -i\boldsymbol{e}^{(3)*} \quad \ldots \tag{727}$$

Equations (723) and (727) therefore represent a closed, cyclically symmetri algebra in which all three space-like components are meaningful. The cyclic commutator basis can be used to build a matrix representation of the three spac like magnetic components of the electromagnetic wave in the vacuum

$$\begin{aligned} B^{(1)} &= iB^{(0)}e^{(1)}e^{i\phi} \\ B^{(2)} &= -iB^{(0)}e^{(2)}e^{-i\phi} \\ B^{(3)} &= B^{(0)}e^{(3)} \end{aligned} \tag{728}$$

from which emerges the commutative Lie algebra equivalent to the vectorial L algebra

$$[B^{(1)}, B^{(2)}] = -iB^{(0)}B^{(3)*} \quad \ldots \tag{729}$$

This algebra can be expressed in terms of the infinitesimal rotation generators the O(3) group [42] in three dimensional space:

$$J^{(1)} = -ie^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{bmatrix}; \quad J^{(2)} = ie^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{bmatrix}$$

$$J^{(3)} = -ie^{(3)}\begin{bmatrix} 0 & -1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{730}$$

The magnetic field matrices and rotation generators are linked by

$$B^{(1)} = -B^{(0)}J^{(1)}e^{i\phi}$$
$$B^{(2)} = -B^{(0)}J^{(2)}e^{-i\phi} \quad (731)$$
$$B^{(3)} = iB^{(0)}J^{(3)}$$

so the commutative algebra of the magnetic fields (729) is part of the Lie algebra of spacetime. The real and physical $\boldsymbol{B}^{(3)}$ component is directly proportional to the rotation generator $J^{(3)}$, which is a fundamental property of spacetime, in which the matrices (730) become

$$J^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad J^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J^{(3)} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (732)$$

It follows that magnetic fields in the vacuum on the O(3) level are directly proportional to rotation generators of the Poincaré group [42], and electric fields are directly proportional to boost generators.

The rotation generators form a commutator algebra of the following type in the circular basis:

$$[J^{(1)}, J^{(2)}] = -J^{(3)*} \quad (733)$$

which becomes

$$[J_X, J_Y] = iJ_Z \quad (734)$$

in the Cartesian basis, and which is, within a factor $\hbar$, identical with the commutator algebra of angular momentum operators in quantum mechanics. This inference provides a simple route to the quantization of the magnetic fields, giving the result

$$B^{(1)} = -B^{(0)}\frac{J^{(1)}}{\hbar}e^{i\phi}; \qquad B^{(2)} = -B^{(0)}\frac{J^{(2)}}{\hbar}e^{-i\phi}; \qquad B^{(3)} = iB^{(0)}\frac{J^{(3)}}{\hbar} \quad (735)$$

where $B^{(i)}$ are now operators of quantum mechanics. Such a quantization scl can exist only on the O(3) level. In particular, the longitudinal $B^{(3)}$ i photomagneton operator, which is a stationary state in quantum mechanic

These results can be generalized to electric fields using boost operators, which in the Poincaré group are also 4 × 4 matrices:

$$E^{(1)} = E^{(0)}K^{(1)}e^{i\phi}$$
$$E^{(2)} = E^{(0)}K^{(2)}e^{-i\phi} \quad ($$
$$iE^{(3)} = iE^{(0)}K^{(3)}$$

Therefore, electric fields are boost generators, whereas magnetic field rotation generators. It follows that the Lie algebra of electric and magnetic f in spacetime is isomorphic with that of the infinitesimal generators o Poincaré group [42]. The latter type of Lie algebra can be summarize follows:

$$[J^{(1)}, J^{(2)}] = -J^{(3)*}\cdots$$
$$[K^{(1)}, K^{(2)}] = -ie^{(3)*}\cdots \quad ($$
$$[K^{(1)}, e^{(2)}] = -iK^{(3)*}\cdots$$
$$[K^{(1)}, J^{(1)}] = 0\cdots$$

This isomorphism is conclusive evidence for the existence of the longitu $\boldsymbol{B}^{(3)}$ in the vacuum.

There is also a relation between polar unit vectors, boost generators, electric fields. An electric field is a polar vector, and unlike the magnetic f cannot be put into matrix form as in Eq. (724). The cross-product of two p unit vectors is however an axial vector $\boldsymbol{k}$, which, in the circular basis, is $\boldsymbol{e}^{($ spacetime, the axial vector $\boldsymbol{k}$ becomes a 4 × 4 matrix related directly to th finitesimal rotation generator $J^{(3)}$ of the Poincaré group. A rotation generat therefore the result of a classical commutation of two matrices that play the of polar vectors. These matrices are boost generators. In spacetime, it is th fore

$$[K_X, K_Y] = -iJ_Z \quad ($$

and cyclic permutations. In the circular basis, this algebra becomes

$$[K^{(1)}, K^{(2)}] = -ie^{(3)*} \quad ($$

Therefore, although polar vectors cannot be put into matrix form in th dimensional space, they correspond to 4 × 4 matrices in spacetime. In th dimensional space, the electric component of the electromagnetic field are oscillatory fields that can be written directly in terms of the unit vectors of the circular basis:

$$\boldsymbol{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}; \qquad \boldsymbol{E}^{(2)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-i\phi} \tag{740}$$

In spacetime, the equivalents are

$$E^{(1)} = E^{(0)}K^{(1)}e^{i\phi}; \qquad E^{(2)} = E^{(0)}K^{(2)}e^{-i\phi} \tag{741}$$

The phase ϕ is a line integral on the O(3) level. The boost generators appearing in Eq. (741) are written in a circular basis

$$K^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \end{bmatrix}; \qquad K^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 \end{bmatrix} \tag{742}$$

and correspond to the complex, polar, unit vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ in Euclidean space.

It is not possible to form a real electric field from the cross-product of $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$, and this is self-consistent with the fact that on the O(3) level there is no real $\boldsymbol{E}^{(3)}$ [42].

The complete Lie algebra of the infinitesimal boost and rotation generators of the Poincaré group can be written as we have seen either in a circular basis or in a Cartesian basis. In matrix form, the generators are

$$K_X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}; \quad K_Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \quad K_Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$J_X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad J_Y = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad J_Z = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{743}$$

The relation between fields and generators in spacetime can be summarized a

$$\begin{aligned} B^{(1)} &= -B^{(0)}J^{(1)}e^{i\phi} = iB^{(0)}e^{(1)}e^{i\phi} \\ B^{(2)} &= -B^{(0)}J^{(2)}e^{-i\phi} = -iB^{(0)}e^{(2)}e^{-i\phi} \\ B^{(3)} &= iB^{(0)}J^{(3)} = B^{(0)}e^{(3)} \\ E^{(1)} &= E^{(0)}K^{(1)}e^{i\phi} \\ E^{(2)} &= E^{(0)}K^{(2)}e^{-i\phi} \\ iE^{(3)} &= iE^{(0)}K^{(3)} \end{aligned} \tag{744$$

leading to the Lie algebra:

$$\begin{aligned} [B^{(1)}, B^{(2)}] &= iB^{(0)}B^{(3)*}\cdots \\ [E^{(1)}, E^{(2)}] &= iE^{(0)2}e^{(3)*}\cdots \\ [E^{(1)}, B^{(2)}] &= iB^{(0)}(iE^{(3)})\cdots \\ [E^{(1)}, B^{(1)}] &= 0\cdots \end{aligned} \tag{745$$

where we have used the notation

$$\begin{aligned} &ie^{(1)} = J^{(1)}; \quad -ie^{(2)} = J^{(2)}; \quad ie^{(3)} = J^{(3)} \\ &ie^{(2)} = J^{(2)}; \quad -ie^{(1)} = J^{(1)}; \quad ie^{(3)} = -J^{(3)} \end{aligned} \tag{746$$

This type of Lie algebra occurs on the O(3) level, but not on the U(1) level. Sinc $\boldsymbol{iE}^{(3)}$ is purely imaginary, it has no physical meaning.

Therefore, the Lie algebra of the magnetic and electric components of th plane waves and spin fields in free space is isomorphic with that of the infini tesimal boost and rotation generators of the Poincaré group in spacetime. Ex perimental evidence (presented in Ref. 3 and in this review) suggests that $\boldsymbol{B}^{(3}$ is real and physical and the theory of electromagnetism in the vacuum is rela tivistically rigorous if and only if the longitudinal fields $\boldsymbol{B}^{(3)}$ (physical) and $\boldsymbol{iE}^{(3}$ (unphysical) are accounted for through the appropriate algebra. If $\boldsymbol{B}^{(3)}$ and $\boldsymbol{iE}^{(3}$ are set to zero, as in the received view [U(1) level], then the isomorphism is lost and electromagnetism becomes incompatible with relativity. If $\boldsymbol{B}^{(3)}$ were zero the rotation generator $J^{(3)}$ would be zero, which is incorrect. Similarly, if $iE^{(3}$ were zero, the boost generator $K^{(3)}$ would be incorrectly zero.

In units of $\hbar$, the eigenvalues of the massless photon are -1 and $+1$, anc those of the photon with mass are -1, 0, and $+1$. In three-dimensional space the latter are obtained from relations such as:

$$J^{(3)}e^{(1)} = +1e^{(1)}$$
$$J^{(3)}e^{(2)} = -1e^{(2)} \quad (747)$$
$$J^{(3)}e^{(3)} = 0e^{(3)}$$

where $J^{(3)}$ is the rotation generator:

$$J^{(3)} = ie^{(3)}\times = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (478)$$

There is no paradox [112] in the use of $e^{(3)}$ as an operator as well as a unit vector. In the same sense [112], there is no paradox in the use of the scalar spherical harmonics as operators. The rotation operators in space are first-rank T operators, which are irreducible tensor operators, and under rotations, transform into linear combinations of each other. The T operators are directly proportional to the scalar spherical harmonic operators. The rotation operators, J, of the full rotation group are related to the T operators as follows

$$T^1_{-1} = iJ^{(1)}; \qquad T^1_1 = iJ^{(2)}; \qquad T^1_0 = iJ^{(3)} \quad (749)$$

and to the scalar spherical harmonic operators by

$$Y^1_{-1} = \frac{i}{r}\left(\frac{3}{4\pi}\right)^{1/2} J^{(1)}; \qquad Y^1_1 = \frac{i}{r}\left(\frac{3}{4\pi}\right)^{1/2} J^{(2)}; \qquad Y^1_0 = \frac{i}{r}\left(\frac{3}{4\pi}\right)^{1/2} J^{(3)} \quad (750)$$

This implies that the fields $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ are also operators of the full rotation group, and are therefore irreducible representations of the full rotation group. Specifically

$$B^{(1)} = B^{(0)} r\left(\frac{4\pi}{3}\right)^{1/2} Y^1_{-1} e^{i\phi}$$
$$B^{(2)} = B^{(0)} r\left(\frac{4\pi}{3}\right)^{1/2} Y^1_1 e^{-i\phi} \quad (751)$$
$$B^{(3)} = B^{(0)} r\left(\frac{2\pi}{3}\right)^{1/2} Y^1_0$$

which shows that $B^{(3)} = ?\ 0$ violates the fundamentals of group theory $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ are all nonzero components of the same rank 1 scalar spl harmonic $Y^1_M; M = -1, 0, 1$. Furthermore, since the operators $J^{(1)}$, $J^{(2)}$, a are components in a circular basis of the spin, or intrinsic, angular momen the vector field representing the electromagnetic field, the fields $B^{(1)}$, $B^{($ $B^{(3)}$ are themselves components of spin angular momentum. It is also cle $J^{(1)}$ is a lowering (annihilation) operator

$$J^{(1)}e^{(2)} = +1e^{(3)}; \qquad J^{(1)}e^{(3)} = -1e^{(1)}; \qquad J^{(1)}e^{(1)} = 0e^{(2)}$$

and that $J^{(2)}$ is a raising (creation) operator:

$$J^{(2)}e^{(2)} = 0e^{(1)}; \qquad J^{(2)}e^{(3)} = -1e^{(2)}; \qquad J^{(2)}e^{(1)} = +1e^{(3)}$$

The total angular momentum J^2 is also an eigenoperator, for example:

$$J^2 e^{(3)} = l(l+1)e^{(3)}; \qquad l = 1$$

The operator $J^{(3)}$ is therefore also an intrinsic spin, and can be identified novel quantization method based on the B cyclic theorem with the intrinsi of a photon with mass, with eigenvalues -1, 0, and $+1$.

For a classical vector field, its intrinsic (spin) angular momentum is id able with its transformation properties [112] under rotations, and within a $\hbar$, the rotation operators are spin angular momentum operators of the sp boson. Recognition of a nonzero $B^{(3)}$ is therefore compatible with the ei lues of both the massive and massless bosons. The vector spherical harn [112] are specific vector fields that are eigenvalues of j^2 and of j_Z where j operator for vector fields of infinitesimal rotations about axis (3). They ha finite total angular momentum and occur in sets of dimension $(2j+1)$ tha in standard form the D representations of the full rotation group, and are fore irreducible tensors of rank j. Defining the total angular momentum sum of the "orbital" angular momentum l and intrinsic (spin) angular m tum J, we have

$$j = l + J$$

and the vector spherical harmonics are compound irreducible tensor ope [112]:

$$\mathbf{Y}^L_{Ml1} \equiv \left[Y^1 \otimes l\right]^L_M$$

They are formed from the scalar spherical harmonics Y^l_M, which form a cor set for scalar functions, and the $e^{(i)}$ operators, which form a complete set f vector in three-dimensional space. Therefore, the vector spherical harmonics form a complete set for the expansion of any arbitrary classical vector field:

$$\boldsymbol{A} = A_X\boldsymbol{i} + A_Y\boldsymbol{j} + A_Z\boldsymbol{k} \tag{757}$$

in a Cartesian basis. For this vector, the I_Z operates on the A_X, A_Y, A_Z and J_Z operates on $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. Thus, I_Z operates on the spatial part of the field and J_Z, on the vector part.

Therefore the operator for infinitesimal rotations about the Z axis contains two "angular momentum" operators, I and J, analogous with orbital and spin angular momentum in the quantum theory of atoms and molecules. The infinitesimal rotation is therefore formally a coupling of a set of spatial fields transforming according to $D^{(1)}$ with a set of three vector fields $[\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}]$, transforming according to $\boldsymbol{D}^{(1)}$. Equation (756) is an expression of this coupling, or combining, of entities in two different spaces to give a total angular momentum. It follows, from these considerations, that the vector spherical harmonics are defined by

$$\boldsymbol{Y}_{Ml1}^{L} = \sum_{mn} \langle l1mn|l1LM\rangle Y_m^l \boldsymbol{e}_m \tag{758}$$

where $\langle l1mn|l1LM\rangle$ are Clebsch–Gordan, or coupling, coefficients [112]. For photons regarded as bosons of unit spin, it is possible to multiply Eq. (758) by $\langle 110M|11LM\rangle$ and to sum over L [112]. Using the orthogonality condition

$$\sum_j \langle j_1 m_1' j_2 m - m_2' | j_1 j_2 j_m\rangle \langle j_1 j_2 j_m | j_1 m_1 j_2 m - m_2\rangle = \delta_{m_1 m_1'} \tag{759}$$

it is found that

$$Y_0^1(\theta, \phi)\boldsymbol{e}_M = \sum_{L=|l-1|}^{l+1} \langle l10M|l1LM\rangle \boldsymbol{Y}_{Ml1}^{L} \tag{760}$$

which is an expression for the unit vectors $\boldsymbol{e}_M$ in terms of sums over vector spherical harmonics, that is, of irreducible compound tensors, representations of the full rotation group.

