CONTENTS

INTRODUCTION

This book is one in a series on the prospect for nonabelian electrodynamics. This book takes a somewhat different path than the previous texts. It starts out with the nature of nonabelian electrodynamics as a classical field theory. In this sense it it similar to previous books on the subject. However, in chapter 6 this book explores the quantum electrodynamics of this nonabelian field. From there matters concerning field theory are explored. These include the unification of this nonabelian field theory with the weak interaction, and further with the impact this may have upon theoretical understanding of superstrings and quantum gravity. These developments are based largely on the theoretical prospect that electromagnetism and the weak interaction share a duality. The subject of duality in physics evolves into a central theoretical topic in chapter 10. This is largely unexplored territory in previous texts written on the subject.

While the classical field is exhibited in the first five chapters, later on the nature of the classical field is questioned. It turns out that there are a number of surprises in store. While no definative conclusion is drawn, the question is raised as to whether the $B^{(3)}$ field truly exists as a classical field. It is possible that it exists purely as an effect due to vacuum symmetries. This conclusion is suggested by the electroweak unification model developed in chapter 8. If this is the case then various nonHamiltonian operators in quantum electrodynamics, such as squeezed state operators, can be demonstrated to have a more fundamental basis in field theory.

One major thrust of this book is that nature may well have surprises

in store that evade current physical understanding. It was commonly stated that classical mechanics was a closed subject until chaos theory came to the forefront in 1979. A similar situation may be the case with electrodynamics which, at least classically, is regarded as a matter of engineering. With this in mind it is suggested that the reader have an open mind to the possibility that nature may have departures from some of our rather canonical knowledge.

There have been a variety of reactions to the suggestion that electrodynamics is more fundamentally nonabelian. These reactions range of hostile shock to excitement. One of the more compelling reasons to consider this possibility is that the standard model of electroweak interactions must be of the form $SU(2) \times SU(2)$ instead of $SU(2) \times U(1)$. This indicates that the electroweak field is a field that involves the duality between two physical fields. This has the exciting prospect that gauge field duality is something that can be experimentally explored. One further development is that soliton wave equations for photons, as with the case of nonlinear optics, can be derived without a strong appeal to phenomenology. While none of this proves the case, for a theory can not be proven with theory, it does give pause for some degree of consideration of the possiblity.

It is then intended that the reader should be one with the willingness to consider physical possibilities outside of the their canonical knowledge. If a reader is unwilling to do this then they might consider not reading beyond this introduction. However, science is filled with instances where nature is found to exhibit structure that is beyond standard theory.

CHAPTER 1
INTERACTION OF ELECTROMAGNETIC RADIATION WITH ONE FERMION

1.1 LORENTZ ELECTRON

The interaction of electromagnetic radiation with one Lorentz electron is classically and nonrelativistically described with the electron kinetic energy,

$$T_0 = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p}, \tag{1.1.1}$$

where $\boldsymbol{p}$ is the momentum and m is the mass of the electron. In a classical electromagnetic field the momentum is modified to make it gauge invariant. This requires that $\boldsymbol{p}$ is replaced by $\boldsymbol{p} - e\boldsymbol{A}$, where e is the charge of the electron and $\boldsymbol{A}$ is the classical vector potential of the electromagnetic field. This vector potential is in general a complex valued quantity[1-3]. The kinetic energy of the classical charged particle in the field is therefore,

$$T = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})\cdot(\boldsymbol{p} - e\boldsymbol{A}^*). \tag{1.1.2}$$

This result is typical of many textbooks on electromagnetic radiation. There are four terms in general in this kinetic energy: the free kinetic energy, two terms that depend on $\boldsymbol{A}$, a first order term,

$$T_1 = -\frac{e}{2m}(\boldsymbol{p}\cdot\boldsymbol{A}^* + \boldsymbol{A}\cdot\boldsymbol{p}), \tag{1.1.3}$$

and a term second order $\boldsymbol{A}$,

$$T_2 = \frac{e^2}{2m}\boldsymbol{A}\cdot\boldsymbol{A}^*. \tag{1.1.4}$$

The average over many cycles in the field leads to $\langle\boldsymbol{p}\cdot\boldsymbol{A}\rangle = 0$ and the average kinetic energy is then

$$T = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \frac{e^2}{2m}\boldsymbol{A}\cdot\boldsymbol{A}^*. \tag{1.1.5}$$

It is standard in electrodynamics to disregard the action of $\boldsymbol{A} \cdot \boldsymbol{A}^*$. Yet in some nonlinear optical physics products of fields exist. Based upon this observation this condition is relaxed and the consequences of this quadratic term in the potentials are examined.

The standard procedure with electromagnetism as a gauge theory posits a relation between the vector potential $\boldsymbol{A}$ and the magnetic induction $\boldsymbol{B}$:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{1.1.6}$$

If the vector potential is written as a plane wave $\boldsymbol{A} = \boldsymbol{A}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$ the term second order in $\boldsymbol{A}$ may be expressed as

$$T_2 = \frac{e^2c^2}{2m\omega^2}\boldsymbol{B} \cdot \boldsymbol{B}^*, \tag{1.1.7}$$

where c is the speed of light and ω is the frequency of the radiation. We define the polarizability as

$$\alpha = \frac{e^2}{2m\omega^2}. \tag{1.1.8}$$

Maxwell's equation in vacuum may be employed to give the usual expression for the energy associated with the electromagnetic field,

$$T_2 = \alpha \boldsymbol{E} \cdot \boldsymbol{E}^*. \tag{1.1.9}$$

With this form of the electromagnetic energy the zeroth Stokes parameter is introduced as

$$S_0 := \boldsymbol{E} \cdot \boldsymbol{E}^*, \tag{1.1.10}$$

so that equation 1.1.9 assumes the form

$$T_2 = \alpha S_0. \tag{1.1.11}$$

This product defines the average interaction energy between a classical charged particle and the classical electromagnetic field.

There are fundamental problems inherent in this theory. For example the interaction energy (1.1.4) is not invariant under the usual classical gauge transformation [4-6]

$$\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\chi, \tag{1.1.12}$$

where χ is a scalar function. Equation (1.1.12) leaves the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, invariant, but shifts the value of the interaction energy and the Stokes coefficient S_0, when the latter is expressed according to $\boldsymbol{A} \cdot \boldsymbol{A}^*$. Paradoxically, if S_0 is expressed as $\boldsymbol{E} \cdot \boldsymbol{E}$ it is trivially seen to be gauge invariant by the invariance of the field $\boldsymbol{E}$ and $\boldsymbol{E}^*$. Yet by $\boldsymbol{E} = \partial\boldsymbol{A}/\partial t$ we see that this Stokes parameter may be associated with $\boldsymbol{A}\cdot\boldsymbol{A}^*$. It is known that this Stokes parameter is a gauge invariant observable proportional to the beam intensity in any polarization; and in circularly polarized radiation we have,

$$S_0 = \pm S_3, \tag{1.1.13}$$

where

$$S_3 := -i|\boldsymbol{E} \times \boldsymbol{E}^*| \tag{1.1.14}$$

is the third Stokes parameter. However, the third Stokes parameter does not appear in equation (1.1.11), and again $\boldsymbol{A} \times \boldsymbol{A}^*$ is not gauge invariant where as $\boldsymbol{E} \times \boldsymbol{E}^*$ and $\boldsymbol{B} \times \boldsymbol{B}^*$ are gauge invariant. This is despite the fact that if $\boldsymbol{A}$ is a plane wave, a solution of the vacuum d'Alembert equation, with $\boldsymbol{A} \times \boldsymbol{A}^*$ proportional to $\boldsymbol{E} \times \boldsymbol{E}^*$ and $\boldsymbol{B} \times \boldsymbol{B}^*$. In the theory the classical Lorentz electron the classical electromagnetic field Newtons's Third Law in not obeyed (1-8). The latter is known as the radiation reaction problem. The replacement of $\boldsymbol{p}$ by $\boldsymbol{p} - e\boldsymbol{A}$ is a method that produces the Lorentz force equation from the Euler-Lagrange equation. This is usually regarded as sufficient empirical proof of its effectiveness. Nonetheless, this linear procedure of adding $e\boldsymbol{A}$ to the linear momentum suffers from the above drawbacks.

In general optical phenomena are nonlinear[9], and in general $\boldsymbol{p}$ is replaced by a Taylor series in all odd powers of $\boldsymbol{A}$. In surmountable problems with the linear theory start to occur at the level of the Stokes parameters, as just demonstrated, because the latter are nonlinear in the field and potential compoments. The Stokes parameters were proposed in 1854, several years before the Maxwell equations were proposed, which implies that Maxwell's equations amount to a linearization of the general theory. To be more consistent the equations of classical electrodynamics appear to be more generally nonlinear. This implies that the background gauge symmetry should be extended from the linear $U(1)$ symmetry to a nonabelian one.

These simple arguments show that all is not well with contemporary electrodynamics, either classical or quantum. The problems all stem from the use of a particular gauge symmetry for electrodynamics; the $U(1)$ gauge symmetry that describes rotations on the complex valued circle[10]. They are insurmountable in $U(1)$ gauge theory, but not in some other gauge symmetry which is nonlinear and nonAbelian. The purpose of this books is to propose that electrodynamics can be developed as an $O(3)_b$ symmetry gauge theory inwhich there exists a fundamental vacuum component, which we label as $\boldsymbol{B}^{(3)}$[11, 12] in the complex circular basis ((1), (2), (3)). Here the field theory is called $O(3)_b$ as a generic name, for the theory is really a broken $SO(3)$ theory. This theory involves the internal $U(1)$ symmetry with an additional axial symmetry. Within nonabelian gauge theories there are a number of electric and magnetic components of the gauge theory, where these components are determined by the Lie algebra of the field symmetry. In a gauge theory with 3 internal components there will exist three vector potentials. Ordinarily one would think that if there are three forms of the electric and magnetic fields that two of these carry electric charges, while one is neutral. This question is addressed later on in this book, where the charged vector potential are demonstrated to for the gauge potentials $\boldsymbol{A}(1)$ and $\boldsymbol{A}(2)$ with $\boldsymbol{A}^{(2)} = \boldsymbol{A}^{(1)*}$. This is later demonstrated to eliminate the existence of charged photons, and to equate the internal gauge directions with the external directions in space. From this result, this gauge symmetry the Stokes parameters S_0 and S_3 are self consistent physical quantities when expressed according to the vector potential, since they transform covariantly under a change of gauge[10]. In this gauge theory the third Stokes parameter becomes direction proportional to $\boldsymbol{B}(3)$, which demonstrates that the latter is a physical observable. The existence of $\boldsymbol{B}$ follows directly from the existence of circularly polarized radiation, as demonstrated by Arago in 1811. The quantized equivalent of $\boldsymbol{B}(3)$ for one photons is known as the photomagneton [13].

The most elementary illustration of the $\boldsymbol{B}(3)$ field is to rewrite equation (1.1.2) according to the Pauli matrices [10,14]. Mathematically this procedure is equivalent to the reexpression of this equation according to the $SU(2)$ Lie algebra homomorphic to, or up to a double covering to, $SO(3)$[14]. Various texts on elementary Lie Algebra theory illustrate this procedure[15-18]. The Pauli matrices then act as unit vectors in $SU(2)$ with an additional rotational degree of freedom that defines the double covering and homomorphism. If we then allow the momentum to have this additional degree of freedom, then the observable energy will not exhibit any change,

$$T = \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p}) = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p}. \qquad (1.1.15)$$

and the two homomorphic representation are indistiguishable physically. As the classical electromagnetic field, with an $SU(2)$ representation, is expressed in a gauge invariant manner the field kinetic energy is

$$T = \frac{1}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A})\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A}^*), \qquad (1.1.16)$$

and new terms exist that are physical observables. These turn out to be very useful for NMR and ESR applications, as they persist from

classical to quantum electrodynamics. As yet the empirical evidence for this is incomplete.

1.2 RADIATION INDUCED ELECTRON AND PROTON SPIN RESONANCE

The simplest development of equation (1.1.16) is the most illustrative. The average kinetic energy from equation (1.1.16), which involves the averaging over cycles and the elimination of first order terms, is

$$\langle T \rangle = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \frac{e^2}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{A}\boldsymbol{\sigma}\cdot\boldsymbol{A}^* \qquad (1.2.17)$$

and the interaction energy determined by the homomorphism with $SU(2)$ is,

$$\langle T \rangle_{SU(2)} = \frac{e^2}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{A}\boldsymbol{\sigma}\cdot\boldsymbol{A}^*. \qquad (1.2.18)$$

The rules employed by Dirac in 1926 [19] determine that this energy term is

$$\langle T \rangle_{SU(2)} = \frac{e^2}{2m}(\boldsymbol{A}\cdot\boldsymbol{A}^* + i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*), \qquad (1.2.19)$$

which demand the existence of a term proportional to $i\sigma\boldsymbol{A}\times\boldsymbol{A}^*$. The classical interaction differs from its homomorphism, as seen in equation (1.1.4), and is expressed as

$$\langle T \rangle_{SU(2)} = \alpha(S_0 - S_3\boldsymbol{\sigma}_z\cdot\boldsymbol{k}). \qquad (1.2.20)$$

The unit vector $\boldsymbol{k}$ is thus defined along the z axis. The third Stokes parameter S_3 then appears as a manifestations of $\boldsymbol{A}\times\boldsymbol{A}^*$, which exists as a result of circularly polarized radiation.

Now the energy term that results from the $SU(2)$ homomorphism, which contains an elementary kinetic energy term plus a self-interaction term involving the field potentials, is:

$$T_{int} = i\frac{e^2}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*. \qquad (1.2.21)$$

This term can been seen to be relevant to ESR and NMR. This interaction term is similar in form to the interaction of a charged spin with a magnetic field, where the magnetic field is determined by $\boldsymbol{A}\times\boldsymbol{A}^*$

The existence of electron spin resonance is deduced by the equality between the interaction term T_{int} with the energy of a photon in a probe beam $\hbar\omega_{res}$,

$$T_{int} = \hbar\omega_{res}. \qquad (1.2.22)$$

By using the equation for electromagnetic intensity or power density $I(watts\ cm^{-2})$:

$$I = \frac{c}{\mu_0}B^{(0)^2}, \qquad (1.2.23)$$

where μ_0 is the vacuum permeability, it is easily deduced that the electron resonance occurs at the frequency[20]:

$$\nu_{res} = \frac{\omega_{res}}{2\pi} = \left(\frac{e^2\mu_0 c}{2\pi\hbar m}\right)\frac{I}{\omega^2} = 1.007\times10^{28}\frac{I}{\omega^2}. \qquad (1.2.24)$$

By adjusting $1/\omega^2$ the resonance frequency can bet tuned to the infrared or visible range compater to the MHz range of magnet based ESR[21]. This leads to an increased resolution of several order of magnitude with great potential advantages. Similarly, the NMR resonance frequency from equation (1.2.22) is

$$\nu_{res} = \frac{g_p e^2\mu_0 c}{g_e 2\pi\hbar m}\frac{I}{\omega^2} = 1.532\times10^{25}\frac{I}{\omega^2}. \qquad (1.2.25)$$

where $g_e = 2.002$ is the electron g factor and $g_p = 5.5857$ is the proton g factor. The increase in resolution acheived by this method is comparable with a similar improvement of the electron microscope over the optical microscope.

1.3 THE $B^{(3)}$ FIELD

The $\boldsymbol{B}^{(3)}$ field is defined from the $O(3)_b$ symmetry gauge theory as

$$\boldsymbol{B}^{(3)} := -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{1.3.26}$$

in the basis ((1), (2), (3))[22,23], whose unit vectors are related to their cartesian counterparts by

$$e^{(1)} = e^{(2)*} = \frac{1}{\sqrt{2}}(i - ij);\ e^{(3)} = k, \tag{1.3.27}$$

and so are natural descriptions of circular polarization. Therefore the radiatively induce frequency from equation (1.3.26) is

$$\nu_{res} = -\frac{e}{4\pi m}\boldsymbol{\sigma}_z \cdot \boldsymbol{B}^{(3)}. \tag{1.3.28}$$

This has the same form as the description of a resonance frequency in a static magnetic field. As previously demonstrated $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is not gauge invariant under global or local $U(1)$ gauge transformations, so $\boldsymbol{B}^{(3)}$ is not a physical observable under this gauge theory. However, $\boldsymbol{A} \times \boldsymbol{A}^*$ is proportional to $\boldsymbol{E} \times \boldsymbol{E}^*$ and equivalently the third Stokes parameter, which is an empirical observable. This means that the $U(1)$ gauge theory that must be replaced by a theory based on a nonabelian symmetry group. Since $SO(3)$ is the rotation group in three dimensions, and the Stokes parameters have a rotational symmetry homomorphic to this, it is natural to consider this group, or broken variants from it, as the generalization of electrodynamics.

1.4 ELECTRODYNAMICS AS A NONABELIAN GAUGE FIELD THEORY

As we have seen, the need for this development springs from a simple consideration of the interaction between the electromagnetic field and the Lorentz electron. One of the fundamental differences between $U(1)$ and $O(3)_b$ gauge field theory is that in the latter the

potentials enter more directly into physical constructs. The potentials enter into products, which result from the nonabelian nature of the field, that define a physical field. The structure of the $O(3)_b$ gauge theory represents a return to the original concepts of Faraday and Maxwell. The natural philosophy of the classical electromagnetic field as developed by Heaviside and others of the late nineteenth century crystallized the theory into four differential equations known as the Maxwell's equations. In S.I units they are:

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.4.29}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{1.4.30}$$

$$\nabla \cdot \boldsymbol{D} = \rho \tag{1.4.31}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{1.4.32}$$

and are respectively the Gauss, Maxwell, Coulomb, and Faraday laws. In this system of equations the electric displacement vector $\boldsymbol{D}$ is related to the electric field $\boldsymbol{E}$, and the magnetic induction $\boldsymbol{B}$ is related to the magnetic field $\boldsymbol{H}$ by,

$$\boldsymbol{D} = \epsilon\boldsymbol{E} = \epsilon_0\boldsymbol{E} + \boldsymbol{P},\ \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}), \tag{1.4.33}$$

where $\boldsymbol{P}$ is the polarization vector defined in a medium, and $\boldsymbol{M}$ is the magnetization of a material. The electric field is defined to have units of *volts* m^{-1}, the electric displacement has units of Cm^{-2}, the charge density ρ Cm^{-3}, and the polarization has the units of the electric displacement. With the other two Maxwell's equations the magnetic field has units of Am^{-1} and the magnetic induction has units of T or $Wb\ m^{-2}$. The electric permativity and magnetic permeability have the values

$$\begin{aligned} \epsilon_0 &= 8.854188 \times 10^{-12} C^2 J^{-1} m^{-1} \\ \mu_0 &= 4\pi \times 10^{-7} Js^2 C^{-2} m^{-1} \end{aligned} \tag{1.4.34}$$

equations (1.4.29) and 1.4.30) are combined into the homogeneous Maxwell equations, and equations (1.4.31) and (1.4.32) into the inhomogeneous equations. The former is a Jacobi identity in the fundamental field $\boldsymbol{E}$ and $\boldsymbol{B}$; which can be expressed according to the

scalar and vector potentials ϕ and $\boldsymbol{A}$ respectively. These equations do not permit the existence of magnetic monopoles. The quantities that appear in the inhomogeneous Maxwell equations are $\boldsymbol{D}$ and $\boldsymbol{H}$. These equations allow for the existence of electric monopoles and currents.