On the U(1) level, the transverse components of $\boldsymbol{e}_M$ are physical but the longitudinal component corresponding to $M = 0$ is unphysical. This asserts two states of transverse polarization in the vacuum: left and right circular. However, this assertion amounts to $\boldsymbol{e}_0 \equiv \boldsymbol{e}^{(3)} = ?\boldsymbol{0}$, meaning the incorrect disappearance of some vector spherical harmonics that are nonzero from fundamental group theory because some irreducible representations are incorrectly set to zero.

This point can be emphasized by expanding $\boldsymbol{B}^{(3)}$ in terms of Wigner 3-j symbols [112], which yields results such as

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)} = 2B^{(0)}\frac{\boldsymbol{Y}_{001}^{1}}{Y_0^1} = \frac{B^{(0)}}{\sqrt{3}}\frac{\sqrt{2}\boldsymbol{Y}_{001}^{2} - \boldsymbol{Y}_{011}^{0}}{Y_0^1} \tag{761}$$

showing that $\boldsymbol{B}^{(3)}$ is nonzero and proportional to the nonzero vector spherical harmonic $\boldsymbol{Y}_{001}^{1}$ on a fundamental level. Therefore, the fundamentals of group theory are obeyed on the O(3) level, but not on the U(1) level.

All three of $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}$ can be expressed in terms of vector spherical harmonics. Thus, in addition to the nonlinear B cyclic theorem, the following linear relations occur

$$\begin{aligned} \boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)} &= \frac{\sqrt{2}}{2} a B^{(0)}(\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)}) + B^{(0)}\boldsymbol{b} \\ &= -\frac{\sqrt{2}}{2} c B^{(0)}(\boldsymbol{e}^{(1)} - \boldsymbol{e}^{(2)}) + B^{(0)}\boldsymbol{d} \end{aligned} \tag{762}$$

where the coefficients are defined by the following combination of scalar and vector spherical harmonics:

$$\begin{aligned} a &= \frac{2}{\sqrt{2}}\left(\frac{Y_0^1}{Y_1^1 - Y_{-1}^1}\right); & c &= -\frac{2}{\sqrt{2}}\left(\frac{Y_0^1}{Y_1^1 + Y_{-1}^1}\right) \\ \boldsymbol{b} &= \sqrt{2}\left(\frac{\boldsymbol{Y}_{111}^1 + \boldsymbol{Y}_{-111}^1}{Y_1^1 - Y_{-1}^1}\right); & \boldsymbol{d} &= \sqrt{2}\left(\frac{\boldsymbol{Y}_{111}^1 - \boldsymbol{Y}_{-111}^1}{Y_1^1 + Y_{-1}^1}\right) \end{aligned} \tag{763}$$

On the O(3) level, therefore, $\boldsymbol{B}^{(3)}$ is nonzero because $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are nonzero.

On the U(1) level, the plane wave is subjected to a multipole expansion in terms of the vector spherical harmonics, in which only two physically significant values of M in Eq. (761) are assumed to exist, corresponding to $M = +1$ and -1, which translates into our notation as follows:

$$\boldsymbol{e}_1 = -\boldsymbol{e}^{(2)}; \qquad \boldsymbol{e}_{-1} = \boldsymbol{e}^{(1)}; \qquad \boldsymbol{e}_0 = \boldsymbol{e}^{(3)} \tag{764}$$

On the O(3) level, the case $M = 0$ is also considered to be physically meaningful. In consequence, there is an additional, purely real, 2^L- pole component of the electromagnetic plane wave in vacuo corresponding to $\boldsymbol{B}^{(3)}$. The vector spherical harmonics $\boldsymbol{Y}_{mL1}^{L}$ with $1 = L$ are no longer transverse fields, and the vector $\boldsymbol{e}^{(3)}$, which is longitudinal, can also be expressed in terms of the $L = 1, M = 0$ vector spherical harmonics as in Eq. (761). The longitudinal $\boldsymbol{B}^{(3)}$, according to Eq. (761), can be expanded for all integer 1 of that equation in terms of vector spherical harmonics. Each value of 1 for $M = 0$ in $\boldsymbol{Y}_{0l1}^{L}$ defines a different nonzero component of $\boldsymbol{B}^{(3)}$. Therefore the $L = 1$ components in the expansion of $\boldsymbol{B}^{(3)}$ are dipolar fields.

As an example of these methods, consider the B cyclic theorem for multipole radiation, which can be developed for the multipole expansion of plane-wave radiation to show that the $\boldsymbol{B}^{(3)}$ field is irrotational, divergentless, and fundamental for each multipole component. The magnetic components of the plane wave are defined, using Silver's notation [112] as

$$\begin{aligned} \boldsymbol{B}_1 &= B^{(0)} e^{i\phi} \boldsymbol{e}_1 \\ \boldsymbol{B}_{-1} &= B^{(0)} e^{-i\phi} \boldsymbol{e}_{-1} \\ \boldsymbol{B}_0 &= B^{(0)} \boldsymbol{e}_0 \end{aligned} \tag{765}$$

where the basis vectors in Silver's spherical representation are related by

$$\boldsymbol{e}_{-1} \times \boldsymbol{e}_1 = -i\boldsymbol{e}_0 \tag{766}$$

in cyclic permutation. The phase factor on the O(3) is a line integral, as argued in this review and elsewhere [44]. The B cyclic theorem in this notation is therefore

$$\boldsymbol{B}_{-1} \times \boldsymbol{B}_1 = -iB^{(0)}\boldsymbol{B}_0 \tag{767}$$

In order to develop Eq. (767) for multipole radiation, we use the following expansions [112]:

$$\begin{aligned} e^{ikZ} &= \sum_l i^l (2l+1) j_l(kZ) P_l(\cos\theta) \\ \boldsymbol{e}_M &= \frac{1}{Y_0^1} \sum_{L=|l-1|}^{l+1} \langle l10M|l1LM\rangle \boldsymbol{Y}_{Ml1}^{L} \end{aligned} \tag{768}$$

where l is the lth multipole moment, j_l the lth modified Bessel function, and P_l is the lth Legendre polynomial. The basis vector $\boldsymbol{e}_m$ $(M = -1, 0, +1)$ is expanded in terms of the Clebsch–Gordan coefficients $\langle l10M|l1LM\rangle$ and the vector spherical harmonics $\boldsymbol{Y}_{Ml1}^{L}$, and normalized with the scalar spherical harmonic Y_0^1.

In deriving Eq. (767), we have used on the left-hand side the conjugate product of phase factors:

$$e^{i\phi} e^{-i\phi} = 1 \tag{769}$$

Using Eqs. (768a) and (769), it is seen that the product is unity if we sum over all multipole components with $1 \to \infty$ in Eq. (768). In all other cases, the B cyclic theorem is

$$\boldsymbol{B}_{-1} \times \boldsymbol{B}_1 = -ixB^{(0)}\boldsymbol{B}_0 \tag{770}$$

where x is different from unity. It is given as follows for the first few multip

$$\begin{aligned} x &= 9j_1^2 P_1^2, \quad \text{for} \quad l = 1 \\ &= 25 j_2^2 P_2^2 \quad \text{for} \quad l = 2 \\ &= 49 j_3^2 P_3^2 \quad \text{for} \quad l = 3 \end{aligned}$$

In this notation

$$\begin{aligned} P_l(\cos\theta) &= (2\pi(2l+1))^{1/2} Y_0^1(\theta) \\ j_l(kr) &= \left(-\frac{r}{k}\right)^l \left(\frac{1}{r}\frac{d}{dr}\right) j_0(kr) \end{aligned}$$

It is important to note that $\boldsymbol{B}_0$ in Eq. (770) is the same as $\boldsymbol{B}_0$ in Eq. (7 phaseless, irrotational, and divergentless. The factor x arises purely from truncation of the infinite series (768a) in individual multipole components discussed by Silver [112], the $\boldsymbol{e}_m$ vectors are polarization vectors for electromagnetic wave, but are also spin angular momentum eigenfuncti Tautologically, therefore, Eq. (767), the B cyclic theorem, is a spin ang momentum equation for the photon, with $M = -1, 0, 1$. The photon w function, therefore, has components $e^{ikZ}\boldsymbol{e}_1$, $e^{-ikZ}\boldsymbol{e}_{-1}$, and $\boldsymbol{e}_0$. The observable this theory are therefore energy and $\boldsymbol{B}_0$. The complete vector fields $\boldsymbol{B}_1, \boldsymbol{B}_{-1}, \boldsymbol{B}$ described in terms of the vector spherical harmonics, and the B cyclic theo indicates the existence of an intrinsic magnetic field $\boldsymbol{B}_0$, which is described by transformation of the frame under rotation. As is well known in classical ang momentum theory, only the $\boldsymbol{B}_0$ component remains sharply defined u rotation. The components $\boldsymbol{B}_1$ and $\boldsymbol{B}_{-1}$ are defined only within an arbitrary p factor. Within $\hbar$, this is the quantum theory of angular momentum [112].

Since $\boldsymbol{B}^{(3)}$ is time-independent, it obeys

$$\boldsymbol{B}^{(3)} = -\nabla\Phi_B$$

where Φ_B is determined by the Laplace equation:

$$\nabla^2 \Phi_B = 0$$

Analogously, a Coulomb field can be expressed as the gradient of a sc potential that obeys the Laplace equation in a source-free region such as vacuum in conventional electrostatics. To find the general form of $\boldsymbol{B}^{(3)}$ multipole expansion, we therefore solve the Laplace equation for Φ_B, evaluate the gradient of this solution

$$\Phi_B = \frac{U(r)}{r} \rho(\theta) Q(\Phi)$$

in spherical polar coordinates. The general solution (775) can be written as

$$\Phi_B = (Ar^l + Br^{-2})Y_{lm}(\theta, \phi) \tag{776}$$

where $Y_{lm}(\theta, \phi)$ are the spherical harmonics and A and B are constants. Here, m and l are integers, with l running from $-m$ to m. The solution of Laplace's equation is therefore obtained as a product of radial and angular functions. The latter are orthonormal functions, the spherical or tesseral harmonics, which form a complete set on the surface of the unit sphere for the two indices l and m. The integer l defines the order of the multipole component; $l = 1$ is a dipole, $l = 2$ is a quadrupole, $l = 3$ is an octopole, and $l = 4$ is a hexadecapole.

The most general form of $\boldsymbol{B}^{(3)}$ from the Laplace equation is therefore

$$\boldsymbol{B}^{(3)} = -\nabla(Ar^l + Br^{-2})Y_{lm}(\theta, \phi) \tag{777}$$

This is the phaseless magnetic field of multipole radiation on the O(3) level. The solution (777) reduces to the simple

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)} = B^{(0)}\boldsymbol{k} \tag{778}$$

when $l = 1$, $m = 0$, $r = Z$, $\theta = 0$, $A = -B^{(0)}$, $B = 0$, and $\nabla = (\partial/\partial Z)\boldsymbol{k}$. More generally, there exist other irrotational forms of $\boldsymbol{B}^{(3)}$:

1. $\boldsymbol{B}^{(3)}$ for dipole radiation: $l = 1,\ m = -1, 0, 1$
2. $\boldsymbol{B}^{(3)}$ for quadrupole radiation: $l = 2,\ m = -2, \ldots, 2$ (779)
3. $\boldsymbol{B}^{(3)}$ for octopole radiation: $l = 3,\ m = -3, \ldots, 3$

The $\boldsymbol{B}^{(3)}$ fields for n-pole fields are irrotational for all n on the O(3) level.

As argued, infinitesimal field generators appear as a by-product of this novel quantization scheme, so that $\boldsymbol{B}^{(3)}$ is rigorously nonzero from the symmetry of the Poincaré group and the B cyclic theorem is an invariant of the classical field. The basics of infinitesimal field generators on the classical level are to be found in the theory of relativistic spin angular momentum [42,46] and relies on the Pauli–Lubanski pseudo-4-vector:

$$W^{\lambda} = -\frac{1}{2}\varepsilon^{\lambda\mu\nu\rho}p_{\mu}J_{\nu\rho} \tag{780}$$

where $\varepsilon^{\lambda\mu\nu\rho}$ (with $\varepsilon^{0123} = 1$) is the antisymmetric unit 4-vector. The antisymmetric matrix of generators $J_{\nu\rho}$ is given by

$$J_{\nu\rho} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{bmatrix} \tag{781}$$

where every element is an element of spin angular momentum in four dime[n]sions. The energy momentum polar 4-vector is defined by

$$p^{\mu} = (p^0, \boldsymbol{p}) = \left(\frac{En}{c}, \boldsymbol{p}\right) \tag{78}$$

The infinitesimal generators can be represented as matrices or as combinations [of] differential operators [46]. The Pauli–Lubanski operator then becomes a produ[ct] of the $J_{\nu\rho}$ and p_{μ} operators. Barut [113] shows that the Lie algebra of the W operators is

$$[W^{\mu}, W^{\nu}] = -i\varepsilon^{\mu\nu\sigma\rho}p_{\sigma}W_{\rho} \tag{78}$$

which is a four-dimensional commutator relation. The theory is relativistical[ly] covariant and, of course, compatible with special relativity. Equation (783) give[s] the Lie algebra [42] of intrinsic spin angular momentum because rotatio[n] generators are angular momentum operators within a factor $\hbar$, and this allow[s] relativistic quantization to be considered. Similarly, translation generators a[re] energy momentum operators within a factor $\hbar$. This development leads [to] Wigner's famous result that every particle is characterized by two Casim[ir] invariants of the Poincaré group, the mass and spin invariants [46].

Our basic ansatz is to assume that this theory applies to the vacuum electr[o]magnetic field, considered as a physical entity of spacetime in the theory of spe[]cial relativity. The intrinsic spin of the classical electromagnetic field is th[e] magnetic flux density $\boldsymbol{B}^{(3)}$. Infinitesimal generators of rotation correspond wit[h] those of intrinsic magnetic flux density in the vacuum. Boost generators corre[]spond with intrinsic electric field strength. Translation generators correspon[d] with the intrinsic, fully covariant, field potential. Thus, the symbols are trans[]muted as follows:

$$J \rightarrow B; \qquad K \rightarrow E; \qquad P \rightarrow A \tag{784}$$

In Cartesian notation, the Pauli–Lubanski vector of particle theory becomes [a] 4-vector of the classical electromagnetic field

$$W^{\lambda} = -\frac{1}{2}\varepsilon^{\lambda\mu\nu\rho}A_{\mu}F_{\nu\rho} \tag{785}$$

and the Lie algebra (783), a Lie algebra of the field.

If it is assumed that the electromagnetic field propagates at c in the vacuum then we must consider the Lie algebra (783) in a light-like condition. The latte[r] is satisfied by a choice of

$$A^{\mu} = (A^0, A_Z)$$
$$A^0 = A_Z \tag{786}$$

The basic ansatz is that there is a field vector analogous to the Pauli–Lubanski vector of particle physics, a field vector defined by

$$W^{\lambda} = \tilde{F}^{\lambda\mu} A_{\mu} \tag{787}$$

where $\tilde{F}^{\lambda\mu}$ is the dual of the antisymmetric field tensor. This vector has the following components:

$$\begin{aligned} W^0 &= -B^1 A_1 - B^2 A_2 - B^3 A_3 \\ W^1 &= B^1 A_0 + E^3 A_2 - E^2 A_3 \\ W^2 &= B^2 A_0 - E^3 A_1 + E^1 A_3 \\ W^3 &= B^3 A_1 + E^2 A_1 - E^1 A_2 \end{aligned} \tag{788}$$

If it assumed that for the transverse components

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{789}$$

that $\boldsymbol{A}$ and $\boldsymbol{B}$ are plane waves

$$\begin{aligned} \boldsymbol{A} &= \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} \\ \boldsymbol{B} &= \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} \end{aligned} \tag{790}$$

and that the longitudinal $\boldsymbol{E}^{(3)}$ is zero, then Eq. (788) reduces to

$$\begin{aligned} W_0 &= A_Z B_Z \\ W_X &= A_0 B_X + A_Z E_Y \\ W_Y &= A_0 B_Y - A_Z E_X \\ W_Z &= A_0 B_Z \end{aligned} \tag{791}$$

These assumptions mean that

$$A^{\mu} = (A^0, 0, 0, A^3); \qquad A^0 = A^3 \tag{792}$$

can be used as an ansatz. Conversely, the use of this definition means that the transverse components are plane waves, and for the transverse components, $\boldsymbol{B} = \nabla = \boldsymbol{A}$.

In the Coulomb gauge, the vector W^{μ} vanishes, meaning that there is no correspondence between the particle and field theory for the Coulomb gauge, or the

received view of transversality in the vacuum. The final result is therefore

$$W^{\mu} = A^0(B_Z, 0, 0, B_Z) \qquad (79$$

which is compatible with the Lie algebra of a light-like particle. This correspond in the particle interpretation to the light-like translation generator:

$$p^{\mu} = (p^0, p_Z); \quad p^0 = p_Z \qquad (79$$

The Pauli–Lubanski pseudovector of the field in this condition is

$$\begin{aligned} W^{\mu} &= (A_Z B_Z,\, A_Z E_Y + A_0 B_X,\; -A_Z E_X + A_0 B_Y,\; A_0 B_Z) \\ &= A_0(B_Z,\, E_Y + B_X,\; -E_X + B_Y,\; B_Z) \end{aligned} \qquad (79$$

and the Lie algebra (783) becomes, in $c = 1$ units:

$$\begin{aligned} [B_X + E_Y,\; B_Y - E_X] &= i(B_Z - B_Z) \\ [B_Y - E_X,\; B_Z] &= i(B_X + E_Y) \\ [B_Z,\; B_X + E_Y] &= i(B_Y - E_X) \end{aligned} \qquad (796$$

which has E(2) symmetry. In the particle interpretation, Eqs. (795) and (796 correspond to

$$W^{\mu} = (p_Z J_Z, p_Z K_Y + p_0 J_X,\; -p_Z K_X + p_0 J_Y,\; p_0 J_Z) \qquad (797$$

and

$$\begin{aligned} [J_X + K_Y,\; J_Y - K_X] &= i(J_Z - J_Z) \\ [J_Y - K_X,\; J_Z] &= i(J_X + K_Y) \\ [J_Z,\; J_X - K_Y] &= i(J_Y - K_X) \end{aligned} \qquad (798$$

In the rest frame of a photon with mass, the field and particle Pauli–Lubansk vectors are respectively

$$W^{\mu} = (0,\; A_0 B_X,\; A_0 B_Y,\; A_0 B_Z) \tag{799}$$

and

$$W^{\mu} = (0,\; p_0 J_X,\; p_0 J_Y,\; p_0 J_Z) \tag{800}$$

The rest frame Lie algebra for field and particle is respectively (normalized $B^{(0)} = 1$ units):

$$[B_X, B_Y] = iB_Z \cdots \tag{801}$$

and

$$[J_X, J_Y] = iJ_Z \cdots \tag{802}$$

The $E^{(2)}$ field algebra is compatible with the vacuum Maxwell equations written for eigenvalues of our novel infinitesimal field operators. This can be demonstrated as follows:

$$B_Y = E_X; \qquad B_X = -E_Y \tag{803}$$

It is assumed that the eigenfunction (χ) operated on by these infinitesimal field generators is such that the same relation (803) holds between eigenvalues of the field. In order for this to be true, the eigenfunction must be the de Broglie wave function, specifically, the phase of the classical electromagnetic field. On the O(3) level, this is a line integral, as we have seen.

The relation (803) interpreted as one between eigenvalues is compatible with the plane-wave solutions

$$\begin{aligned} \boldsymbol{E}^{(1)} &= \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi} \\ \boldsymbol{B}^{(1)} &= \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} \end{aligned} \tag{804}$$

which are special cases of the O(3) invariant electrodynamics defined by

$$\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}); \qquad \boldsymbol{e}^{(3)} = \boldsymbol{k} \tag{805}$$

It follows that the same analysis can be applied to the particle interpretation, giving

$$\partial_\mu J^{\mu\nu} = \partial_\mu \tilde{J}^{\mu\nu} = 0 \tag{806}$$

in the vacuum. This is a possible conservation equation (relation between spins) that is compatible with the $E^{(2)}$ symmetry of the little group of the Poincaré group. This is the little group for a massless particle. On the U(1) level, therefore, it is concluded that the vacuum Maxwell equations for the field correspond with Eq. (806) for the particle, an equation that asserts that the spin angular momentum matrix is divergentless. In vector notation, we obtain from Eqs. (803)–(806) the familiar U(1) equations

$$\begin{aligned} &\nabla \cdot \boldsymbol{B} = 0; && \nabla \cdot \boldsymbol{E} = 0 \\ &\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}; && \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{0} \end{aligned} \tag{807}$$

and the less familiar relation between eigenvalues of spin angular momentum in four dimensions:

$$\begin{aligned} &\nabla \cdot \boldsymbol{J} = 0; && \nabla \cdot \boldsymbol{K} = 0 \\ &\nabla \times \boldsymbol{J} + \frac{\partial \boldsymbol{K}}{\partial t} = \boldsymbol{0}; && \nabla \times \boldsymbol{K} - \frac{\partial \boldsymbol{J}}{\partial t} = \boldsymbol{0} \end{aligned} \tag{808}$$

On the O(3) level, particular solutions of the $E^{(2)}$ Lie algebra (796) give a total of six commutator relations. Three of these form the B cyclic theorem ($B^{(0)} = 1$ units):

$$\begin{aligned} [B_X,\ B_Y] &= iB_Z \\ [B_Y,\ B_Z] &= iB_X \\ [B_Z,\ B_X] &= iB_Y \end{aligned} \tag{809}$$

and the other three are

$$\begin{aligned} [E_X,\ E_Y] &= -iB_Z \\ [B_Z,\ E_X] &= iE_Y \\ [E_Y,\ B_Z] &= iE_X \end{aligned} \tag{810}$$

In the particle interpretation, these are part of the Lie algebra of rotation and boost generators of the Poincaré group:

$$\begin{aligned} &[J_X,\ J_Y] = iJ_Z && [K_X,\ K_Y] = -iJ_Z \\ &[J_Y,\ J_Z] = iJ_X && [J_Z,\ K_X] = iK_Y \\ &[J_Z,\ J_X] = iJ_Y && [K_Y,\ J_Z] = iK_X \end{aligned} \tag{811}$$

From these relations, we can obtain

$$\begin{aligned} \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} &= iB^{(0)}\boldsymbol{B}^{(3)*} \\ \boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*} \\ \boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*} \end{aligned} \tag{812}$$

$$\begin{aligned} \boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} &= ic^2B^{(0)}\boldsymbol{B}^{(3)*} \\ \boldsymbol{B}^{(3)} \times \boldsymbol{E}^{(1)} &= icB^{(0)}\boldsymbol{E}^{(2)*} \\ \boldsymbol{B}^{(3)} \times \boldsymbol{E}^{(2)} &= -icB^{(0)}\boldsymbol{E}^{(1)*} \end{aligned} \tag{813}$$

where $\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(0)}\boldsymbol{e}^{(3)}$. Similarly, in the particle interpretation, and switching from rotation generators to spin angular momentum, we obtain:

$$\begin{aligned} \boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)} &= i\hbar \boldsymbol{J}^{(3)*} \\ \boldsymbol{J}^{(2)} \times \boldsymbol{J}^{(3)} &= i\hbar \boldsymbol{J}^{(1)*} \\ \boldsymbol{J}^{(3)} \times \boldsymbol{J}^{(1)} &= i\hbar \boldsymbol{J}^{(2)*} \end{aligned} \tag{814}$$

where $\hbar$ is the quantum of spin angular momentum.

In the rest frame of a photon or particle with mass, we obtain, for field and particle, respectively, Eqs. (812) and (813); that is, there are no boost generators.

From this analysis, it is concluded that the $\boldsymbol{B}^{(3)}$ component is identically nonzero, otherwise all the field components vanish in the B cyclic theorem (812) and Lie algebra (809). If we assume Eq. (803) and at the same time assume that $\boldsymbol{B}^{(3)}$ is zero, then the Pauli–Lubanski pseudo-4-vector vanishes for all A_0. Similarly, in the particle interpretation, if we assume the equivalent of Eq. (803) and assume that $\boldsymbol{J}^{(3)}$ is zero, the Pauli–Lubanski vector W^{μ} vanishes. This is contrary to the definition of the helicity of the photon. Therefore, for finite field helicity, we need a finite $\boldsymbol{B}^{(3)}$.

The precise correspondence between field and photon interpretation developed here indicates that E(2) symmetry does not imply that $\boldsymbol{B}^{(3)}$ is zero, any more than it implies that $\boldsymbol{J}^{(3)} = \boldsymbol{0}$. The assertion $\boldsymbol{B}^{(3)} = \boldsymbol{0}$ is counterindicated by a range of data reviewed here and in Ref. 44, and the B cyclic theorem is Lorentz-covariant, as it is part of a Lorentz-covariant Lie algebra. If we assume the particular solutions (809) and (810) and use in it the particular solution (803), we obtain the cyclics (809) from the three cyclics Eq. (810); thus we obtain

$$\begin{aligned} [B_Y,\ -B_X] &= iB_Z \\ [B_Z,\ B_Y] &= -iB_X \\ [B_Z,\ -B_X] &= -iB_Y \end{aligned} \tag{815}$$

This is also the relation obtained in the hypothetical rest frame. Therefore, the B cyclic theorem is Lorentz-invariant in the sense that it is the same in the rest frame and in the light-like condition. This result can be checked by applying the Lorentz transformation rules for magnetic fields term by term [44]. The equivalent of the B cyclic theorem in the particle interpretation is a Lorentz-invariant construct for spin angular momentum:

$$\boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)} = i\hbar \boldsymbol{J}^{(3)*} \tag{816}$$

It is concluded that the $\boldsymbol{B}^{(3)}$ component in the field interpretation is nonzero in the light-like condition and in the rest frame. The B cyclic theorem is a Lorentz-invariant, and the product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is an experimental observable [44]. In this representation, $\boldsymbol{B}^{(3)}$ is a phaseless and fundamental field spin, an intrinsic property of the field in the same way that $\boldsymbol{J}^{(3)}$ is an intrinsic property of the photon. It is incorrect to infer from the Lie algebra (796) that $\boldsymbol{B}^{(3)}$ must be zero for plane waves. For the latter, we have the particular choice (803) and the algebra (796) reduces to

$$i(B_Z - B_Z) = 0 \tag{817}$$

which does not indicate that B_Z is zero any more than the equivalent particle interpretation indicates that J_Z is zero.

In order to translate a Cartesian commutator relation such as

$$[B_X, B_Y] = iB^{(0)}B_Z \tag{818}$$

to a ((1),(2),(3)) basis vector equation such as

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \tag{819}$$

consider firstly the usual vector relation in the Cartesian frame:

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \tag{820}$$

The unit vector $\boldsymbol{i}$, for example, is defined by

$$\boldsymbol{i} = u_X \boldsymbol{i} \tag{821}$$

where u_x is a rotation generator, in general a matrix component [46]. Therefore

$$u_X = i(J_X)_{YZ} \tag{822}$$

The cross-product $\times \boldsymbol{j}$ therefore becomes a commutator of matrices

$$[J_X, J_Y] = iJ_Z \tag{823}$$

that is

$$\frac{1}{i}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \frac{1}{i}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \frac{1}{i}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{1}{i}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{824}$$

This can be extended straightforwardly to angular momentum operators and infinitesimal magnetic field generators. Therefore, a commutator such as Eq. (818) is equivalent to a vector cross-product. If we write $B^{(0)}$ as the scalar magnitude of magnetic flux density, the commutator (818) becomes the vector cross-product

$$(B^{(0)}\boldsymbol{i}) \times (B^{(0)})\boldsymbol{j} = B^{(0)}(B^{(0)}\boldsymbol{k}) \tag{825}$$

which can be written conveniently as

$$(B_X B_Y)^{1/2}\boldsymbol{i} \times (B_X B_Y)^{1/2}\boldsymbol{j} = iB^{(0)}B_Z\boldsymbol{k} \tag{826}$$

However, the Cartesian basis can be extended to the circular basis using relations between unit vectors developed in this review chapter. So Eq. (826) can be written in the circular basis as

$$(B_X B_Y)^{1/2}\boldsymbol{e}^{(1)} \times (B_X B_Y)^{1/2}\boldsymbol{e}^{(2)} = -B^{(0)}B_Z\boldsymbol{e}^{(3)*} \tag{827}$$

which is equivalent to

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \tag{828}$$

where we define

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)} = (B_X B_Y)^{1/2}\boldsymbol{e}^{(1)}; \qquad \boldsymbol{B}^{(3)} = B_Z\boldsymbol{e}^{(3)} \tag{829}$$