In manifestly covariant notation, with indices $\mu, \nu \in \{0, 1, 2, 3\}$ that range over the spacetime variables $t = x^0$ and x^i, the differential operators have the form

$$\frac{\partial}{\partial x^\mu} := \partial_\mu = (\partial_t, \partial_i) = (\partial_0, \partial_i), \tag{1.4.35}$$

and with the metric $g^{\mu\nu}$, with signature $[1, -1, -1, -1]$, the differential operator $\partial^\mu = g^{\mu\nu}\partial_\nu$ is defined

$$\frac{\partial}{\partial x_\mu} := \partial^\mu = (\partial_t, -\partial_i) = (\partial^0, \partial^i). \tag{1.4.36}$$

The fields are determined in this abelian case from the vector potentials by

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu, \tag{1.4.37}$$

with components of the four vector potential defined as $(\phi, \boldsymbol{A})$. This then defines elements of the antisymmetric field tensor $F^{\mu\nu}$. In this manifestly Lorentz form the Maxwell equations assume the forms,

$$\partial_\mu F^{\mu\nu} = 0 \tag{1.4.38}$$

$$\partial_\mu H^{\mu\nu} = j^\nu. \tag{1.4.39}$$

The two field strength tensors are related by the Hodge dual star operator $*$. This determines the tensor on dual 2-chains in four dimensions by

$$H^{\mu\nu} = *F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{1.4.40}$$

Here $\epsilon^{\mu\nu\rho\sigma}$ is a totally antisymmetric symbol. The magnetic fields and electric fields are respectively defined by the spacial components

and mixed temporal and spacial components of this field strength tensor by

$$B_i = \epsilon_{ijk}F^{jk}, \; E_i = \epsilon_{i0j}F^{0j}. \tag{1.4.41}$$

Thus the field strength tensor $F^{\mu\nu}$ and its dual $*F^{\mu\nu}$ have the components

$$[F^{\mu\nu}] = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & cB^3 & -cB^2 \\ E^2 & -cB^3 & 0 & cB^1 \\ E^3 & cB^2 & -cB^1 & 0 \end{pmatrix} \tag{1.4.42}$$

$$[*F^{\mu\nu}] = \begin{pmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{pmatrix}. \tag{1.4.43}$$

By the Hodge star duality equation (1.4.39) is the dual of the equation $\partial_\mu F^{\mu\nu} = 0$ which is the Jacobi identity

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} = 0. \tag{1.4.44}$$

Further, by employing the rules of Einstein summation over repeated indices then equations (1.4.29) and (1.4.30); and equations (1.4.36) reproduce equations (1.4.31) and (1.4.32) with the definition

$$[H^{\mu\nu}] = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3/c & H^2/c \\ D^2 & H^3/c & 0 & -H^1/c \\ D^3 & -H^2/c & H^1/c & 0 \end{pmatrix}. \tag{1.4.45}$$

This manifestly covariant formalism then is equivalent to the $U(1)$ gauge field theory that is Maxwell's equations of electromagnetism.

1.5 LIMITATION OF THE U(1) THEORY

A vast amount of empirical data has been described by equations (1.4.29-32). The classical limit of the theory is also well known. Ritz, for example, as argued that Maxwell's ether, a medium posited by Maxwell, is mechanically unstable [24] and that the third law

of Newton's mechanics is not consistent with the radiation reaction problem. There are intrinsic assumptions behind the comtemporary Maxwell equations which are exemplified by the fact that they are vector differential equations with no stated boundary conditions. These equations are not due to Maxwell, but by Heaviside and his contemporaries. This relegated the vector potential to a subsidiary mathematical role. Within this view the vector potential, or products thereof, do not contribute directly to physical quantities. It is rather their "coboundaries," which are independent of these potentials, which are physical. Maxwell on the other hand used quaternionic $SU(2)$ algebra with twenty equations where potentials entered into physical quantities directly, based on Faraday's argument for a physical potential he named the electronic state. This allows the potentials to be rescaled at will while the fields remain invariant. On the other hand, if one takes Schwarzschild's standpoint [24], the modern Maxwell equations may be replaced completely by delayed action at a distance equations that involve only the classical potentials. These were argued to be just as physical as fields. The $U(1)$ electromagnetic theory, in contrast, employs potentials that do not contribute directly to physical observables. However, nonabelian field theory have commutators between potentials that contribute to physical observables, such as the Gauss' law for the self interaction of gluons in quarks physics. This leads to a compulsion to develop a nonabelian form of electromagnetism, where the potentials enter into commutators, such as is the case with the $\boldsymbol{B}^{(3)}$ field, that are physical observables.

Barret [25] has demonstrated that the interpretation of several physical effects relies upon a physical contribution from the classical vector potential A^μ. This does not obtain in the Heaviside formalism of the Maxwell electromagnetic field theory. There exist quantum effects that indicate the existence of the potentials such as the effects of Aharonov and Bohm, de Haas and van Alphen, Ehrenberg and Siday, and the quantum Hall effect. It is well known that in quantum mechanics that the potential plays a role with the "reality" of wave

functions: something that is not seen in classical $U(1)$ electromagnetism. This means that $U(1)$ electromagnetism has a larger gap between the physical contribution of potentials than one has with nonabelian Quantum ElectroDynamics (QED).

The four Maxwell equations are linear in the fields and differential operators. They may be derived from a Lagranigian density with Abelian symmetry through the Euler-Lagrange equations of motion[36]. The Abelian rotation group $SO(2)$ is the group of a rotations in two real dimensions that is homomorphic to the $U(1)$ group of rotations, of the form $e^{i\phi}$, on the complex plane. Similarly the rotation group $SO(3)$ is homomorphic with $SU(2)$, the group of unitary 2×2 complex valued matrices. In contemporary gauge field theory the electromagnetic field is the gauge field that guarantees invariance under $U(1)$ transformations. Local gauge transformations require that the differential operator assume a covariant form

$$\mathcal{D}_\mu = \partial_\mu + i\frac{e}{\hbar}A_\mu := \partial_\mu + igA_\mu, \tag{1.5.46}$$

which is the minimal prescription expressed in coordinate representation rather that the momentum representation

$$p_\mu \rightarrow p_\mu - eA_\mu. \tag{1.5.47}$$

This covariant description of the differential operator and the momentum operator are required to preserve the invariance of the momentum as a wave function is phase shifted by local gauge transformations $\psi \rightarrow exp(-ie\int A_\mu dx^\mu)\psi$. This entry of the electromagnetic field into momentum and differential operators is seen with the fundamental quantum ansatz

$$P_\mu = i\hbar\partial_\mu,\ E = i\hbar\frac{\partial}{\partial t},\ \boldsymbol{p} = \frac{\hbar}{i}\nabla \tag{1.5.48}$$

and the application of this momentum operator onto the phase shifted wave function, due to a local gauge transformation, demonstrates that the momentum remains invariant. This means that the

potentials attain a measure of physical reality as measured in a phase shift of a quantum wave function. This is demonstrated by the Aharonov-Bohm effect. This demonstrates that the potentials exhibit a subtle measure of physical reality here clearly demonstrated by $U(1)$ classical electrodynamics. In gauge theories there are products or commutators of potentials enter into observables that are measurable with physical units.

Here a formalism of electromagnetism is advanced where the potentials enter into quantities that are, at least in principle, classical physical observables. This is done in high energy physics, where covariant differentials have nonzero commutators, and where commutators of potentials can define charge densities and currents associated with the gauge theory. In this case they define the $B^{(3)}$ field. Here the field tensor $G_{\mu\nu}$ is defined both on Minkowski space of four dimensions plus an internal gauge space of three dimensions. The field tensor and vector potentials are written according to an internal space superimposed on spacetime, and where the Lagrangian is nonAbelian. The relations between physical field and the potentials are nonlinear. Gauge transformations are defined on various charts on the spacetime so that parameterized motion through spacetime is associated with a continuous gauge transformation. This results in a commutator of differential operators that associated with a closed loop in spacetime[10]. This commutator is then evaluated on the area enclosed by the loop according to the general Stokes' law $\int_{S=\partial\mathcal{M}}\omega = \int_{\mathcal{M}} d\omega$. A similar process leads to a Jacobi identity for any group symmetry, an identity first derived by Feynman[10]:

$$\mathcal{D}^{\mu}\tilde{G}_{\mu\nu} = 0. \tag{1.5.49}$$

The internal symmetry space((1), (2), (3)) introduces commutator terms into the equations of Gauss and Faraday, where for indices (1) and (2) we identify terms in a commutator as the fundamental magnetic field:

$$\boldsymbol{B}^{(3)} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{1.5.50}$$

where g is the coupling constant of the field and $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \boldsymbol{A} \times \boldsymbol{A}^{*}$, with $\boldsymbol{A}^{*}$ as the complex conjugate of $\boldsymbol{A}$. The magnetic field $\boldsymbol{B}^{(3)}$ is observed as the skew symmetric product of these two vector potentials in magneto-optical experiments and, as previously demonstrated, as the third Stokes parameter. This result suffices to demonstrate clearly the advantages of the $O(3)_b$ gauge field theory, here after referred to as simply the "$O(3)_b$ theory." Within the $U(1)$ gauge field theory of electrodynamics, the "$U(1)$ theory," such a nonlinear skew symmetric product term is identically zero. Nonlinearity in optics is phenomenologically treated as due to atomic effects, but ultimately these interactions are due to complex electrodynamic interactions. It would then appear plausible to state that electrodynamics must be nonabelian if it is a gauge theory that can exhibit these nonlinear effects in optical systems.