To complete the derivation, we multiply both sides of Eq. (828) by the phase factor $e^{i\phi}e^{-i\phi}$ to obtain the B cyclic theorem. The latter is therefore equivalent to a commutator relation of the Poincaré group between infinitesimal magnetic field generators. Similarly

$$[E_X,\ E_Y] = ic^2B^{(0)}B_Z \tag{830}$$

is equivalent to

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = ic^2B^{(0)}\boldsymbol{B}^{(3)} \tag{831}$$

XIV. O(3) AND SU(3) INVARIANCE FROM THE RECEIVED FARADAY AND AMPÈRE–MAXWELL LAWS

The received Faraday and Ampère–Maxwell laws [111] in the vacuum asserts that there are fields without sources, so the laws become respectively

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0} \tag{832}$$

$$\nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{0} \tag{833}$$

These laws are useful but represent cause without effect, that is, fields prop ing without sources, and the Maxwell displacement current is an emp construct, one that happens to be very useful. These two laws can be classifi U(1) invariant because they are derived from a locally invariant U(1) Lagra as discussed already. Majorana [114] put these two laws into the form of a D Weyl equation (Dirac equation without mass)

$$\begin{aligned} W\psi_1 - ip_2\psi_3 - ip_3\psi_2 &= 0 \\ W\psi_2 - ip_3\psi_1 - ip_1\psi_3 &= 0 \\ W\psi_3 - ip_1\psi_2 - ip_2\psi_1 &= 0 \end{aligned}$$

in which a combination of fields (SI units) acts as a wave function

$$\psi_i = \frac{1}{c}E_i - iB_i; \qquad i = 1, 2, 3$$

and in which the quantum ansatz

$$\boldsymbol{p} = -i\hbar\nabla; \qquad i\hbar\frac{\partial}{\partial t} \to En \equiv W$$

has been used. It is shown in this section that the Majorana equations are invariant, so the received view is self-contradictory. There is something hi inside the structure of the Faraday and Ampère–Maxwell laws that removes U(1) invariance [44]. It can be checked straightforwardly that Eqs. (835) (834) lead back to Eqs. (833) and (832). In condensed notation, the Majc equations (834) have the form of the Dirac–Weyl equation:

$$(W + \boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\Psi} = \boldsymbol{0}$$

The structure of the Dirac–Weyl equation itself is [46]

$$(\gamma^0 p_0 + \gamma^i p_i)\psi = 0$$

In Eq. (837), however, the $\boldsymbol{\alpha}$ matrix is an O(3) rotation generator matrix components

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}; \qquad \alpha_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}; \qquad \alpha_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

obeying the O(3) invariant commutator equation

$$[\alpha_i, \alpha_k] = -i\varepsilon_{ikl}\alpha_l, \qquad (i, k, l = 1, 2, 3)$$

which is within a factor $\hbar$, the O(3) invariant commutator equation for angular momentum [42,44,46]. Therefore, the Majorana form of Eqs. (832) and (833), namely, Eq. (837), is O(3) invariant, not U(1) invariant. The determinant condition

$$\begin{vmatrix} W & -ip_3 & ip_2 \\ ip_3 & W & -ip_1 \\ -ip_2 & ip_1 & W \end{vmatrix} = 0 \tag{841}$$

gives the relation between energy and momentum for a massless photon, but at the same time, the Majorana equation (837) can be written as a Schrödinger equation

$$H\boldsymbol{\Psi} = W\boldsymbol{\Psi} \tag{842}$$

$$H \equiv -\boldsymbol{\alpha}\cdot\boldsymbol{p} \tag{843}$$

which is usually a nonrelativistic equation for a particle with mass. This is another self-inconsistency of the received Faraday and Ampère–Maxwell laws: the latter ought to be a law for a particle with mass and ought to account for the Lehnert current, as argued already. Operators such as

$$\sum = -i\boldsymbol{\alpha} \times \boldsymbol{\alpha} \tag{844}$$

are intended for the intrinsic spin of the photon, which however, must have eigenvalues $-1, 0, +1$ in order to be consistent with the O(3) angular momentum commutator equation (840). The received view [42,44,46] produces eigenvalues -1 and $+1$ only, which is another self-inconsistency.

Equation (837) can be put into the form of an O(3) covariant derivative acting on the wave function $\boldsymbol{\Psi}$

$$(\partial_0 - igA_0)\boldsymbol{\Psi} \equiv D_0\boldsymbol{\Psi} = \mathbf{0} \tag{845}$$

where

$$gA_0 = \boldsymbol{\alpha}\cdot\frac{\boldsymbol{p}}{\hbar} = \boldsymbol{\alpha}\cdot\boldsymbol{\kappa}, \qquad g = \frac{\kappa}{A^{(0)}} \tag{846}$$

So the simplest form of the Majorana equation is

$$D_0\boldsymbol{\Psi} = \mathbf{0} \tag{847}$$

and is the time-like part of an O(3) covariant derivative acting on the wav function $\boldsymbol{\Psi}$. The form of Eq. (847) is not, however, fully covariant. The full covariant form of the vacuum O(3) field equations, as argued already, i collectively Eqs. (318) and (323), which have a Yang–Mills structure. Therefore the Majorana equation is part of an approximation to the O(3) invariant fiel equations (318) and (323). As argued already, these latter equations give photo mass through the Higgs mechanism. It does not seem possible to introduc photon mass into the Majorana equation (837), revealing that it is an approxi mation. This implies that the received Faraday and Ampère–Maxwell laws in th vacuum are also incomplete [42,44] and that U(1) invariant electrodynamics i incomplete. The latter is seen dramatically in interferometry, as argued in thi review and elsewhere [44]. For example, a U(1) invariant electrodynamics canno describe Sagnac interferometry, with platform either at rest or in motion; an cannot describe Michelson interferometry. An O(3) invariant electrodynamic describes both effects self consistently. Oppenheimer [115] derived the sam equation as Majorana independently a few years later.

The Majorana equation (837) can also be put in the form

$$\bar{\boldsymbol{\Psi}}(W + \boldsymbol{\alpha}\cdot\boldsymbol{p}) = \mathbf{0} \tag{848}$$

which is analogous with the corresponding equation for Dirac–Weyl adjoin wave function. The notation of Eq. (848) means that

$$\boldsymbol{p} \equiv i\frac{\overleftarrow{\partial}}{\partial x}; \qquad \bar{\boldsymbol{\Psi}} = (\boldsymbol{\Psi}^*)^T \tag{849}$$

The symmetric energy-momentum tensor $(T_{\mu\nu})$ of electromagnetism in the vacuum can be defined from the Majorana equation using the matrices

$$2\alpha_{00} = 1; \quad 2\alpha_{01} = \alpha_1; \quad 2\alpha_{02} = \alpha_2; \quad 2\alpha_{03} = \alpha_3$$

$$2\alpha_{11} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad 2\alpha_{12} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2\alpha_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \quad \alpha_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{850}$$

$$2\alpha_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}; \quad 2\alpha_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where

$$\alpha_{\mu\nu} = \alpha_{\nu\mu} \qquad (\mu, \nu = 0, 1, 2, 3) \tag{851}$$

to give the result:

$$T_{\mu\nu} = \bar{\Psi}\alpha_{\mu\nu}\Psi \tag{852}$$

Only eight of the nine matrices (850) are independent, and they form a basis for the SU(3) group, which is used for strong-field theory [46]. Therefore, the energy-momentum tensor is SU(3) invariant.

Therefore, if we start from a traditionally U(1) invariant pair of equations (832) and (833), we find that they can be put into an O(3) invariant form, and that the concomitant energy-momentum tensor is SU(3) invariant. It is therefore interesting to speculate that an SU(3) invariant electrodynamics can be constructed self-consistently, and is more general than the O(3) invariant form developed here and elsewhere [44]. To view electrodynamics in the vacuum as a U(1) invariant theory is highly restrictive, self-inconsistent [44], and in contradiction with ordinary data such as those from ordinary interferometry and ordinary physical optical effects such as normal reflection [44]. Analyses by Majorana, and later Oppenheimer, show that invariance symmetries can be transmuted among each other for the same set of equations, and so it seems that there is no limit to the internal structural symmetry of electrodynamics on both classical and quantum levels. It is necessary to check each set of equations empirically as the theory is developed. The O(3) invariant electrodynamics [44], for example, has been checked extensively with interferometry and other forms of data [47–62] by several leading specialists. Broad agreement has been reached as to the fact that a paradigm shift has occurred, and that the Maxwell–Heaviside electrodynamics have been replaced by one where there can be invariance under symmetry groups different from U(1). This paradigm shift has extensive consequences throughout physics and the ontology of physics, in chemistry, and in cosmology. The next section, for example, shows that the dark matter in the universe can be thought of as being made up of photons with mass slowed to their rest frame by the Higgs mechanism. The Dirac equation itself is SU(2) invariant [46], and therefore a model of the electron must be either SU(2) or O(3) invariant. Vigier has recently developed an O(3) invariant model of the electron [116] based on the development of an O(3) invariant electrodynamics [42,45,47–62]. The Dirac equation is the relativistically correct form of the Schrödinger equation, and an example of an O(3) invariant Schrödinger equation appears in Eq. (842). We argued earlier that the phase of the Schrödinger equation must be O(3) invariant in general. Taking this line of argument to its logical conclusion, then, Newtonian dynamics are also O(3) invariant. The latter is clear from the fact that Newtonian dynamics takes place in the space of three dimensions described by the rotation group O(3). Another insight is obtained from the fact that the angular momentum commutator relations of quantum mechanics [68] are O(3) invariant.

The O(3) invariance of the Majorana equation (837) can be demonstrated clearly by the use of plane waves

$$\begin{aligned} \boldsymbol{A} &= \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} \\ \boldsymbol{B} &= \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} \\ \boldsymbol{E} &= \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi} \end{aligned} \tag{853}$$

whereon

$$-i\boldsymbol{B} + \frac{1}{c}\boldsymbol{E} = -2\kappa i\boldsymbol{A} \tag{854}$$

Therefore, Eqs. (834) reduce to

$$\begin{aligned} W\frac{B_1}{\kappa} &= -ip_2A_3 + ip_3A_2 \\ W\frac{B_2}{\kappa} &= -ip_3A_1 + ip_1A_3 \\ W\frac{B_3}{\kappa} &= -ip_1A_2 + ip_2A_1 \end{aligned} \tag{855}$$

Using the four equations

$$\begin{aligned} W &= p^0 = \hbar g A^0 \\ p_i &= \hbar g A_i; \qquad i = 1, 2, 3 \end{aligned} \tag{856}$$

we recover the O(3) invariant definition of the $\boldsymbol{B}^{(3)}$ field and two other similar equations that are equations of the O(3) invariant field tensor as argued already:

$$B_Z = -i\frac{\kappa}{A^{(0)}}(A_X A_Y^* - A_Y A_X^*) \tag{857}$$

$$\cdots$$

These equations reduce in turn [42,44,47–62] to the B cyclic theorem:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \tag{858}$$

$$\cdots$$

showing that the Majorana equations are the B cyclic theorem. The latter is therefore O(3) gauge-invariant and Lorentz-covariant because the Majorana equations are equivalent to equations with these properties.

XV. SELF-CONSISTENCY OF THE O(3) ANSATZ

A three-way cross-check of the self-consistency of the O(3) ansatz can be carried out starting from Eq. (459), in which $\boldsymbol{A}$ is complex because the electromagnetic field in O(3) electrodynamics carries a topological charge $\kappa/A^{(0)}$. The vector field $\boldsymbol{A}$ in the internal space of O(3) symmetry must depend on x^μ by special relativity and can be written as

$$\boldsymbol{A} = \boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)} \tag{859}$$

where

$$\begin{aligned} \boldsymbol{A}^{(1)} &= \boldsymbol{A}^* = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i\phi} = \boldsymbol{A}^{(2)*} \\ \boldsymbol{A}^{(3)} &= A^{(0)}\boldsymbol{k} \end{aligned} \tag{860}$$

It is now possible to check whether Eq. (459), with its extra vacuum current, is compatible with Eq. (106) of Ref. 44, which is

$$\nabla \times \boldsymbol{B}^{(3)} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} - ig(\boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)} - \boldsymbol{A}^{(2)} \times \boldsymbol{B}^{(1)}) - \frac{g}{\mu_0} D^\mu \boldsymbol{A}^* \times \boldsymbol{A} \tag{861}$$

It follows, from the structure adopted for $\boldsymbol{A}$ in Eq. (860), that

$$\begin{aligned} D^3\boldsymbol{A}^* \times \boldsymbol{A} &= \partial^3\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(1)} + \partial^3\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \\ &= i\kappa\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} - i\kappa\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(1)} \\ &= ig\boldsymbol{B}^{(1)} \times \boldsymbol{A}^{(2)} - ig\boldsymbol{B}^{(2)} \times \boldsymbol{A}^{(1)} \end{aligned} \tag{862}$$

and so we obtain

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0} \tag{863}$$

which is self-consistent with the fact that $\boldsymbol{B}^{(3)}$ is irrotational and that $\boldsymbol{E}^{(3)}$ is zero.

Another consequence of Eq. (459) is that it gives a vacuum polarization

$$\nabla \cdot \boldsymbol{P}^{(3)} = \rho(\text{vac}) = -\frac{g}{\mu_0} D^0 \boldsymbol{A}^* \times \boldsymbol{A} \tag{864}$$

where $\rho(\text{vac})$ is the vacuum charge density. The vacuum polarization $\boldsymbol{P}^{(3)}$ does not appear from the field tensor [42], but appears from the vacuum charge current density term on the right-hand side of Eq. (459). This vacuum charge current density term must always be present from fundamental gauge principles on the O(3) level. So we have identified the concept of a vacuum charge density as the divergence of a vacuum polarization.

The concepts of O(3) electrodynamics developed in this review and in Ref. 44 scratch the surface of what is possible. The field equations must be solved numerically to obtain all the possible solutions, and checked against empirical data at each stage. Numerical solution of this nature has not yet been attempted. The concept of radiatively induced fermion resonance [44], which might lead to nuclear magnetic resonance and electron spin resonance without the need for permanent magnets, is one obviously useful spinoff of O(3) electrodynamics that has not been explored. These are two of several major advances that could be made within the near future. On the high-energy scale, the concept of higher-symmetry electrodynamics has led to the Crowell boson, which has been detected empirically, and, as reviewed by Crowell in this edition, leads to a novel grand unified theory. The development of O(3) electrodynamics also gives better insight into the energy inherent in the vacuum, and shows beyond reasonable doubt that all optical phenomena are O(3) invariant, a major advance in the 400-year subject of physical optics. During the course of this development, it has been shown that there are several internal inconsistencies [44] in the U(1) invariant electrodynamics, and several instances, in particular interferometry, where the U(1) theory fails. Two typical examples are the Sagnac and Michelson effects. The O(3) invariant electrodynamics succeeds in describing both effects with precision from first principles because of the use of a non-Abelian Stokes theorem for the electromagnetic phase, a theorem that shows that all interferometry is topological in nature and depends on the Evans–Vigier field $\boldsymbol{B}^{(3)}$. The O(3) invariant electrodynamics carries a topological charge $\kappa/A^{(0)}$ in the vacuum, a charge that also acts as the coupling constant of the O(3) covariant derivative. The concept of vacuum charge current density has been established self-consistently on the O(3) invariant level from the first principles of gauge field theory. These are some of several major advances.

Therefore, the empirical and theoretical evidence for the superiority of an O(3) invariant over a U(1) invariant electrodynamics is overwhelming. It is clear that the process of development can be continued, for example, in quantum electrodynamics, electroweak theory, and grand unified theory, and the ontology of these developments can also be studied in parallel.

XVI. THE AHARONOV–BOHM EFFECT AS THE BASIS OF ELECTROMAGNETIC ENERGY INHERENT IN THE VACUUM

The Aharonov–Bohm effect shows that the vacuum is configured or structured, and that the configuration can be described by gauge theory [46]. The result of this experiment is that, in the structured vacuum, the vector potential $\boldsymbol{A}$ can be nonzero while the electric field strength $\boldsymbol{E}$ and magnetic flux density $\boldsymbol{B}$ can be zero. This empirical result is developed in this section by defining an inner space for the gauge theory, and by summarizing some of the results proposed earlier in this review in light of the Aharonov–Bohm effect. Therefore the non-simply connected U(1) vacuum is described by a scalar internal gauge space, and the non-simply connected O(3) vacuum, by a vector internal gauge space. The core of the idea being presented in this section is that the Aharonov–Bohm effect is a local gauge transformation of the true vacuum, where

$$A_\mu = 0 \tag{865}$$

This type of gauge transformation produces a vector potential from the true vacuum. Components of this vector potential are used for the internal gauge space whose Lagrangian is globally invariant. A local gauge transformation of this Lagrangian produces the topological charge

$$g = \frac{\kappa}{A^{(0)}} \tag{866}$$

the electromagnetic field, which carries energy, and the vacuum charge current density first proposed empirically by Lehnert [49] and developed by Lehnert and Roy [45]. These authors have also demonstrated that the existence of a vacuum charge current density implies the existence of photon mass. Empirical evidence for the existence of the vacuum charge current density is available from total internal reflection [45,49]. The source of the energy inherent in vacuo is therefore spacetime curvature introduced through the use of a covariant derivative:

$$D_\mu \equiv \partial_\mu - igA_\mu \tag{867}$$

The product gA_μ in the covariant derivative is, within a factor $\hbar$, an energy momentum. Therefore, photon mass is produced by spacetime curvature because, in a covariant derivative, the axes vary from point to point and there is spacetime curvature. Similarly, mass is produced by spacetime curvature in general relativity. Therefore, spacetime curvature in the configured vacuum implied by the Aharonov–Bohm effect is the source of electromagnetic energy momentum in the vacuum. There is no theoretical upper bound to the magnitude of this electromagnetic energy momentum, which can be picked up by devices, as reviewed in this series by Bearden and Fox (Part 2, Chapters 11 and 12; this part, Chapter 11). Therefore, devices can be manufactured, in principle, to take an unlimited amount of electromagnetic energy from the configured vacuum as defined by the Aharonov–Bohm effect, without violation of Noether's theorem.

The gauge theory developed earlier in this review is summarized for the U(1) and O(3) non-simply connected vacua using the appropriate internal gauge spaces. The earlier calculations are summarized in this section. It has been demonstrated in this series, that there are several advantages of O(3) gauge theory applied to electrodynamics over U(1) gauge theory applied to electrodynamics, but the latter can be used to illustrate the method and to produce the vacuum Poynting theorem that is an expression of Noether's theorem for the structured vacuum. The theory being used is standard gauge theory, so the Noether theorem is conserved; that is, the laws of energy/momentum and charge current conservation are obeyed. The magnitude of the energy momentum is not bounded above by gauge theory, so the Poynting theorem (law of conservation of electromagnetic energy) in the configured vacuum indicates this fact through the presence of a constant of integration whose magnitude is not bounded above. This suggests that the magnitude of the electromagnetic energy in the structured classical vacuum is, in effect, limitless.

The non-simply connected U(1) vacuum is considered first to illustrate the method as simply as possible. This is defined as earlier in this review by the globally invariant Lagrangian density

$$\mathscr{L} = \partial_\mu A \partial^\mu A^* \tag{868}$$

where A and A^* are considered to be independent complex scalar components of the vector potential obtained by gauge transformation of the true vacuum, where $A_\mu = 0$ [46]. The potentials A and A^* are complex because they are associated with a topological charge g, which appears in the covariant derivative when the Lagrangian (868) is subjected to a local gauge transformation. The topological charge g should not be confused with the point charge e on the proton. In the classical structured vacuum, g exists but e does not exist. The two scalar fields are therefore defined as complex conjugates:

$$A = \frac{1}{\sqrt{2}}(A_1 + iA_2) \tag{869}$$

$$A^* = \frac{1}{\sqrt{2}}(A_1 - iA_2) \tag{870}$$

The two independent Euler–Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial A} = \partial_\nu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu A)}\right); \qquad \frac{\partial \mathscr{L}}{\partial A^*} = \partial_\nu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu A^*)}\right) \tag{871}$$

produce the independent d'Alembert equations of the structured vacuum:

$$\Box A = 0; \qquad \Box A^* = 0 \tag{872}$$

The Lagrangian (868) is invariant under a global gauge transformation:

$$A \to e^{-i\Lambda}A; \qquad A^* \to e^{i\Lambda}A^* \tag{873}$$

where Λ is a number. Under a local gauge transformation, however

$$A \to e^{-i\Lambda(x^\mu)}A; \qquad A^* \to e^{i\Lambda(x^\mu)}A^* \tag{874}$$

where Λ becomes a function of the spacetime coordinate x^μ by special relativity. Under a local gauge transformation [46] of the structured U(1) vacuum defined by the Lagrangian (868), the latter is changed to

$$\mathscr{L} = D_\mu A D^\mu A^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{875}$$

as argued earlier in this review. Here, $F_{\mu\nu}$ is the U(1) invariant electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{876}$$

where the covariant derivatives are defined by

$$D_\mu A = (\partial_\mu + igA_\mu)A \tag{877}$$

$$D^\mu A^* = (\partial^\mu - igA^\mu)A^* \tag{878}$$

Here, A_μ is the vector 4-potential introduced in the vacuum as part of the covariant derivative, and therefore introduced by spacetime curvature. The electromagnetic field and the topological charge g are the results of the invariance of the Lagrangian (868) under local U(1) gauge transformation, in other words, the results of spacetime curvature.

By using the Euler–Lagrange equation

$$\partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu A_\mu)}\right) = \frac{\partial\mathscr{L}}{\partial A_\mu} \tag{879}$$

with the Lagrangian (875), we obtain the field equation of the U(1) structured vacuum

$$\partial_\nu F^{\mu\nu} = -igc(A^* D^\mu A - AD^\mu A^*) \tag{880}$$

a field equation that identifies the vacuum charge current density

$$J^\mu(\text{vac}) \equiv -igc\varepsilon_0(A^* D^\mu A - AD^\mu A^*) \tag{881}$$

first introduced and developed by Lehnert et al. [45,49]. Equation (880) is an inhomogeneous field equation of the configured U(1) vacuum, and gives rise to the inherent energy of the configured vacuum

$$En = \int J^\mu(\text{vac})A_\mu\, dV \tag{882}$$

and rate of doing work by the configured vacuum

$$\frac{dW}{dt} = \int \boldsymbol{J}(\text{vac})\cdot\boldsymbol{E}\, dV \tag{883}$$

where $\boldsymbol{E}$ is the electric field strength of the field tensor $F_{\mu\nu}$. The volume V is arbitrary, and standard methods of U(1) invariant electrodynamics give the Poynting theorem of the U(1) configured vacuum:

$$\frac{dU(\text{vac})}{dt} + \nabla\cdot\boldsymbol{S}(\text{vac}) = -\boldsymbol{J}(\text{vac})\cdot\boldsymbol{E} \tag{884}$$

Here, $\boldsymbol{S}$(vac) is the Poynting vector of the U(1) configured vacuum, representing electromagnetic energy flow, and is defined by

$$\nabla\cdot\boldsymbol{S}(\text{vac}) = -\boldsymbol{J}(\text{vac})\cdot\boldsymbol{E} \tag{885}$$

Integrating this equation gives

$$\boldsymbol{S}(\text{vac}) = -\int \boldsymbol{J}(\text{vac})\cdot\boldsymbol{E}\, dr + \text{constant} \tag{886}$$

where the constant of integration is not bounded above. The electromagnetic energy flow inherent in the U(1) configured vacuum is not bounded above, meaning that there is an unlimited amount of electromagnetic energy flow available in theory, for use in devices. Some of these devices are reviewed in this edition by Bearden and Fox [chapters given above, in text following Eq. (867)]. Sometimes, the constant of integration is referred to as the "Heaviside component of the vacuum electromagnetic energy flow," and the detailed nature of this component is not restricted in any way by gauge theory. The Poynting theorem (884) is, of course, the result of gauge theory.

In the non-simply connected O(3) vacuum, the internal gauge space is a vector space rather than the scalar space of the U(1) vacuum. Therefore, we can summarize and collect earlier results of this review using the concept of an O(3) symmetry internal gauge space, a space in which there exist complex vectors A and A^*. The globally invariant Lagrangian density for this internal space is

$$\mathscr{L} = \partial_\mu A \cdot \partial^\mu A^* \tag{887}$$

and the two independent Euler–Lagrange equations are

$$\frac{\partial \mathscr{L}}{\partial A} = \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial_\nu A} \right); \qquad \frac{\partial \mathscr{L}}{\partial A^*} = \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial_\nu A^*} \right) \tag{888}$$

giving the d'Alembert equations

$$\Box A = 0; \qquad \Box A^* = \mathbf{0} \tag{889}$$

Under the local O(3) invariant gauge transformation

$$A \to e^{iJ_i \Lambda_i} A; \qquad A^* \to e^{-iJ_i \Lambda_i} A^* \tag{890}$$

the Lagrangian (887) becomes, as we have argued earlier

$$\mathscr{L} = D_\mu A \cdot D^\mu A^* - \frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu} \tag{891}$$

and using the Euler–Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial A_\mu} = \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial (\partial_\nu A_\mu)} \right) \tag{892}$$

the inhomogeneous O(3) invariant field equation is obtained

$$D_\nu G^{\mu\nu} = -g D^\mu A^* \times A \tag{893}$$

as shown in detail earlier. The term on the right-hand side is the O(3) invariant vacuum charge current density that is the non-Abelian equivalent of the right-hand side of Eq. (880). In general, Eq. (893) must be solved numerically, but the presence of a vacuum charge current density gives rise to the energy of the O(3) configured vacuum

$$En\,(\text{vac}) = \int j^\mu(\text{vac}) \cdot A_\mu \, dV \tag{894}$$

whose source is curvature of spacetime introduced by the O(3) covariant derivative containing the rotation generators J_i of the O(3) group. The curvature

of spacetime is also the source of photon mass, in analogy with general relativity, where curvature of spacetime occurs in the presence of mass or a gravitating object.

Therefore, in summary, the empirical basis of the development in this section is that the Aharonov–Bohm effect shows that, in regions where $\boldsymbol{E}$ and $\boldsymbol{B}$ are both zero, $\boldsymbol{A}$ can be nonzero. Therefore, the Aharonov–Bohm effect can be regarded as a local gauge transformation of the true vacuum, defined by $A_\mu = 0$, and the Aharonov–Bohm effect shows that a nonzero A_μ can be generated by a local gauge transformation from regions in which A_μ is zero. Therefore, in a structured vacuum, it is possible to construct a gauge theory whose internal space is defined by components of A_μ in the absence of an electromagnetic field. The latter is generated by a local gauge transformation of components of an A_μ which was generated originally by a local gauge transformation of the true vacuum where $A_\mu = 0$. This concept is true for all gauge group symmetries. It is well known that contemporary gauge theories lead to richly structured vacua whose properties are determined by topology [46]. The Yang–Mills vacuum, for example, is infinitely degenerate [46]. Therefore local gauge transformation can produce electromagnetic energy, a vacuum charge current density, a vacuum Poynting theorem, and photon mass, all interrelated concepts. We reach the sensible conclusion, that in the presence of a gravitating object (a photon with mass), spacetime is curved. The curvature is described through the covariant derivative for all gauge group symmetries. The energy inherent in the vacuum is contained in the electromagnetic field, and the coefficient g is a topological charge inherent in the vacuum. For all gauge group symmetries, the product gA_μ is energy momentum within a factor $\hbar$, indicating clearly that the covariant derivative applied in the vacuum contains energy momentum produced on the classical level by spacetime curvature. This energy momentum, as in general relativity, is not bounded above, so the electromagnetic energy inherent in the classical structured vacuum is not bounded above. There appear to be several devices available that extract this vacuum energy, and these are reviewed in this compilation by Bearden and Fox. In theory, the amount of energy appears to be unlimited.

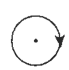
The Aharonov–Bohm effect depends on the group space of the internal space used in the gauge theory. If this internal space is U(1), the group space is a circle, which is denoted in topology [46] by S^1. This group space is not simply connected because a path that goes twice around a circle cannot be continuously deformed, while staying on the circle, to one that goes around only once [46]. A curve going around the solenoid n times cannot be shrunk to one around m times, where $m \neq n$. The configuration space of the vacuum is therefore not simply connected, and this allows a gauge transform of the pure vacuum, to create what is known as a "pure gauge vacuum" [46]. In U(1) gauge theory, the mathematical reason for the Aharonov–Bohm effect is that the configuration space of the null field (pure gauge vacuum) is a ring, denoted by $S^1 \times R$ in topology [46]. The vector potential in the pure gauge vacuum is derived from a gauge function that maps the gauge space in to the configuration space. These mappings are not all deformable to a constant gauge function χ, which would give a zero $\nabla\chi$ in the pure gauge vacuum and a null Aharonov–Bohm effect. This, then, is the conventional U(1) invariant explanation of the Aharonov–Bohm effect.

The O(3) invariant explanation, as we have seen, uses an internal gauge space that is the physical space O(3). This space is doubly connected [46]. The group space of O(3) is obtained by identifying opposite points on the 3-space S^3, which is the topological description of the unit sphere in four-dimensional Euclidean space, denoted E^4. Opposite points on the 3-space S^3 correspond to the same O(3) transformation. It is possible to show that this space is doubly connected by considering closed curves S^1 in the group space of O(3). One can consider paths [46] that may be shrunk to (are homotopic to) a point and to a straight line. These are the two types of closed path S^1 in the group space of O(3), with the implication that there is one nontrivial vortex in an O(3) gauge theory.