The nature of a gauge transformation is changed fundamentally under this nonAbelian generalization. In the Abelian view a local transformation means that the momentum operator is transformed as,

$$p_{\mu} \rightarrow p_{\mu} - eA_{\mu} + e\partial_{\mu}\Phi. \tag{1.5.51}$$

The same gauge transformation in quantum mechanics can be written as,

$$(\partial_{\mu} + i\frac{e}{\hbar}A_{\mu})\psi \rightarrow \partial_{\mu}(\psi' + e\Phi) + i\frac{e}{\hbar}A_{\mu}\psi', \tag{1.5.52}$$

where the wave function exhibits a phase shift $\psi' = \psi e^{-i\theta}$. The phase factor is just $\theta = e\int \partial_{\mu}\Phi dx^{\mu}$ evaluated along the path the local gauge transformation occurs. This leads to the occurrence of the term $-\partial_{\mu}\Phi$ that cancels out the gauge term in the definition of the differential. From this the basic equations of quantum mechanics, in particular the Dirac equation and Klein-Gordon equations are then gauge invariant.

Canonical quantization in the Lorentz gauge $\partial^\mu A_\mu = 0$ imposes the elimination of longitudinal photons with the Gupta-Bleuler condition:

$$\langle\psi|\partial^\mu A_\mu|\psi\rangle = 0. \tag{1.5.53}$$

This is equivalent to stating that the longitudinal field is annulled on the space of physical fields by

$$\partial^\mu A_\mu|\psi\rangle = 0. \tag{1.5.54}$$

yet in another gauge, such as the radiation gauge or the Landau gauge, this choice is different. In effect the choice of the gauge is equivalent to a particular choice of conjugate observables. This is imposed on the quantization procedure by the addition of a gauge fixing term to the Lagrangian density

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2, \tag{1.5.55}$$

where the parameter λ is purely arbitrary and is set to fix various gauge conditions: $\lambda = 1$ for the Lorentz gauge and $\lambda = 0$ for the Landau gauge. Yet there is something rather troubling about this term. While it is unphysical it does have the effect of determining the existence of longitudinal modes. This appears to create a subtle discrepancy with the notion that the field potentials are unobservable.

The $O(3)_b$ gauge theory echoes the original intent of Yang and Mills [10] to generalize the Maxwell equations. The minimal prescription is extended to give

$$[\boldsymbol{A}^{(1)}{}_\mu, \boldsymbol{A}^{(2)}{}_\nu] = \frac{i}{g}\boldsymbol{B}^{(3)}{}_{\mu\nu}, \tag{1.5.56}$$

with the commutators

$$[A^{(1)}{}_\mu, A^{(1)}{}_\nu] = [A^{(2)}{}_\mu, A^{(2)}{}_\nu] = 0. \tag{1.5.57}$$

Now in the $O(3)_b$ case the gauge condition reads, for the internal indices $a, b \in \{1, 2\}$ as

$$\mathcal{D}_\mu A^{a\mu} = 0 = \partial_\mu A^{a\mu} + ige^b A^b{}_\mu A^{a\mu}, \tag{1.5.58}$$

where the quadratic term in the potentials can be replaced with $A^{a\mu} \rightarrow (1/\omega)E^{a\mu}$. We are left with a gauge choice that gives longitudinal modes. The existence of longitudinal modes is not dependent upon the gauge choice due to the nonAbelian term that enters into the gauge fixing term.

In this $O(3)_b$ gauge theory the covariant derivative and minimal prescriptions are structure in such as way that equation (1.5.49) becomes

$$\mathcal{D}_\mu \tilde{G}^a_{\mu\nu} = 0;\ a \in \{1, 2, 3\}, \tag{1.5.59a}$$

and since $\tilde{G}^a_{\mu\nu} = \epsilon^{abc}[A^b{}_\nu, A^c{}_\mu]$ the Hodge star dual operation gives the Jacobi identity

$$[A^a{}_\mu, [A^b{}_\nu, A^c{}_\rho]] + [A^b{}_\nu, [A^c{}_\rho, A^a{}_\mu]] + [A^c{}_\rho, [A^a{}_\mu, A^b{}_\nu]] = 0. \tag{1.5.59b}$$

Further with the prescription that $\boldsymbol{A}^3 = 0$, an issue discussed at greater length later, we have the associated equation for the $\boldsymbol{B}^{(3)}$ field and the cyclicity condition:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)}. \tag{1.5.59c}$$

Equations (1.5.59) are the basic set of field equations for $O(3)_b$ nonAbelian electrodynamics. It can be seen that the cyclicity condition equation (1.5.59c) is identical to equation (1.5.57). Further, equations (1.5.59) for $a = 3$ determine that $\boldsymbol{B}^{(3)}$ is an irrotational and time independent field

$$\nabla \times \boldsymbol{B}^{(3)} = 0,\ \frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = 0. \tag{1.5.59d}$$

Equation (1.5.59a) with $a = 1$ or 2 gives equations (1.4.29) and (1.4.30).

There is in summary a fundamental philosophical difference between $U(1)$ and $O(3)_b$ electrodynamics. This can be seen in the nature of gauge transformations. Essentially, there is no dependence upon the existence or removal of longitudinal modes by a gauge choice in nonAbelian electrodynamics. Further, since the algebraic indices of $O(3)_b$ electrodynamics are associated with the Stokes parameters, they become identified with a spacial rotations. The process by which this occurs is discussed in greater detail later. This means that local gauge differences in $O(3)_b$ electrodynamics are identified with a frame rotations. Within this framework the $\boldsymbol{B}^{(3)}$ field is a covariant longitudinal field that can not be eliminated by a convenient choice of gauge.

1.6 CLASSICAL RELATIVISTIC NONABELIAN ELECTRODYNAMICS

In special relativity the basic invariant is the interval $ds^2 = dt^2 - d\sigma^2$, where $d\sigma^2$ is an infinitesimal distance in Euclidean space. This is often expressed in a more compact for with the metric tensor as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. This interval is a measure of the proper time of an observer in spacetime. The corresponding invariant in momentum-energy space is the rest mass of a particle. This invariant is stated as

$$p^\mu p_\mu = m^2c^2, \tag{1.6.60}$$

where $p^\mu = (E, \boldsymbol{p})$ is the relativistic four momentum composed of the energy E, conjugate to the timelike direction plus the momentum conjugate to spacial direcitons $\boldsymbol{p}$. The four momentum is covariant with respect to the Lorentz transformations

$$p'^\mu = \Lambda^\mu{}_\nu p^\nu, \tag{1.6.61a}$$

that may be expressed explicitly as the matrix operation

$$\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}, \tag{1.6.61b}$$

for a boost in the z direction. Here $\beta = v/c$, for v as the boosting velocity, and $\gamma = 1/\sqrt{1-\beta^2}$. This leads to the rotation between the p_z and E components with

$$E' = \gamma(E - \beta p_z),\ p'_z = \gamma(p_z - \beta E) \tag{1.6.61c}$$

This transformation is one that leaves the square of the four momentum invariant, and defines the rest mass of the particle.

We now turn to electromagnetic theory where to preserve gauge invariance the momentum assumes the form $p^\mu \to p^\mu - eA^\mu$. This means that the momentum invariant is

$$(p^\mu - eA^\mu)(p_\mu - eA^*{}_\mu) = m^2c^2. \tag{1.6.62}$$

To derive the RFR term, we introduce $SU(2)$ internal indices and the Dirac matrices γ^μ, that produces the momentum invariant

$$\gamma^\mu p_\mu \gamma^\nu p_\nu = m^2c^2, \tag{1.6.63a}$$

where the Dirac matrices are represented as,

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{1.6.63b}$$

In the Feynman slash notation [29] becomes

$$\not{p}\not{p} = m^2c^2. \tag{1.6.64}$$

When the electromagnetic field is introduces this momentum invariant is

$$(\not{p} - e\not{A})(\not{p} - e\not{A}^*) = m^2c^2. \tag{1.6.65}$$

This is the classical relativistic expression for the interaction of a charged particle with the classical electromagnetic field. It will be

demonstrated later in this chapter that the quantized version of this equation is the van der Waerden equation [30] according to a two component spinor, rather than the four component spinor as the Dirac equation. The term $e^2 A' A'^*$ is a classical quantity that in the van der Waerden equation multiplies a wave function as a simple multiplication operator. Its real expectation value is therefore $e^2 A' A'^*$ and appears as a classical type of variable, just as the potential does in the Schrödinger equation written in the position representation.

The term $e^2 \not{A}' \not{A}'^*$ is expanded according to Dirac matrices as

$$e^2 \not{A}' \not{A}'^* = e^2 \gamma^\mu A_\mu \gamma^\nu A_\nu{}^*. \tag{1.6.66}$$

Now decompose the inner product according to its temporal and spacial parts,

$$e^2 \gamma^\mu A_\mu \gamma^\nu A_\nu{}^* = e^2 (\gamma^0 A_0 - \boldsymbol{\gamma} \cdot \boldsymbol{A})(\gamma^0 A_0{}^* - \boldsymbol{\gamma} \cdot \boldsymbol{A}^*). \tag{1.6.67}$$

The anticommutation between Dirac matrices

$$\{\gamma^\mu, \gamma^\nu\} = \frac{1}{2} g^{\mu\nu}, \tag{1.6.68}$$

and the representation of the Dirac matrices in equation (1.6.63b) are used to decompose this product into

$$e^2 \not{A}' \not{A}'^* = e^2 (A_0 A_0{}^* - \boldsymbol{A} \cdot \boldsymbol{A}^* - i\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \boldsymbol{A}^*). \tag{1.6.69}$$

From this classical description of the relativistic motion of a charged particle in a magnetic field we arrive at the RFR term; which is the same as what obtains in the nonrelativistic limit,

$$-ie^2 \boldsymbol{\sigma} \cdot \boldsymbol{A} \times \boldsymbol{A}^* = e^2 A^{(0)2} \sigma_z. \tag{1.6.70}$$

1.7 RELATIVISTIC QUANTUM DESCRIPTION

We start with the standard correlation between classical variables and quantum operators:

$$p^\mu \rightarrow i\hbar \partial^\mu, \tag{1.7.71}$$

where these operators replace the classical variables in equation (1.6.63). This results in the van der Waerden equation of motion [30]. By our previous discussion of the correlation between classical variables and quantum operators, we see that the momentum operator has the component form

$$p^\mu = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right), \; p_\mu = i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right), \tag{1.7.72}$$

when these quantum operators replace the classical variables in equation (1.6.63) we arrive at the wave equation

$$(i\gamma^\mu \partial_\mu)(i\gamma^\nu \partial_\nu)\psi = \frac{m^2 c^2}{\hbar^2}\psi. \tag{1.7.73}$$

When written in component form this wave equation becomes the equation for the two component spinor ψ as decribed by Sakurai[30]

$$\left(\frac{i}{c}\frac{\partial}{\partial t} + i\boldsymbol{\sigma} \cdot \nabla\right)\left(\frac{i}{c}\frac{\partial}{\partial t} + i\boldsymbol{\sigma} \cdot \nabla\right)\psi = \frac{m^2 c^2}{\hbar^2}\psi. \tag{1.7.74}$$

The electromagnetic field is then introduced into equation (1.7.73) with covariant differentials. With this procedure we regain the classical term $e^2 \not{A}' \not{A}'^*$ as deduced by the Dirac equation in ref. [31]. The Dirac equation may be produced from the van der Waerden equation by standard methods [30], and this classical term emerges within the Dirac equation.