The simplest example of the O(3) invariant Aharonov–Bohm effect is the equation of interferometry

$$\oint \boldsymbol{A}^{(3)} \cdot d\boldsymbol{r} = \int \boldsymbol{B}^{(3)} \cdot d\boldsymbol{S} \tag{895}$$

used in the region outside the solenoid where the vector potential sketched below is nonzero:



$\boldsymbol{A}^{(3)}$ (896)

The line integral is defined over the circular path, exactly as in the O(3) invariant explanation of the Sagnac effect discussed earlier in this review and in Vol. 114, part 2. The key difference between the O(3) and U(1) invariant explanations of the Aharonov–Bohm effect is that, in the former, there is a magnetic field $\boldsymbol{B}^{(3)}$ present at the point of contact with the electrons. Agreement with the empirical data is obtained because

$$|\boldsymbol{B}^{(3)}| = |\boldsymbol{B}| \tag{897}$$

that is the total magnetic flux inside the area S must be generated by the static magnetic field $\boldsymbol{B}$ of the solenoid. The fact that we are using an O(3) gauge theory means that the configuration space of the vacuum is doubly connected. As discussed in the technical appendix, the vector potential $\boldsymbol{A}^{(3)}$ in Eq. (895) can be

regarded as having been generated by an O(3) gauge transformation that leaves $\boldsymbol{B}^{(3)}$ invariant. Equation (895) is the consequence of a round trip in spacetime using parallel transport with O(3) covariant derivatives. Therefore, the simplest O(3) invariant explanation of the Aharonov–Bohm effect simply means that it is an interferometric effect, very similar in nature to the O(3) invariant explanation of the Sagnac effect or Michelson interferometry.

The simplest example of the generation of energy from a pure gauge vacuum is to consider the case of an electromagnetic potential plane wave defined by

$$\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} \tag{898}$$

The pure gauge vacuum is then defined by

$$\boldsymbol{A} \neq 0; \qquad \boldsymbol{E} = \boldsymbol{0}; \qquad \boldsymbol{B} = \boldsymbol{0} \tag{899}$$

and a Lagrangian density can be constructed which is proportional to

$$\mathscr{L} = \partial_\mu \boldsymbol{A} \cdot \partial^\mu \boldsymbol{A} \tag{900}$$

A global gauge transformation of $\boldsymbol{A}$ in the pure gauge vacuum is equivalent to a rotation of $\boldsymbol{A}$ through an angle Λ [46], producing a conserved quantity Q as the result of the invariance of the action under the global gauge transformation. It can be shown as follows that Q is proportional to conserved electromagnetic kinetic energy

$$En = \frac{1}{\mu_0}\int B^{(0)2}\, dV \tag{901}$$

generated by the global gauge transformation of the pure gauge vacuum, which, in turn, is generated from the pure vacuum by a local gauge transformation.

For plane waves

$$A_1 = i\frac{A^{(0)}}{\sqrt{2}}e^{-i(\omega t - \kappa Z)}; \qquad A_2 = \frac{A^{(0)}}{\sqrt{2}}e^{-i(\omega t - \kappa Z)} \tag{902}$$

In a U(1) invariant theory, the pure gauge vacuum is defined by a scalar internal gauge space in which there exist the independent complex scalar fields:

$$A = \frac{1}{\sqrt{2}}(A_1 + iA_2); \qquad A^* = \frac{1}{\sqrt{2}}(A_1 - iA_2) \tag{903}$$

These are complex scalar fields because there is an invariant topological charge present, defined by

$$g = \frac{\kappa}{A^{(0)}} \tag{904}$$

The Lagrangian density produced by these scalar fields is, as we have seen

$$\mathscr{L} = \partial_\mu A \partial^\mu A^* \tag{905}$$

and the global gauge transformation is defined by

$$A \to e^{-i\Lambda} A; \qquad A^* \to e^{i\Lambda} A^* \tag{906}$$

This type of transformation is not dependent on spacetime and is purely internal [46] in Noether's theorem. Under a global gauge transformation, Noether's theorem gives the conserved current

$$J^\mu = igc(A^* \partial^\mu A - A \partial^\mu A^*) \tag{907}$$

with a vanishing 4-divergence and a conserved topological charge:

$$Q = \int J^0 \, dV \tag{908}$$

From Eq. (907), the conserved topological charge Q is

$$Q = \frac{2c}{A^{(0)}} \int \kappa^2 A^{(0)2} \, dV = \frac{2c}{A^{(0)}} \int B^{(0)2} \, dV \tag{909}$$

which can be written as

$$Q = \frac{2c\mu_0}{A^{(0)}} En \tag{910}$$

where

$$En = \frac{1}{\mu_0} \int B^{(0)2} \, dV \tag{911}$$

is a conserved kinetic electromagnetic energy. For a monochromatic plane wave in the vacuum, the quantity g is also conserved because κ and $A^{(0)}$ do not change. Therefore it has been demonstrated that, in a pure gauge vacuum defined by the plane wave $\boldsymbol{A}$, conserved electromagnetic energy density is generated by a global gauge transformation, which is a rotation of $\boldsymbol{A}$ through the angle Λ.

This is the simplest example of the generation of kinetic electromagnetic energy by a gauge transformation of a pure gauge vacuum defined initially by a nonzero $\boldsymbol{A}$ and zero $\boldsymbol{E}$ and $\boldsymbol{B}$. The more complete description of energy generated from the pure gauge vacuum is given by a local gauge transformation, as argued already in this review on the U(1) and O(3) levels. It is to be noted that the conserved quantity Q has the following properties:

1. It is time independent.
2. It does not depend on the charge on the proton.
3. It is a classical quantity.
4. It is not integer-valued and when $\boldsymbol{A}$ is real it vanishes.

It can be shown as follows that the transition from a pure vacuum to a pure gauge vacuum is described by the spacetime translation generator of the Poincaré group. The pure vacuum on the U(1) invariant level is described by the field equations:

$$\partial_\mu \tilde{F}^{\mu\nu} \equiv 0 \tag{912}$$

$$\partial_\mu F^{\mu\nu} = 0 \tag{913}$$

with

$$\tilde{F}^{\mu\nu} = 0; \qquad F^{\mu\nu} \equiv 0 \tag{914}$$

So the kinetic electromagnetic energy term in the Lagrangian

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{915}$$

is zero. In the pure gauge vacuum, the ordinary derivative is replaced by the covariant derivative, so the field equations (912) and (913) become

$$\partial_\mu \tilde{F}^{\mu\nu} = -iA_\mu \tilde{F}^{\mu\nu} \tag{916}$$

$$\partial_\mu F^{\mu\nu} = -iA_\mu F^{\mu\nu} \tag{917}$$

where A_μ is defined by

$$A_\mu = -\frac{i}{g}(\partial_\mu S) S^{-1} \tag{918}$$

but where the fields $\tilde{F}^{\mu\nu}$ and $F^{\mu\nu}$ are still zero. Therefore

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = 0 \tag{919}$$

and the contribution of the field to the energy in a pure gauge vacuum is zero. However, there occurs an energy change from a pure vacuum to a pure gauge vacuum, an energy change proportional to gA_μ. The origin of this energy change is topological; that is, the energy change can be traced to the replacement of the ordinary derivative ∂_μ by the covariant derivative D_μ.

Essentially, this replacement means that the spacetime changes from one that is conformally flat to one that is conformally curved; in other words, the axes vary from point to point whenever a covariant derivative is used for any gauge group symmetry. It is this variation of the axes that introduces energy into a pure gauge vacuum. The covariant derivative in the latter is

$$D_\mu = \partial_\mu - igA_\mu \tag{920}$$

which can be written using the rule $i\partial_\mu = \kappa_\mu$ as

$$\kappa_\mu \rightarrow \kappa_\mu + \kappa'_\mu \tag{921}$$

This expression is equivalent [42] to

$$P_\mu \rightarrow P_\mu + P'_\mu \tag{922}$$

where P_μ is the spacetime translation generator of the Poincaré group. Within a factor $\hbar$, the spacetime translation generator is the energy-momentum 4-vector. It becomes clear that the use of a covariant derivative introduces energy momentum into the vacuum, in this case a pure gauge vacuum. Lagrangians, consisting of components of A_μ in the pure gauge vacuum when subjected to a local gauge transformation, give the electromagnetic field and its source, the vacuum charge/current density, first introduced empirically by Lehnert [49].

In the final part of this section, the method of local gauge transformation is outlined in detail to show how the electromagnetic field and conserved vacuum charge current density emerge from the local gauge transformation of the pure gauge vacuum. The illustration is given for convenience in a U(1) invariant theory, and leans heavily on the excellent account given by Ryder [46, pp. 94ff.]. We therefore consider a local gauge transformation of a pure gauge vacuum with scalar components A and A^*:

$$\begin{aligned} A &\rightarrow \exp(-i\Lambda(x^\mu))A \\ A^* &\rightarrow \exp(-i\Lambda(x^\mu))A^* \end{aligned} \tag{923}$$

For $\Lambda \ll 1$

$$\delta A = -i\Lambda A \tag{924}$$

and

$$\partial_\mu A \rightarrow \partial_\mu A - i(\partial_\mu \Lambda)A - i\Lambda(\partial_\mu A) \tag{925}$$

Therefore

$$\delta(\partial_\mu A) = -i\Lambda(\partial_\mu A) - i(\partial_\mu \Lambda)A \tag{926}$$

and

$$\begin{aligned} \delta A^* &= i\Lambda A^* \\ \delta(\partial_\mu A^*) &= i\Lambda(\partial_\mu A^*) + i(\partial_\mu \Lambda)A^* \end{aligned} \tag{927}$$

The effect of the local gauge transform is to introduce an extra term $\partial_\mu \Lambda$ in the transformation of the derivatives of fields. Therefore, $\partial_\mu A$ does not transform covariantly, that is, does not transform in the same way as A itself. These extra terms destroy the invariance of the action under the local gauge transformation because the change in the Lagrangian is

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial A}\delta A + \frac{\partial\mathscr{L}}{\partial(\partial_\mu A)}\delta(\partial_\mu A) + (A \rightarrow A^*) \tag{928}$$

where $(A \rightarrow A^*)$ denotes the two additional terms in A^*. Substituting the Euler-Lagrange equation (888) into the first term, and using Eqs. (924)–(926), gives

$$\begin{aligned} \partial\mathscr{L} &= \partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu A)}(-i\Lambda A) + \frac{\partial\mathscr{L}}{\partial(\partial_\mu A)}(-i\Lambda\partial_\mu A - iA\partial_\mu \Lambda) \\ &= -i\Lambda\partial_\mu \frac{\partial\mathscr{L}A}{\partial(\partial_\mu A)} - i\frac{\partial\mathscr{L}}{\partial(\partial_\mu A)}(\partial_\mu \Lambda)A + (A \rightarrow A^*) \end{aligned} \tag{929}$$

The first term is a total divergence, so the corresponding change in the action is zero. Using

$$\mathscr{L} = (\partial_\mu A)(\partial^\mu A^*) - m^2 A^* A \tag{930}$$

for the Lagrangian then gives

$$\delta\mathscr{L} = i\partial_\mu \Lambda(A^*\partial^\mu A - A\partial^\mu A^*) = J^\mu \partial_\mu \Lambda \tag{931}$$

where the (SI) current is given by Eq. (907), in reduced units

$$J^\mu = i(A^*\partial^\mu A - A\partial^\mu A^*) \tag{932}$$

The action is therefore not invariant under local gauge transformation. To restore invariance the four potential, A_μ must be introduced into the pure gauge vacuum to give the Lagrangian

$$\mathscr{L}_1 = -gJ^\mu A_\mu \\ = -ig(A^*\partial^\mu A - A\partial^\mu A^*)A_\mu \tag{933}$$

where g is the topological charge in the vacuum, defined in such a way that gA_μ has the same SI units as κ_μ. On the U(1) level, local gauge transformation means that

$$A_\mu \rightarrow A_\mu + \frac{1}{g}\partial_\mu \Lambda \tag{934}$$

so that

$$\delta\mathscr{L}_1 = -g(\delta J^\mu)A_\mu - gJ^\mu(\delta A_\mu) \\ = -g(\delta J^\mu)A_\mu - J^\mu\partial_\mu\Lambda \tag{935}$$

The action is still not invariant under a local gauge transformation, however, because of the presence of the term $-g(\delta J^\mu)A_\mu$ on the right-hand side of Eq. (935), a term in which

$$\delta J^\mu = i\delta(A^*\partial^\mu A - A\partial^\mu A^*) \\ = 2A^*A\partial^\mu\Lambda \tag{936}$$

so that

$$\delta\mathscr{L} + \delta\mathscr{L}_1 = -2gA_\mu(\partial^\mu\Lambda)A^*A \tag{937}$$

Therefore, another term must be added to the Lagrangian $\mathscr{L}$:

$$\mathscr{L}_2 = e^2 A_\mu A^\mu A^* A \tag{938}$$

Using Eq. (934), we find that

$$\delta\mathscr{L}_2 = 2g^2 A_\mu \delta A^\mu A^* A = 2gA_\mu(\partial^\mu\Lambda)A^*A \tag{939}$$

so that

$$\delta\mathscr{L} + \delta\mathscr{L}_1 + \delta\mathscr{L}_2 = 0 \tag{940}$$

The total Lagrangian $\mathscr{L} + \mathscr{L}_1 + \mathscr{L}_2$ is now invariant under the local gau transformation because of the introduction of the 4-potential A_μ, which coup to the current J_μ of the complex A of the pure gauge vacuum. The field A_μ a contributes to the Lagrangian, and since $\mathscr{L} + \mathscr{L}_1 + \mathscr{L}_2$ is invariant, an ex term $\mathscr{L}_3$ appears, which must also be gauge-invariant. This can be so only if electromagnetic field is introduced

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (94$$

so that

$$\mathscr{L}_3 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad (94$$

The total invariant Lagrangian is therefore

$$\mathscr{L}_{\text{tot}} = \mathscr{L} + \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3 \\ = (\partial_\mu A + igA_\mu A)(\partial^\mu A^* - igA^\mu A^*) - m^2 A^* A - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad (94$$

The Lehnert field equation is obtained from this Lagrangian using the Eule Lagrange equation

$$\frac{\partial\mathscr{L}}{\partial A_\mu} - \partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu A_\mu)}\right) = 0 \qquad (94$$

giving in SI units

$$\partial_\nu F^{\mu\nu} = -igc(A^*D^\mu A - AD^\mu A^*) \qquad (94$$

It is noted that the Lehnert charge current density

$$J^\mu = -i\varepsilon_0 gc(A^*D^\mu A - AD^\mu A^*) \qquad (94$$

gauge-covariant and also conserved, and thus cannot be gauged to zero by ar ethod of gauge transformation. It is the direct result of a local gau ansformation on a pure gauge vacuum and acts as the source of the vacuu ectromagnetic field $F^{\mu\nu}$, as discussed already. The covariant current (946) onserved because

$$\partial_\mu J^\mu = 0 \qquad (94$$

XVII. INTRODUCTION TO THE WORK OF PROFESSOR J. P. VIGIER

We append what we believe to be a comprehensive listing of the publications of Professor Jean-Pierre Vigier. They represent a wide range of topics from the interpretation of quantum mechanics, particle physics, cosmology, and relativistic physics. What is remarkable about this list is not just the breadth of topics, but the philosophical consistency that underlies the physics. Firmly rejecting the orthodox interpretation of quantum mechanics, particles of all types are, at all times, regarded as objectively existing entities with their own internal structure. Particles are guided by pilot waves, so the dualism of orthodoxy is firmly rejected in favor of realist ontology.

What follows is a brief account of Professor Vigier's life and career as related to one of us (S. Jeffers) in a series of conversations held in Paris during the summer of 1999. A more complete version of these conversations will appear in a book being compiled by Apeiron Press and the Royal Swedish Academy to mark the 80th birthday of Professor Vigier. A comprehensive biography of this remarkable man, whose life has witnessed major revolutions both in physics and in politics (his twin passions), remains to be written.

"Great physicists fight great battles"—so wrote Professor Vigier in an essay he wrote in a tribute to his old friend and mentor Louis de Broglie. However, this phrase could be applied equally well to Vigier himself. He has waged battle on two fronts—within physics and within politics. Now almost 80, he still continues to battle.

He was born on January 16, 1920 to Henri and Françoise (née Dupuy) Vigier. He was one of three brothers, Phillipe (deceased) and François, currently Professor of Architecture at Harvard University. His father was Professor of English at the École Normale Supérieure—hence Vigier's mastery of that language. He attended an international school in Geneva at the time of the Spanish civil war. This event aroused his intense interest in politics, as most of his school friends were both Spanish and Republicans. Vigier was intensely interested in both physics and mathematics, and was sent by his parents to Paris in 1938 to study both subjects. For Vigier, mathematics is more like an abstract game, his primary interest being in physics as it rests on two legs, the empirical and the theoretical.

All the young soldiers were sent to Les Chantiers de la Jeunesse, and it was there that he joined the Communist Party. The young radicals were involved in acts of sabotage near the Spanish border, such as oiling the highways to impede the progress of the fascists. At that time, the French Communist Party was deeply split concerning the level of support to be given to the Résistance. A few leaders went immediately to the Résistance, while others, like Thorez, wavered. In the period before the Nazi attack on the Soviet Union, the party equivocated with respect to the Résistance. At that time, Vigier was in a part of France

controlled by the famous communist leader, Tillion, who had participated in the revolt of the sailors in the Black Sea in 1918. Tillion immediately organized groups of resistance fighters called the *Organisation Spéciale*. Vigier was involved in bombing campaigns against both the Nazis and Vichy collaborators in the Free Zone.

In Geneva, Vigier was involved in communicating between the French military communist staff and Russia, until he was arrested at the French border in the spring of 1942 and taken to Vichy. There, the French police interrogated him as he was carrying coded documents. Two police officers brought him by train from Vichy to Lyon to be delivered into the hands of the notorious Klaus Barbie. Fortunately, the train was bombed by the English, and Vigier managed to jump through a window, escaped to the mountains, and resumed his activities with the Résistance until the end of the war. He became an officer in the FTP movement (Francs-Tireurs et Partisans, meaning sharpshooters and supporters). When De Gaulle returned to France, part of the Résistance forces were converted to regular army units. The cold war started almost immediately after the defeat of the Germans. Vigier was still a member of the French General Staff while completing the requirements for a Ph.D. in mathematics in Geneva. Then the communists were kicked out of the General Staff and Vigier went to work for Joliot-Curie. He, in turn, lost his job for refusing to build an atomic bomb for the French government. Vigier became unemployed for a while and then learned, through an accidental meeting with Joliot-Curie, that Louis de Broglie was looking for an assistant. When he met De Broglie, the only questions asked were "Do you have a Ph.D. in mathematics?" and "Do you want to do physics?" He was hired immediately in 1948, and with no questions asked about his political views. Although Secretary of the French Academy of Science, de Broglie was marginalized within physics circles given his well-known opposition to the Copenhagen interpretation of quantum mechanics. Notwithstanding his Nobel prize, de Broglie had difficulty in finding an assistant. Vigier entered the CNRS (Centre national de la recherche scientifique) and worked with De Broglie until his retirement. Vigier's political involvement at that time included responsibility for the French communist student movement.

In 1952, a visiting American physicist named Yevick, gave a seminar at the CNRS on the recent ideas of David Bohm. Vigier reports that upon hearing of this work, De Broglie became radiant and commented that these ideas were first considered by himself a long time ago. Bohm had gone beyond De Broglie's original ideas however. De Broglie charged Vigier with reading all of Bohm's works in order to prepare a seminar. De Broglie went back to his old ideas, and both he and Vigier started working on the causal interpretation of quantum mechanics. At the 1927 Solvay Congress, de Broglie had been shouted down, but now, following the work of Bohm, there was renewed interest in his idea that wave and particle could coexist, eliminating the need for dualism. Vigier recalls that at that time, the catholic archbishop of Paris who exclaimed that everyone knew that Bohr was right, upbraided de Broglie, and how de Broglie could possibly believe otherwise. Although a devout Christian, he was inclined to materialist philosophy in matters of physics.

Vigier comments on his days with de Broglie that he was a very timid man who would meticulously prepare his lectures in written form—in fact, his books are largely compendia of his lectures. He also recalls one particular incident that illustrates de Broglie's commitment to physics. Vigier was in the habit of meeting weekly with de Broglie to take direction as to what papers he should be reading, and what calculations he should be focusing on. On one of these occasions, he was waiting in an anteroom for his appointment with de Broglie. Also waiting was none other than the French Prime Minister Edgar Faure who had come on a courtesy visit in order to discuss his possible membership in the French Academy. When the door finally opened, de Broglie called excitedly for Mr. Vigier to enter as he had some important calculations for him to do, and as for the prime minister, well he could come back next week! For De Broglie, physics took precedence over politicians, no matter how exalted.

De Broglie sent Vigier to Brazil to spend a year working on the renewed causal interpretation of quantum mechanics with David Bohm. Thereafter, Yukawa got in touch with de Broglie, with the result that Vigier went to Japan to work with him for a year. Vigier comments that about the only point of disagreement between him and de Broglie was over nonlocality. De Broglie never accepted the reality of nonlocal interactions, whereas Vigier himself accepts the results of experiments such as Aspect's that clearly imply that such interactions exist.

His response to the question "Why do we do science?" is that, in part, it is to satisfy curiosity about the workings of nature, but it is also to contribute to the liberation from the necessity of industrial labor. With characteristic optimism, he regards the new revolution of digital technology as enhancing the prospects for a society based on the principles enunciated by Marx, a society whose members are freed from the necessity of arduous labor—this, as a result of the application of technological advances made possible by science.

TECHNICAL APPENDIX A: CRITICISMS OF THE U(1) INVARIANT THEORY OF THE AHARONOV–BOHM EFFECT AND ADVANTAGES OF AN O(3) INVARIANT THEORY

In this appendix, the U(1) invariant theory of the Aharonov–Bohm effect [46] is shown to be self-inconsistent. The theory is usually described in terms of a holonomy consisting of parallel transport around a closed loop assuming values in the Abelian Lie group U(1) [50] conventionally ascribed to electromagnetism. In this appendix, the U(1) invariant theory of the Aharonov–Bohm effect is

criticized in several ways with reference to the well-known test of the effect verified empirically by Chambers [46] and a holonomy consisting of parallel transport with O(3) covariant derivatives is applied to the Aharonov–Bohm effect, eliminating the self-inconsistencies of the U(1) invariant theory. Close similarities between the O(3) invariant theories of the Aharonov–Bohm and Sagnac effects are revealed.

It is well known that the change in phase difference of two electron beams in the Aharonov–Bohm effect is described in the conventional U(1) invariant theory by

$$\Delta\delta = \frac{e}{\hbar}\int \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = \frac{e}{\hbar}\int \boldsymbol{B} \cdot d\boldsymbol{S} \tag{A.1}$$

where the magnetic flux density $\boldsymbol{B}$ of the solenoid is related to the vector potential $\boldsymbol{A}$ by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{A.2}$$

Outside the solenoid, however

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \boldsymbol{0} \tag{A.3}$$

which means that the change in phase difference in Eq. (A.1) is zero, and that there is no Aharonov–Bohm effect, contrary to the observation. In the U(1) theory, an attempt is made to remedy this self-inconsistency by using the fact that $\boldsymbol{A}$ is not zero outside the solenoid, and so can be represented by a function of the type

$$\boldsymbol{A} = \nabla\chi \tag{A.4}$$

The Aharonov–Bohm effect is then described by [46]

$$\Delta\delta = \frac{e}{\hbar}\oint \nabla\chi \cdot d\boldsymbol{r} = \frac{e}{\hbar}[\chi]_0^{2\pi} = \frac{e}{\hbar}\int \boldsymbol{B} \cdot d\boldsymbol{S} \tag{A.5}$$

using the assertion that χ is not single-valued. The analytical form of χ is

$$\chi = \frac{BR^2}{2}\phi \tag{A.6}$$

where B is the magnitude of the flux density $\boldsymbol{B}$ inside the solenoid, R is the radius of the solenoid, and ϕ is an angle, the ϕ component in cylindrical polar coordinates.

However, the interpretation in (A.5) is self-inconsistent in several ways:

1. Outside the solenoid, $B = 0$, so $\chi = 0$ from Eq. (A.5), and there is no Aharonov–Bohm effect, contradicting Eq. (A.5).
2. For any function χ, a basic theorem of vector analysis states that

$$\nabla \times (\nabla \chi) \equiv \mathbf{0} \tag{A.7}$$

This theorem is also valid for a periodic function, so outside the solenoid

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \mathbf{0} \tag{A.8}$$

for χ, and from Eq. (A.1), the Aharonov–Bohm effect again disappears. For example, if we take the angle

$$\chi = \sin^{-1}\frac{x}{a}, \qquad (|x| < a) \tag{A.9}$$

then:

$$\nabla \chi = (a^2 - x^2)^{-1/2}\boldsymbol{i} \tag{A.10}$$

and

$$\nabla \times (\nabla \chi) \equiv \mathbf{0} \tag{A.11}$$

or if we take the periodic function

$$\chi = \cos x; \qquad \nabla \chi = -\sin x\, \boldsymbol{i} \tag{A.12}$$

then

$$\nabla \times (\nabla \chi) = \mathbf{0} \tag{A.13}$$

Another criticism of Eq. (A.5) is that the empirical result is obtained only if $\chi \to \chi + 2\pi$, whereas in general, $\chi \to \chi + 2\pi n$ for a periodic function. So the value of n has to be artificially restricted to $n = 1$ to obtain the correct analytical and empirical result.

The basic problem in a U(1) invariant description of the Aharonov–Bohm effect is that the field $\boldsymbol{B}$ is zero outside the solenoid, so outside the solenoid, $\nabla \times \boldsymbol{A}$ is zero, whereas $\boldsymbol{A}$ is not zero [46]. At the same time, the U(1) Stokes theorem states that

$$\int \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{r} \tag{A.14}$$

so that the holonomy $\oint \boldsymbol{A} \cdot d\boldsymbol{r}$ is zero and the effect again disappears for $\boldsymbol{A}$ outside the solenoid because the left-hand side in Eq. (A.14) is zero.

In the O(3) invariant theory of the Aharonov–Bohm effect, the holonomy consists of parallel transport using O(3) covariant derivatives and the internal gauge space is a physical space of three dimensions represented in the basis ((1),(2),(3)). Therefore, a rotation in the internal gauge space is a physical rotation, and causes a gauge transformation. The core of the O(3) invariant explanation of the Aharonov–Bohm effect is that the Jacobi identity of covariant derivatives [46]

$$\sum_{\sigma,\mu,\nu} [D_\sigma, [D_\mu, D_\nu]] \equiv 0 \tag{A.15}$$

is identical for all gauge group symmetries with the non-Abelian Stokes theorem:

$$\oint D_\mu \, dx^\mu + \frac{1}{2}\int [D_\mu, D_\nu]\, d\sigma^{\mu\nu} \equiv 0 \tag{A.16}$$

for any covariant derivative in any gauge group symmetry. In the O(3) invariant theory, the following three identities therefore exist

$$\oint \boldsymbol{A}^{(i)} \cdot d\boldsymbol{r} = \int \boldsymbol{B}^{(i)} \cdot d\boldsymbol{S}; \qquad i = 1, 2, 3 \tag{A.17}$$

that is, one for each of the three internal indices (1), (2), and (3). The quantities in Eq. (A.17) are linked by the following vacuum definition:

$$\boldsymbol{B}^{(3)*} \equiv -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{A.18}$$

The vector potential $\boldsymbol{A}^{(3)}$ and the longitudinal flux density $\boldsymbol{B}^{(3)}$ are both phaseless, so Eq. (A.17) with $i = 3$ is the invariant equation needed for a description of the Aharonov–Bohm effect

$$\oint \boldsymbol{A}^{(3)} \cdot d\boldsymbol{r} = \int \boldsymbol{B}^{(3)} \cdot d\boldsymbol{S} \tag{A.19}$$

The Aharonov–Bohm effect is therefore caused by a gauge transformation in a vacuum whose configuration space is O(3). The effect is a gauge transformation of Eq. (A.19) into the region outside the solenoid because the left- and right-hand sides of Eq. (A.19) exist only inside the solenoid. In general field theory, gauge transformations of the potential and of the field are defined through the rotation operator

$$S = \exp(iM^a \Lambda^a(x^\mu)) \tag{A.20}$$

where M^a are the group rotation generators and Λ^a are angles that depend on the 4-vector χ^μ. Under a general gauge field transformation

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1} \tag{A.21}$$

$$G'_{\mu\nu} = SG_{\mu\nu}S^{-1} \tag{A.22}$$

In the O(3) invariant expression (A.19), the vector potential transforms according to

$$\boldsymbol{A}^{(3)} \rightarrow \boldsymbol{A}^{(3)} + \frac{1}{g}\frac{\partial\alpha}{\partial Z}\boldsymbol{e}^{(3)} \tag{A.23}$$

and the magnetic field transforms as

$$\boldsymbol{B}^{(3)} \rightarrow \boldsymbol{B}^{(3)} \tag{A.24}$$

At the point of contact with the electrons, therefore, in the region outside the solenoid, the Aharonov–Bohm effect is caused by

$$\frac{1}{g}\oint \frac{\partial\alpha}{\partial Z}\boldsymbol{e}^{(3)}\cdot d\boldsymbol{r} = \int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{S} \tag{A.25}$$

in other words, there is a magnetic field present at the point of contact with the electrons and the left-hand side of Eq. (A.25) is physically significant. The reason for this is that the O(3) symmetry internal space of the theory is the physical space of three dimensions: the vacuum with configuration space O(3), a nonsimply connected configuration space. Therefore, none of the self-inconsistencies present in the U(1) invariant theory are present in the O(3) invariant theory of the Aharonov–Bohm effect. Agreement with the empirical data is obtained through the O(3) invariant equation:

$$\Delta\delta = \frac{e}{\hbar}\int \boldsymbol{B}^{(3)}\cdot d\boldsymbol{S} \tag{A.26}$$

and this analysis clearly demonstrates the simplicity with which the novel O(3) electrodynamics removes the self-inconsistencies of the U(1) description.