1.8 NONRELATIVISTIC QUANTUM DESCRIPTION

Before demonstrating the RFR term within this domain it is instructive to consider the nonrelativistic Schrödinger equation with a constant, or flat, potential $\boldsymbol{A}$, and to derive NMR and ESR. It is not necessary to use relativistic quantum physics to derive NMR or ESR as demonstrated in the following discussion. The Schrödinger equation for a free particle is:

$$H\psi \;=\; i\hbar\frac{\partial\psi}{\partial t} \tag{1.8.75a}$$

which is the eigenvalued equation for a stationary wave function

$$H\psi \;=\; E\psi. \tag{1.8.75b}$$

The Hamiltonian operator is

$$H \;=\; \frac{p^2}{2m} \;+\; V, \tag{1.8.76}$$

under the substitution of the classical momentum variable by quantum operator $\boldsymbol{p} \;\rightarrow\; -i\hbar\nabla$, and where V is the potential energy in position coordinates. In the presence of a static magnetic field, represented by a time independent and real valued magnetic vector potential $\boldsymbol{A}$, the Hamiltonian becomes[27]:

$$H \;=\; \frac{1}{2m}(\boldsymbol{p} \;+\; e\boldsymbol{A})^2 \;+\; V \;+\; eA_0. \tag{1.8.77}$$

The cross term in the quadratic part of the Hamiltonian operates on a wave function according to,

$$\boldsymbol{p}\cdot\boldsymbol{A}\psi \;=\; \frac{\hbar}{i}\nabla\cdot(\boldsymbol{A}\psi) \;=\; \frac{\hbar}{i}\big((\nabla\cdot\boldsymbol{A})\psi \;+\; \boldsymbol{A}\cdot(\nabla\psi)\big). \tag{1.8.78}$$

If the Coulomb gauge

$$\nabla\cdot\boldsymbol{A} \;=\; 0, \tag{1.8.79}$$

is employed this cross term gives

$$\boldsymbol{p}\cdot\boldsymbol{A}\psi \;=\; \frac{\hbar}{i}\boldsymbol{A}\cdot(\nabla\psi) \;=\; \boldsymbol{A}\cdot\boldsymbol{p}\psi. \tag{1.8.80}$$

For constant magnetic field we may write the vector potential as

$$\boldsymbol{A} \;=\; \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}, \tag{1.8.81}$$

which for a constant magnetic field oriented in the x direction means that

$$\boldsymbol{A} \;=\; \frac{1}{2}B(x\boldsymbol{j} \;-\; y\boldsymbol{i}). \tag{1.8.82}$$

The Hamiltonian terms $H^{(1)}$ and $H^{(2)}$ may be written for $\gamma_e \;=\; -e/2m$ and $\boldsymbol{L} \;=\; \boldsymbol{r}\times\boldsymbol{p}$ as

$$H^{(1)} \;=\; -\gamma_e\boldsymbol{L}\cdot\boldsymbol{B} \;=\; -\boldsymbol{m}\cdot\boldsymbol{B}$$

$$H^{(2)} \;=\; \frac{e^2}{2m}A^2. \tag{1.8.83}$$

Here $\boldsymbol{L}$ is the orbital angular momentum and γ_e is the geomagnetic ratio, and $\boldsymbol{m}$ is the magnetic dipole moment. The Hamiltonian $H^{(1)}$ gives the orbital Zeeman effect and $H^{(2)}$ is a second order susceptability term.

The above static magnetic field calculation, and as a result A^2 is not proportional to $1/\omega^2$, which implies that the ground state perturbation term is not zero[27]. In general the magnetic vector potential is complex valued. A repeat of the above calculation for a complex valued $\boldsymbol{A}$ we obtain two well know terms: the oscillating magnetic dipole term and the term responsible for the optical Stark effect. The Hamiltonian operator is the following:

$$H \;=\; \frac{1}{2m}(\boldsymbol{p} \;-\; e\boldsymbol{A})\cdot(\boldsymbol{p} \;-\; \boldsymbol{A}^*) \;+\; V \;+\; eA_0, \tag{1.8.84}$$

which may be broken out into distinct terms as:

$$H \;=\; \frac{1}{2m}p^2 \;-\; 2\mathcal{R}e\boldsymbol{A}\cdot\boldsymbol{p} \;+\; \frac{e^2}{2m}\boldsymbol{A}\cdot\boldsymbol{A}^* \;+\; V, \tag{1.8.85}$$

where

$$\mathcal{R}e\boldsymbol{A} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^*). \tag{1.8.86}$$

The oscillating dipole term is

$$H^{(1)} = -\boldsymbol{m}\cdot\boldsymbol{B}, \tag{1.8.87}$$

and the optical Stark effect is given by the second order phase independent term:

$$H^{(2)} = \frac{e^2}{2m}\boldsymbol{A}\cdot\boldsymbol{A}^*. \tag{1.8.88}$$

With phase averaging over many cycles of the electromagnetic field the first order term disappears, but $H^{(2)}$ is real valued and nonzero. The optical Stark effect is well know empirically [32]. In perturbation theory the ground state term

$$E^{(2)} = \langle 0|H^{(2)}|0\rangle, \tag{1.8.89}$$

which is again nonzero, and the optical Stark effect term are proportional to $1/\omega^2$.

1.9 SCHRÖDINGER EQUATION WITH INTRINSIC SPIN

To describe intrinsic spin of the electron or proton in nonrelativistic quantum mechanics, the method described by Sakurai [30] is used. The Hamiltonian in the absence of the electromagnetic field is:

$$H = \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{p}) + V. \tag{1.9.90}$$

The equation of motion is the nonrelativistic Schrödinger-Pauli equation:

$$H\psi = E\psi. \tag{1.9.91}$$

In the presence of a static magnetic field the Hamiltonian assumes the form

$$H = \frac{1}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A})\boldsymbol{\sigma}\cdot(\boldsymbol{p} + e\boldsymbol{A}^*) + V, \tag{1.9.92}$$

which can be expanded as

$$H = \frac{1}{2m}(\boldsymbol{p}\cdot\boldsymbol{p} + i\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{p}) +$$

$$\frac{e}{2m}\big(\boldsymbol{p}\cdot\boldsymbol{A}^* + \boldsymbol{A}\cdot\boldsymbol{p} + i(\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{A}^* + \boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{p})\big) +$$

$$\frac{e^2}{2m}\big(\boldsymbol{A}\cdot\boldsymbol{A}^* + i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*\big). \tag{1.9.93}$$

The effect of the magnetic field on the half integer spin of the electron or proton, the source of ESR and NMR, is seen through the following term:

$$H'\psi = i\frac{e}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p}\times\boldsymbol{A} + \boldsymbol{A}\times\boldsymbol{p})\psi$$

$$= i\frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\big((\nabla\times\boldsymbol{A})\psi + (\nabla\psi)\times\boldsymbol{A} + \boldsymbol{A}\times\nabla\psi\big) \tag{1.9.94}$$

$$= \frac{e\hbar}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{B})\psi.$$

This is the famous "half integral spin" term usually attributed to the quantum relativistic Dirac equation in its nonrelativistic limit. The existence of ESR and NMR therefore depends on the correspondence between classical variables and quantum operators, and so it is a purely quantum effect. For one electron is the part of the Zeeman effect[30], and gives the nonzero ground state energy:

$$E^{(1)} = \frac{e\hbar}{2m}\langle 0|\boldsymbol{\sigma}\cdot\boldsymbol{B}|0\rangle \tag{1.9.95}$$

in perturbation theory[27].

The nonrelativistic Schrödinger-Pauli equation (1.9.91) therefore gives an explanation for the Stern-Gerlach experiment and the anomalous Zeeman effect in the nonrelativistic limit by essentially

replacing the momentum p with $\boldsymbol{\sigma} \cdot \boldsymbol{p}$ by the method given by Sakurai[30].