TECHNICAL APPENDIX B: O(3) ELECTRODYNAMICS FROM THE IRREDUCIBLE REPRESENTATIONS OF THE EINSTEIN GROUP

In Part 1 of this three-volume set, Sachs [117] has demonstrated that electromagnetic energy is available from curved spacetime by using the irreducible representations of the Einstein group. The metric is expressed using a quaternion-valued 4-vector, q^μ, with 16 components. If we define the scalar components of q^μ as

$$q^\mu = (q^0, q^1, q^2, q^3) \tag{B.1}$$

the quaternion-valued 4-vector is defined as

$$\sigma^\mu q_\mu = (q^0\sigma^0, q_1\sigma^1, q_2\sigma^2, q_3\sigma^3) \tag{B.2}$$

In the flat spacetime limit, the q^μ is replaced by the 4-vector made up of Pauli matrices:

$$\sigma^\mu = (\sigma^0, \sigma^1, \sigma^2, \sigma^3) \tag{B.3}$$

The field tensor given by Sachs in his Eq. (4.19) contains, in general, longitudinal as well as transverse components under all conditions, including the vacuum defined as Riemannian spacetime. Sachs' Eq. (4.16) shows that the electromagnetic canonical energy-momentum tensor (T^μ) is spacetime curvature in precisely the same way that gravitational canonical energy momentum is spacetime curvature. Therefore, code must be developed to solve Sachs' Eqs. (4.16) and (4.18) in order to understand electromagnetic phenomena in general relativity for any given situation. Sachs' Eq. (4.16) shows that electromagnetic energy is available in the vacuum, defined as Riemannian curved spacetime, and can be used to power devices.

The electromagnetic field propagating through the curved spacetime vacuum always has a source, part of whose structure is the quaternion-valued T^μ. This source is the most general form of the Lehnert vacuum 4-current [45,49]. General relativity [117] also shows that there is no electromagnetic field if there is no curvature, so a field cannot propagate through the flat spacetime vacuum of Maxwell–Heaviside theory. The latter's notion of transverse plane waves propagating in the vacuum without a source is therefore inconsistent with both general relativity and causality, because there cannot be cause without effect (i.e., field without source).

In general, all the off-diagonal elements of the quaternion-valued commutator term [the fifth term in Sachs' Eq. (4.19)] exist, and in this appendix, it is shown, by a choice of metric, that one of these components is the $\boldsymbol{B}^{(3)}$ field discussed in the text. The $\boldsymbol{B}^{(3)}$ field is the fundamental signature of O(3) electrodynamics discussed in Vol. 114, part 2. In this appendix, we also give the most general form of the vector potential in curved spacetime, a form that also has longitudinal and transverse components under all conditions, including the vacuum. In the Maxwell–Heaviside theory, on the other hand, the vector potential in the vacuum is generally considered to have transverse components only in the radiation zone, a result that is inconsistent with general relativity, O(3) electrodynamics, and Lehnert's extended electrodynamics.

In Vol. 114, part 1, Sachs has shown that the most general form of the electromagnetic field tensor is

$$F_{\rho\gamma} = Q\left(\frac{1}{4}(\kappa_{\rho\lambda}q^{\lambda}q^{*}_{\gamma} + q_{\gamma}q^{\lambda *}\kappa_{\rho\lambda} + q^{\lambda}\kappa^{+}_{\rho\lambda}q^{*}_{\gamma} + q_{\gamma}\kappa^{+}_{\rho\lambda}q^{\lambda *}) + \frac{1}{8}(q_{\rho}q^{*}_{\gamma} - q_{\gamma}q^{*}_{\rho})R\right) \tag{B.4}$$

where $\kappa_{\rho\lambda}$ is the curvature tensor defined in terms of the spin–affine connection [117]

$$\kappa_{\rho\lambda} \equiv \partial_{\rho}\Omega_{\lambda} - \partial_{\lambda}\Omega_{\rho} - \Omega_{\lambda}\Omega_{\rho} + \Omega_{\rho}\Omega_{\lambda} \tag{B.5}$$

where $Q \equiv \Phi^{(0)}$ has the SI units of magnetic flux (Weber), and where R is the scalar curvature in inverse square meters. The asterisk in Eq. (B.4) denotes quaternion conjugate, which entails [117] reversing the sign of the time component of the quaternion-valued q^{μ}. Thus, if

$$q^{\mu} = (q^{0}, q^{1}, q^{2}, q^{3}) \tag{B.6}$$

then

$$q^{\mu *} = (-q^{0}, q^{1}, q^{2}, q^{3}) \tag{B.7}$$

The metric in the irreducible representation of the Einstein group is proportional to [117]

$$q^{\mu}q^{\nu *} + q^{\nu}q^{\mu *} \neq 0 \tag{B.8}$$

and replaces the familiar metric $g^{\mu\nu}$ generated by the reducible representations of the Einstein group and used to describe gravitation. Therefore, the replacement of reducible by irreducible representations unifies the gravitational and electromagnetic fields inside the structure of one Lie group: the Einstein group. This important result shows that electromagnetic energy is available from curved spacetime in the same way that gravitational energy is available from curved spacetime, a well-accepted concept.

The demonstration by Sachs [117] that electromagnetic energy is available from the vacuum (Riemannian curved spacetime) generates the most precise classical electromagnetic theory available. Its notable successes [42] include

the ability to reproduce the Lamb shift in hydrogen without renormalization; the ability to produce the Planck distribution of blackbody radiation classically: the correct prediction of the lifetime of the muon state and electron–muon mass splitting. The Sachs theory also shows the existence of physical longitudinal and time-like components of the vector potential in the vacuum, predicts a small but nonzero neutrino and photon mass, and establishes grounds for charge quantization. These precise predictions firmly establish the possibility of obtaining electromagnetic energy from the vacuum, and firmly establish the existence of the $\boldsymbol{B}^{(3)}$ field as one of the possible longitudinal components of the tensor (B.4) in the vacuum (Riemannian curved spacetime). It follows that O(3) electrodynamics is also a theory of curved spacetime, and that the extended electrodynamics of Lehnert is a transitional theory in flat spacetime, but one that has several notable advantages over the Maxwell–Heaviside theory, as reviewed by Lehnert in Part 2 of Vol. 114. The Lehnert theory also gives the $\boldsymbol{B}^{(3)}$ field in the vacuum.

Equation (B.4) shows that the electromagnetic field in general relativity is non-Abelian, and acts as its own source. The gravitational field also acts as its own source, in that the gravitational field is a source of energy that, in turn, is gravitation. In gravitational theory, the Einstein curvature tensor is equated with the canonical energy-momentum tensor. In electromagnetic theory, the same applies, as in Sachs' Eq. (4.16). Gravitation is therefore an obvious manifestation of energy from the vacuum; electromagnetic energy from the vacuum is also available in nature, a result that has been confirmed experimentally to the precision of the Lamb shift. Therefore, there is an urgent need to develop code to solve the Sachs field equations for any given experimental setup. This code will show precisely the amount of electromagnetic energy that is available in the vacuum (Riemannian curved spacetime).

The quaternion-valued metric q^{μ} can be written as

$$q^{\mu} = \begin{bmatrix} q_0 + q_Z & q_X - iq_Y \\ q_X + iq_Y & q_0 - q_Z \end{bmatrix} \tag{B.9}$$

Therefore

$$q_X = \begin{bmatrix} 0 & q_X \\ q_X & 0 \end{bmatrix}; \qquad q_Y = \begin{bmatrix} 0 & -iq_Y \\ iq_Y & 0 \end{bmatrix} \tag{B.10}$$

and

$$q_X q_Y - q_Y q_X = i(q_X q_Y + q_Y q_X)\sigma_Z \tag{B.11}$$

Similarly

$$q_X q_Y + q_Y q_X = i(q_X q_Y - q_Y q_X)\sigma_Z \tag{B.12}$$

In order for both $q_X q_Y + q_Y q_X$ and $q_X q_Y - q_Y q_X$ to have real-valued parts, the individual scalar components q_X and q_Y must be complex-valued in general.

We recover the structure of O(3) electrodynamics in quaternion-valued form by a choice of metric

$$q_X = \frac{A_X^{(1)}}{A^{(0)}} = -ie^{i\phi}; \qquad q_Y = \frac{A_Y^{(2)}}{A^{(0)}} = e^{-i\phi} \tag{B.13}$$

where ϕ is an electromagnetic phase factor and where $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is part of the vector potential of O(3) electrodynamics as described in the text, and whose phase factor is a Wu–Yang phase factor as developed in Vol. 114, part 2. The choice of metric in Eq. (B.13) leads to

$$q_X q_Y - q_Y q_X = 2\sigma_Z \tag{B.14}$$

giving the phaseless and longitudinally directed $\boldsymbol{B}^{(3)}$ field of O(3) electrodynamics

$$B^{(3)} = \pm\frac{1}{4}\Phi^{(0)}R \tag{B.15}$$

where $\Phi^{(0)}$ is a magnetic flux in webers. The two signs in Eq. (B.15) represent left and right circular polarization. Within a factor of $\frac{1}{4}$, the result (B.15) is the same as that obtained [42] using a unification scheme based on an antisymmetric Ricci tensor.

It can therefore be inferred that O(3) electrodynamics is a theory of Riemannian curved spacetime, as is the homomorphic SU(2) theory of Barrett [50]. Both O(3) and SU(2) electrodynamics are substructures of general relativity as represented by the irreducible representations of the Einstein group, a continuous Lie group [117]. The $\boldsymbol{B}^{(3)}$ field in vector notation is defined in curved spacetime by

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{B.16}$$

while in the flat spacetime of Maxwell–Heaviside theory it vanishes:

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A} \times \boldsymbol{A} = \boldsymbol{0} \tag{B.17}$$

From general relativity, it may therefore be inferred that the $\boldsymbol{B}^{(3)}$ field must exist, and that it is a physically meaningful magnetic flux density in the vacuum. The phaseless $\boldsymbol{B}^{(3)}$ component is one of an infinite set of longitudinal, and in general oscillatory, components of the field tensor (B.4). This result has been tested experimentally to the precision of the Lamb shift.

In general, all the off-diagonal elements of the commutator term in Eq. (B.4) exist and are nonzero. For example

$$q_0 q_Z^* - q_Z q_0^* = 2q_0 q_Z \sigma_Z \tag{B.18}$$

which is a real and physical, longitudinally directed, electric field component in the vacuum. Such a component is in general phase-dependent. If the metric is chosen so that

$$q_0 = q_Z = \frac{A^{(3)}}{A^{(0)}} = 1 \tag{B.19}$$

we recover the longitudinal and phaseless electric field component:

$$E^{(3)} = \pm\frac{1}{4}c\Phi R \tag{B.20}$$

There is in-built parity violation in the Sachs theory [76], so the distinction between axial and polar vector is lost. This is the reason why the Sachs theory allows a phaseless $\boldsymbol{E}^{(3)}$ to exist while O(3) electrodynamics does not. There is no parity violation in O(3) electrodynamics. The question arises as to what is the interpretation of the phaseless $\boldsymbol{E}^{(3)}$ in general relativity. The empirical evidence for a radiated $\boldsymbol{B}^{(3)}$ field is reviewed in Vol. 114, Part 2 and in the text of this review chapter. An example is the inverse Faraday effect, which is magnetization produced by circularly polarized radiation. However, there is no electric equivalent of the inverse Faraday effect; that is, there is no polarization produced by a circularly polarized electromagnetic field. The phaseless $\boldsymbol{E}^{(3)}$ present in the vacuum in general relativity may, however, be interpretable as the Coulomb field between two charges in the radiation zone. The Coulomb field is missing in Maxwell–Heaviside theory, where the electric field is pure transverse, and as pointed out by Dirac [42], this result cannot be a proper description of the fact that there a longitudinal and phase-free Coulomb field between transmitter and receiver must always be present.

The most general form of the vector potential can be obtained by writing the first four terms of Eq. (B.4) as

$$F_{\rho\gamma,1} \equiv \partial_\rho A_\gamma^* - \partial_\gamma A_\rho^* \tag{B.21}$$

The vector potential is therefore obtained as

$$A_\gamma^* = \frac{Q}{4}\int (\kappa_{\rho\lambda}q^\lambda + q^\lambda\kappa_{\rho\lambda}^+)q_\gamma^* dx^\rho \tag{B.22}$$

and can be written as

$$A_\gamma^* = q_\gamma^* \left(\frac{Q}{4} \int (\kappa_{\rho\lambda} q^\lambda + q^\lambda \kappa_{\rho\lambda}^+) \, dx^\rho \right) \tag{B.23}$$

In order to prove that

$$\int q_\gamma^* \, dx^\rho = q_\gamma^* \int dx^\rho \tag{B.24}$$

we can take examples, giving results such as

$$\int q_1^* \, dx^2 = - \int q_X \, dY = -q_X \int dY \tag{B.25}$$

because q_X has no functional dependence on Y. The overall structure of the field tensor is therefore the quaternion-valued

$$F_{\rho\gamma} = C(\partial_\rho q_\gamma^* - \partial_\gamma q_\rho^*) + D(q_\rho q_\gamma^* - q_\gamma q_\rho^*) \tag{B.26}$$

where C and D are coefficients:

$$C \equiv \frac{Q}{4} \int (\kappa_{\rho\lambda} q^\lambda + q^\lambda \kappa_{\rho\lambda}^+) \, dx^\rho \qquad D \equiv \frac{QR}{8} \tag{B.27}$$

Equation (B.26) has the structure of a quaternion-valued non-Abelian gauge field theory. If we denote

$$\frac{D}{C^2} = -ig \tag{B.28}$$

Eq. (B.26) becomes

$$F_{\rho\gamma} = \partial_\rho A_\gamma^* - \partial_\gamma A_\rho^* - ig(A_\rho A_\gamma^* - A_\gamma A_\rho^*) \tag{B.29}$$

which is a general gauge field theory where A_γ^* is quaternion-valued. The rules of gauge field theory developed in the text and in part 2 of Vol. 114 can be applied to Eq. (B.29); for example, Eq. (B.29) is derived from a holonomy in curved spacetime.

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TOPOLOGICAL ELECTROMAGNETISM WITH HIDDEN NONLINEARITY

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