With complex valued $\boldsymbol{A}$ in equation (1.9.92) the RFR term is obtained:

$$H_{RFR} = i\frac{e^2}{2m}\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \boldsymbol{A}^*. \tag{1.9.96}$$

This is classical and real valued as it is obtained with out the replacement $\boldsymbol{p} \rightarrow -i\hbar\nabla$. Using the SI relation between beam intensity and $|\boldsymbol{A}|^2$, we obtain from equation (1.9.96):

$$H_{RFR} = \frac{\mu_0 c e^2}{2m}\frac{I}{\omega^2}\sigma_z, \tag{1.9.97}$$

which gives the familiar result for the RFR term as the matrix valued term along σ_z and proportional to $1/\omega^2$. In perturbation theory there exists the nonzero ground state term:

$$H_{RFR} = i\frac{e^2}{2m}\langle 0|\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \boldsymbol{A}^*|0\rangle. \tag{1.9.98}$$

The fundamental nonAbelian relation of $O(3)_b$ gauge theory

$$\boldsymbol{B}^{(3)} = -i\frac{e}{\hbar}\boldsymbol{A} \times \boldsymbol{A}^* \tag{1.9.99}$$

means that the RFR term may be rewritten as:

$$E^{(1)} = \frac{e\hbar}{2m}\langle 0|\boldsymbol{\sigma} \cdot \boldsymbol{B}^{(3)}|0\rangle, \tag{1.9.100}$$

which defines the fundamental interaction between the fermion half integral spin and the $\boldsymbol{B}^{(3)}$ field.

1.10 RESONANCE CONDITIONS IN RFR

Resonance occurs under the condition:

$$\hbar\omega_{res} = \frac{e^2c^2B^{(0)2}}{2m\omega^2}, \tag{1.10.101}$$

where ω_{res} is the angular frequency of the probe electromagnetic field. For an electron the resonance frequency for the electron in Hertz is therefore given by[31]:

$$\omega_{res} = 1.007 \times 10^{28}\frac{I}{\omega^2}. \tag{1.10.102}$$

For a proton the g factor is 5.5857, while for the electron it is 2.002; and the mass of the proton is 1836 times that of the electron. The proton resonance in RFR is therefore[20]

$$\omega_{res} = 1.532 \times 10^{25}\frac{I}{\omega^2}. \tag{1.10.103}$$

The theory can be tested under ideal condition with the use of an electron or proton beam, and the dependence of the resonance frequency of $1/\omega^2$ should be detected empirically. In the absence of such data there are few experiments in the literature which can be used to test equation (1.10.101). Pershan et al [33] in their empirical demonstration of the inverse Faraday effect show the presence of a term proportional to $i\boldsymbol{\sigma} \cdot \boldsymbol{E} \times \boldsymbol{E}^*$ in perturbation theory; a term which was observed empirically in rare earth doped glasses at low temperature. Warren et al. [34] report sub Hertzian shifts in molecular liquids subjected to irradiation with a circularly polarized argon laser at $528.7nm$, $488nm$, and $476.5nm$ with a power density of $10w/cm^2$. Under these condition the resonance frequencies from equation (1.10.103) are 0.12, 0.10 and $0.098Hz$. However, the data by Warren et al. [34] are probably artifactual, or at best heavily influenced on their own admission by heating or noise, because they did not find the expected alignment of spins by a circularly polarized laser[35]. The latter phenomenon has been dramatically confirmed by Brown et al. [36] with a krypton ion laser of about the same intensity as Warren et al. [24]. Brown et al. [36] report clear evidence for alignment of spins by a circularly polarized laser through the Fermi contact mechanism in III V semiconductors by tuning the laser according to optical selection rules. Other RFR mechanisms have recently been suggested theoretically [37]: these are essentially variations of the original theme [38]. Under the right condition, as defined by Brown et al.[36], Optical NMR (ONMR) produces an

enormous increase in sensitivity of many order of magnitude over ordinary NMR. A similar dramatic improvement in resolution is expected from RFR[31], utilizing the $1/\omega^2$ dependence. To test these effects electron and proton beam data are needed to test the simple theory developed here without the complicating noise effects due to sample choice or heating effects.

1.11 CHEMICAL SHIFTS IN NMR

Another potentially useful feature of RFR is that it is site selective. The chemical shift pattern is however dependent on a different molecular property tensor than NMR. This was found by Harris and Tinoco [39] with time independent perturbation theory. The spectral fingerprint in RFR is therefore usefully different from NMR, and provides in principle a high resolution, site specific resonance technique. Brown et al. [36] have already demonstrated site specificity in ONMR [40-43] down to the level of quantum dots. This is a dramatic demonstration of the enhancement possible with the use of a circularly polarized laser in NMR and ESR, as originally discussed [31].

The calculation of the chemical shift in RFR is straightforward [39]. Harris and Tinoco considered the second order perturbation energy (SI units):

$$E = \sum_n \frac{\langle 0|H|n\rangle\langle n|H|0\rangle}{\hbar\omega_{on}}, \tag{1.11.104}$$

with the perturbation Hamiltonian:

$$H = \frac{1}{2m}(\boldsymbol{p} + e(\boldsymbol{A} + \boldsymbol{A}_N))^2 + V, \tag{1.11.105}$$

where

$$\boldsymbol{A}_N = \frac{\mu_0}{4\pi r^3}\boldsymbol{m}_N \times \boldsymbol{r} \tag{1.11.106}$$

is the vector potential [27] due to the nuclear magnetic dipole moment $\boldsymbol{m}_N$. The perturbation term relevant to the RFR chemical

shift is the single photon off resonance population term [39], which is by far the dominant chemical shift term:

$$E = i\frac{e^3}{m^2\hbar\omega_{on}}\sum_n \langle 0|\boldsymbol{p}\cdot\boldsymbol{A}|n\rangle\langle n|\boldsymbol{A}_N\cdot\boldsymbol{A}^*|0\rangle + c.c.. \tag{1.11.107}$$

The electric dipole moment is defined [39] by:

$$\langle 0|\boldsymbol{\mu}|n\rangle = \frac{e}{m\omega_{on}}\langle 0|\boldsymbol{p}|n\rangle. \tag{1.11.108}$$

The vector relations:

$$i(\boldsymbol{\mu}\times(\boldsymbol{m}_m\times\boldsymbol{r}))\cdot(\boldsymbol{A}\times\boldsymbol{A}^*) = i(\boldsymbol{\mu}\cdot\boldsymbol{A})((\boldsymbol{m}_N\times\boldsymbol{r})\cdot\boldsymbol{A}^*) - i(\boldsymbol{\mu}\cdot\boldsymbol{A}^*)((\boldsymbol{m}_N\times\boldsymbol{r})\cdot\boldsymbol{A}), \tag{1.11.109}$$

and

$$\boldsymbol{\mu}\times(\boldsymbol{\mu}_N\times\boldsymbol{r}) = (\boldsymbol{\mu}\cdot\boldsymbol{r})\boldsymbol{m}_N - (\boldsymbol{\mu}\cdot\dot{\boldsymbol{m}}_N)\boldsymbol{r} \tag{1.11.110}$$

demonstrate that expression (1.10.107) may be written a:

$$E = \zeta\left(i\frac{e^2}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*\right) \tag{1.11.111}$$

where

$$\zeta = \frac{g_N e\mu_0}{8\pi m}\sum_n \langle 0|\boldsymbol{\mu}|n\rangle\cdot\langle n|\frac{\boldsymbol{r}}{r^3}|0\rangle. \tag{1.11.112}$$

Here $\boldsymbol{m}_N = g_N(e/4m)\hbar\boldsymbol{\sigma}$, and equation (1.11.111) This defines the RFR chemical shift factor or shielding constant. It depends upon the novel molecular tensor in equation (1.11.112), where this is novel in the sense that it is not the same tensor that defines the chemical shift in NMR [27] through the well known Lamb shift formula. Its order of magnitude in the first approximation is around 10^{-6}; roughly the same as in NMR. The complete RFR spectrum from the protons in atoms and molecules is therefore, from equations (1.11.111) and (1.11.112):

$$T_{int} = i\frac{e^2}{2m}(1+\zeta)\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*, \tag{1.11.113}$$

and is site specific by the site specific nature of ζ.

1.12 CLASSICAL DERIVATION OF THE INVERSE FARADAY EFFECT

Consider the effect of an electromagnetic field on a proton or electron, treated here as classical particles, in a circular orbit with

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}, \ \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{1.12.114}$$

where $\boldsymbol{\omega}$ is the angular velocity, $\boldsymbol{r}$ is the radius of the orbit and $\boldsymbol{v}$ is the tangential velocity. The tangential momentum is $\boldsymbol{p}$ and the orbital angular momentum is $\boldsymbol{L}$. The energy of the particle is:

$$E = \frac{1}{2}\omega|\boldsymbol{L}| = \frac{\omega}{2}|\boldsymbol{r} \times \boldsymbol{p}|, \tag{1.12.115}$$

and so

$$\omega|\boldsymbol{L}| = \frac{p^2}{m} = \omega r p, \ r = \frac{p}{m\omega}. \tag{1.12.116}$$

If we use the minimal prescription in the form [44] then:

$$p = eA, \ p^* = eA^*, \tag{1.12.117}$$

from which we obtain:

$$r = \frac{eA}{m\omega} = \frac{eE}{m\omega^2}, \tag{1.12.118}$$

This is equation (1) of reference [44] in the nonrelativistic limit, and is the simplest way to derive the orbital inverse Faraday effect. Its angular momentum magnitude is:

$$L = \frac{p^2}{m\omega} = \frac{e^2E^2}{m\omega^3}, \tag{1.12.119}$$

and the energy is

$$E = \frac{1}{2}\omega L = \frac{e^2E^2}{2m\omega^2}. \tag{1.12.120}$$

As a result equation (1.12.119) may be written as

$$\boldsymbol{L} = i\frac{e^2}{m\omega}\boldsymbol{A} \times \boldsymbol{A}^* \tag{1.12.121a}$$

in complex notation. The vacuum definition of $\boldsymbol{B}^{(3)}$ we arrive at the orbital angular momentum as:

$$\boldsymbol{L} = -\frac{e\hbar}{m\omega}\boldsymbol{B}^{(3)}. \tag{1.12.121b}$$

The energy of the inverse Faraday effect for one orbiting classical electron or proton is then:

$$E = -\frac{e\hbar}{2m}|\boldsymbol{B}^{(3)}|. \tag{1.12.122}$$

The form of this energy is the same as that of the Zeeman effect, but the spin component is missing. The latter can be reinstate by using the classical interaction energy:

$$E = \frac{1}{2m}\boldsymbol{\sigma} \cdot (\boldsymbol{p} + e\boldsymbol{A})\boldsymbol{\sigma} \cdot (\boldsymbol{p} + e\boldsymbol{A}^*)$$

$$= i\frac{e^2}{2m}\boldsymbol{A} \times \boldsymbol{A}^* \cdot \boldsymbol{\sigma} + \ldots = -\frac{e\hbar}{2m}\boldsymbol{\sigma} \cdot \boldsymbol{B}^{(3)} + \ldots. \tag{1.12.123}$$

$$= \frac{e}{m}\boldsymbol{s} \cdot \boldsymbol{B}^{(3)} + \ldots$$

The total energy is thus

$$E = -\frac{e}{2m}(\boldsymbol{L} + 2\boldsymbol{s}) \cdot \boldsymbol{B}^{(3)}, \tag{1.12.124}$$

which is the same as the expression for the complete Zeeman effect, most often due to a static external magnetic field.

Finally, relativistic correction can be applied to this calculation. As an example the relativistic momentum

$$\boldsymbol{p} = \gamma m\boldsymbol{v}, \tag{1.12.125}$$

and the relativistic expression for the kinetic energy

$$T - E = \gamma m v^2 \tag{1.12.126}$$

The relativistic corrected orbital angular momentum is then:

$$J = \frac{1}{\omega}(T - E) = \frac{e^2E^2}{m\gamma\omega^3} \qquad (1.12.127)$$

which is given in reference [44].

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CHAPTER 2
THE FIELD EQUATIONS OF CLASSICAL $O(3)_b$ ELECTRODYNAMICS

2.1 INTRODUCTION

In this chapter the classical field equations of $O(3)_b$ electrodynamics are given in the basis $((1),(2),(3))$ suggested by the empirical existence of circularly polarized radiation at the foundational level. The structure of the equations is such that the covariant derivative of the rotation group, $O(3)_b$, replaces the ordinary derivative in the Maxwell field equations ($U(1)$ electrodynamics) and vectors and tensors in Minkowski spacetime are written as vectors in the internal gauge space $((1),(2),(3))$. As a result the equations become intrinsically nonlinear, and give the fundamental field component $\boldsymbol{B}^{(3)}$ in the vacuum [1-8]. The classical field equations of electrodynamics become structured within a nonAbelian field theory [9,10] with an internal $O(3)_b$ gauge symmetry. These equations support a physical interpretation of field potentials similar to Faraday's original concept of the electronic state [11]. The $U(1)$ field theory given in chapter one is a particular solution of the $O(3)_b$ theory, which therefore generalizes the former to a nonAbelian gauge field theory, one which is far richer in structure and physical content [12].

2.2 THE $O(3)_b$ FIELD EQUATIONS

The basic field equations of classical $O(3)_b$ electrodynamics are (chapter one):

$$\mathcal{D}_\mu \tilde{G}^{\mu\nu} = 0 \tag{2.2.1}$$

$$\mathcal{D}_\mu \tilde{H}^{\mu\nu} = J^\nu \tag{2.2.2}$$

where the four current components are defined by:

$$J^{a\mu} = (\rho^{(a)}, J^{(a)}/c), a \in \{1,2,3\} \qquad (2.2.3)$$

and where the field tensors have SI units as defined in chapter one. The bold symbols in eqns. (2.2.1) and (2.2.2) are vectors in the internal gauge space with $O(3)_b$ symmetry. The algebra of these vectors is developed as in ordinary vector algebra, where there are dot and cross products defined by differential operators in three dimensions. Therefore a symbol such as J^μ in eqn. (2.2.2) is a three vector in the internal $O(3)_b$ symmetry gauge space. Each component of the vector is itself a four-vector in Minkowski spacetime as given in eqn. (2.2.3). The algebra of the four-vector components is developed as in ordinary four-vector algebra of special relativity, using covariant-contravariant algebra and Einstein summation over repeated indices [13,14].

2.3 BASIC S. I. UNITS

The basic S.I. units of the quantities appearing in $O(3)_b$ electrodynamics are the same as in $U(1)$ electrodynamics, and some of the most commonly used units and fundamental quantities are given as follows, in which the bold symbols are the observable vector quantities as in $U(1)$ electrodynamics.

Electric field strength, $\boldsymbol{E} : Vm^{-1} = JsC^{-1}m^{-1}$
Electric displacement, $\boldsymbol{D} : Cm^{-2}$
Magnetic flux density, $\boldsymbol{B} : T = Wm^{-2} = JsC^{-1}m^{-2}$
Magnetic Field Strength $\boldsymbol{H} : Am^{-1} = Cs^{-1}m^{-1}$
Magnetic vector potential $\boldsymbol{A} : JsC^{-1}m^{-1}$
Polarization $\boldsymbol{P} : Cm^{-2}$
Magnetization $\boldsymbol{M} : Am^{-1} = Cs^{-1}m^{-1}$
Charge density $\rho : Cm^{-3}$
Current density $\boldsymbol{J} : Am^{-2} = Cs^{-1}m^{-2}$
Vacuum permittivity $\epsilon_0 := 8.854188 \times 10^{-12} J^{-1}C^2m^{-1}$

Vacuum permeability $\mu_0 := 4\pi \times 10^{-7} Js^2C^{-2}m^{-1}$
Electronic g factor $g := 2.002319314$
Dirac constant $\hbar := 1.05459 \times 10^{-34} Js$
Fine structure constant $\alpha := \frac{e^2}{\hbar c} = 0.007297351$
Speed of light in vacuo, $c := 2.997925 \times 10^8 ms^{-1}$
Elementary charge $e := 1.60219 \times 10^{-19} C$
Electron mass $m_e := 9.10953 \times 10^{-31} kg$
Proton mass $m_p := 1.67265 \times 10^{27} kg$
Bohr magneton $\mu_B := \frac{e\hbar}{2m_e} = 9.27408 \times 10^{-24} JT^{-1}$

2.4 THE $O(3)_b$ VACUUM EQUATIONS

The homogeneous $O(3)_b$ field equation in the vacuum is the Feynman Jacobi identity [12], eqn. (2.2.1), an identity which can be expanded as:

$$(\partial_\mu + g[A_\mu,]) \circ \tilde{G}^{\mu\nu} = 0 \qquad (2.4.4)$$

where the vector potential is defined as in chapter one, and where g is a factor with the units of $1/Wb$ (inverse of the magnetic flux); a factor determined [12] by dimensionality and not to be confused with the electron's g factor, which is dimensionless. Therefore:

$$\mathcal{D}_\mu = \partial_\mu + g[A_\mu,] \qquad (2.4.5)$$

is the covariant derivative in an $O(3)_b$ symmetry internal gauge space [12]. It is a derivative that is Lorentz covariant in special relativity, or equivalently flat Minkowski spacetime. This derivative stems from Yang-Mills field theory as applied to elementary particles [15,16]; the most successful formulation of gauge field theory in the twentieth century. The prediction and empirical verification of quarks, for example, is based on a Yang-Mills theory with internal gauge symmetry $SU(3)$, a nonAbelian symmetry like $O(3)_b$ [12].

2.4.1 THE COEFFICIENT g IN THE VACUUM

If we define the coefficient g in the vacuum to be the inverse of the elementary fluxon (magnetic flux of one photon):

$$g = \frac{e}{\hbar} = \frac{\kappa}{A^{(0)}} \qquad (2.4.1.6)$$

we recover the B Cyclic Theorem in the vacuum [1-8]:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = B^{(0)}\boldsymbol{B}^{(3)*} \; et \; cyclicum \tag{2.4.1.7}$$

where $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are components of the vacuum magnetic flux density of the electromagnetic field; and where $B^{(0)}$ is the magnitude of $\boldsymbol{B}^{(3)}$; and we also recover the quantization condition[1-8]:

$$p = \hbar\kappa = eA^{(0)} \tag{2.4.1.8}$$

where p is the linear momentum magnitude of the photon. Therefore eqn. (2.4.1.8) is also a minimal prescription in which p is replaceable by $eA^{(0)}$, where $A^{(0)} = B^{(0)}/\kappa$, with κ the wave-vector magnitude.

The charge quantization condition is a straightforward result of Planck quantization as follows. The energy of one photon is, from the correspondence principle [17-19]:

$$\hbar\omega = \frac{1}{\mu_0}{B^{(0)}}^2 V, \tag{2.4.1.9}$$

where V is the quantization volume. The magnetic flux density carried by the photon is:

$$B^{(0)} = \frac{1}{\mathcal{A}}\frac{\hbar}{e}, \tag{2.4.1.10}$$

where $\mathcal{A}$ is the quantization area and where $\hbar/e$ is the magnetic flux carried by one photon, the fluxon in Wb and a fundamental constant. From equations (2.4.1.9) and (2.4.1.10):

$$\alpha = \frac{1}{4\pi}\left(\frac{V}{\kappa\mathcal{A}^2}\right), \tag{2.4.1.11}$$

where α is the fundamental fine structure constant of quantum electrodynamics [12]. The ratio on the right hand side of equation (2.4.1.11) is therefore a constant of nature. If we consider the area $\mathcal{A}$ to be the determinant of the wave-vector [20-23]:

$$\mathcal{A} = \frac{1}{\kappa^2} \tag{2.4.1.12}$$

we recover equations (2.4.1.7) and (2.4.1.8), which conversely imply equation (2.4.1.12). The photon is the quantum of energy $\hbar\omega$, which is proportional [1-8] to e^2 and is C positive, where $C(e) = -e$ is the charge conjugation operator. The presence of e in the vacuum means that the electromagnetic field in $O(3)_b$ electrodynamics is C negative, and so influences a charge in a receiver. The charge e also occurs in the vacuum in $U(1)$ electrodynamics as discussed by Jackson [14] for example. An electron accelerated to approach c produces an electromagnetic field, and so both electron and field must be C negative. In elementary particle physics, the photon is a C positive particle, and this is the same in $O(3)_b$ and $U(1)$ electrodynamics, both classical and quantum.

The B Cyclic Theorem [1-8], equation (2.4.1.9), does not occur in $U(1)$ electrodynamics and is a non-linear, nonAbelian vacuum relation between field components. Within a factor $B^{(0)}$ it is a Lorentz covariant angular momentum relation [1-8] in relativistic quantum and classical field theory. Using equations (2.4.1.8) and (2.4.1.11) it is equivalent to the fundamental definition of $\boldsymbol{B}^{(3)}$ in $O(3)_b$ electrodynamics:

$$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{2.4.1.13}$$

and so $\boldsymbol{B}^{(3)}$ does not occur in $U(1)$, or field theory according to Maxwell's equations[1-8].

2.4.2 THE $O(3)_b$ FIELD TENSOR IN THE VACUUM

Equation (2.4.1.16) is part of the definition [1-8] of the $O(3)_b$ field tensor in the vacuum:

$$\frac{1}{c}G^{(1)\mu\nu} = \partial^\nu A^{(1)\mu} - \partial^\mu A^{(1)\nu} - ig[A^{(2)\mu}, A^{(3)\nu}] \tag{2.4.2.14}$$

with cyclic permutation of indices (a), $a = 1, 2, 3$. The magnetic part of equation (2.4.2.17) is [1-8]:

$$\frac{1}{c}\boldsymbol{G}^{(1)} = \mu_0\boldsymbol{H}^{(1)*} = \nabla \times \boldsymbol{A}^{(1)*} - i\frac{e}{\hbar}\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(3)}$$

$$\frac{1}{c}\boldsymbol{G}^{(2)} = \mu_0 \boldsymbol{H}^{(2)*} = \nabla \times \boldsymbol{A}^{(2)*} - i\frac{e}{\hbar}\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \quad (2.4.2.15)$$

$$\frac{1}{c}\boldsymbol{G}^{(3)} = \mu_0 \boldsymbol{H}^{(3)*} = \nabla \times \boldsymbol{A}^{(3)*} - i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$$

The meaning of the vector potential $\boldsymbol{A}^{(3)}$ is made clear through consideration of $SU(2) \times SU(2)$ electroweak theory, developed in chapter 8. It is important at very high energies, those in a large hadron collider, because it produces a very massive boson which is characteristic of the $SU(2) \times SU(2)$ theory. In $O(3)_b$ electromagnetism it is a constant, producing a classical field momentum $eA^{(3)}$. There can therefore be radiation reaction in $O(3)_b$ electrodynamics and in consequence a classical Compton effect.

Conservation of linear momentum in the interaction of field and Lorentz electron means that $eA^{(0)}$ is changed, and since $A^{(0)} = B^{(0)}/\kappa$, the frequency of the field is changed, as observed in the Compton effect [25]. The correspondence principle means that $eA^{(0)} = \hbar\kappa$, where $A^{(0)}$ is the magnitude of an effective $A^{(3)}$ that can be defined to give $\boldsymbol{B}^{(3)} = \nabla \times \boldsymbol{A}^{(3)}_{eff}$. It will be demonstrated later that the $\boldsymbol{A}^{(3)}$ field corresponds to a massive photon at high energy that does not contribute to physics at lower energies. In $U(1)$ electrodynamics there is no $A^{(3)}$, no classical Compton effect and no correspondence between the quantized photon momentum $\hbar\kappa$ and the classical field momentum, $eA^{(0)}$.

In $O(3)_b$ electrodynamics the relation between the magnetic field and the magnetic vector potential is therefore non linear in the vacuum. If we take a particular solution (Section (2.5)) of the nonabelian field equation (2.2.1) in such as way as to give Maxwell's equations for indices (1) and (2) we find that:

$$\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)} - i\frac{e}{\hbar}\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(3)} \quad (2.4.2.16)$$

$$\boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)} - i\frac{e}{\hbar}\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \quad (2.4.2.17)$$

$$\boldsymbol{B}^{(3)} = \nabla \times \boldsymbol{A}^{(3)} - i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \quad (2.4.2.18)$$

However, since $\boldsymbol{A}^{(3)}$ is latter demonstrated to correspond to a massive field that exists on only very short lengths, its curl is zero, and so the $\boldsymbol{B}^{(3)}$ field is given by equation (2.4.1.13), which is non-zero. Plane wave solutions for $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ self consistently leads back to the B Cyclic relation (2.4.1.7) and to equation (2.4.1.13). It is emphasized that this is a particular solution, in general, the vacuum field tensor is always non-linear with respect to the vector potentials in $O(3)_b$ electrodynamics.

The self magnetization in the vacuum:

$$\boldsymbol{M}^{(3)} = \frac{1}{\mu_0}\boldsymbol{B}^{(3)} \quad (2.4.2.19)$$

is absent from $U(1)$ electrodynamics and together with the self polarization, equation (2.4.2.28), leads to a minute correction in the g factor of the electron, the Lamb shift, and the Casimir effect in $O(3)_b$ quantum electrodynamics (chapter 6): corrections which start to occur at fifth power in the fine structure constant. Therefore the precision of quantum electrodynamics will have to be increased substantially before we can observe these minute effects. The current precision of quantum electrodynamics (QED) cannot be used to assert, as has unfortunately become customary, that QED is a $U(1)$ gauge theory. On the classical level $O(3)_b$ electrodynamics contains a vacuum magnetization (2.4.2.20) which does not occur in $U(1)$ electrodynamics:

$$\boldsymbol{M}^{(3)}{}_{vacuum} = -i\frac{e}{\mu_0\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}. \quad (2.4.2.20)$$

When the $O(3)_b$ field interacts with one electron equation (2.4.2.21) becomes the inverse Faraday effect, in which g of the vacuum changes its value to a factor g', obviously with the same units. The gauge

symmetry of the theory remains the same: $O(3)_b$. The inverse Faraday effect is therefore:

$$\boldsymbol{M}^{(3)}{}_{IFE} = -i\frac{g'}{\mu_0}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \qquad (2.4.2.21)$$

where g' can be defined as in chapter one from the minimal prescription in $O(3)_b$, using the complex vector potentials $\boldsymbol{A} = \boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^* = \boldsymbol{A}^{(2)}$ (Section (1.12)).

The discerning reader will have realized that as soon as $\boldsymbol{A}$ becomes complex, the theory becomes an $O(3)_b$ gauge field theory, and there is a link between this deduction and the existence of ordinary circular polarization, discovered in 1811 by Arago.

The inverse Faraday effect, which has been observed on numerous occasions [26], shows that when there is field matter interaction electrodynamics is an $O(3)_b$ gauge theory, or alternatively a sub theory of a higher order nonAbelian gauge field theory such as one with $SU(3)$ symmetry. The reason is simple, $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is identically zero in a $U(1)$ gauge field theory, which is Abelian and linear [12]. The process of field matter interaction is therefore summarized simply by:

$$g_{vacuum} \rightarrow g'_{fields/matter}. \qquad (2.4.2.22)$$

Clearly, $\boldsymbol{B}^{(3)}$ of the vacuum is an empirical observable of the inverse Faraday effect when there is field matter interaction. As discussed in chapter one, the third Stokes parameter of the vacuum field is proportional to $\boldsymbol{B}^{(3)}$ and to $\boldsymbol{M}^{(3)}$ in the vacuum (in the absence of matter and material sources), so in $O(3)_b$ electrodynamics, the existence of ordinary circular polarization is sufficient to demonstrate the existence of $\boldsymbol{B}^{(3)}$ in the vacuum through the fundamental definition of the $O(3)_b$ gauge field tensor in equation (2.4.2.14). In vacuum

electrodynamics considered as a $U(1)$ gauge field theory, all nonlinearities such as $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ are axiomatically zero, so the third Stokes parameter is then defined by auxilliary assumptions, because it is proportional directly to $\boldsymbol{A} \times \boldsymbol{A}^*$ in the vacuum (chapter one). The $U(1)$ gauge field theory of electrodynamics therefore does not allow the existence of circular polarization in as consistent a manner. Circular polarization was discovered by Arago in 1811 and explained by Stokes in 1852 several years before the proposal by Maxwell of a set of twenty quaternionic equations.

These became the "Maxwell equations" of chapter one only after considerable simplification by Heaviside and his contemporaries. The Heaviside Maxwell equations were put into covariant form at the turn of this century, during the development of special relativity, a form linear in the potential. In 1955, Yang and Mills developed a gauge field theory in which the field tensor can be non-linear in the potential, and the electrodynamics developed in this book shows that the conjugate product $\boldsymbol{A} \times \boldsymbol{A}^*$ needed to describe ordinary circular polarization appears automatically from gauge principles if and only if the internal gauge symmetry is not $U(1)$. In this book we evaluate some of the numerous consequences of using an $O(3)_b$ internal gauge symmetry.

Specifically, the third Stokes parameter is:

$$S_3 = i|\boldsymbol{E} \times \boldsymbol{E}^*|, \qquad (2.4.2.23)$$

and using the scalar vacuum relations:

$$E^{(0)} = cB^{(0)} = \omega A^{(0)} \qquad (2.4.2.24)$$

we obtain:

$$S_3 = i\omega^2|\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}| = \omega^2\frac{\hbar}{e}|\boldsymbol{B}^{(3)*}| \qquad (2.4.2.25)$$

in $O(3)_b$ electrodynamics. In $U(1)$ electrodynamics, $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero and so in this setting we would have:

$$S_3 = 0 \tag{2.4.2.26}$$

axiomatically. However, S_3 is an observable, it is the basic signature of everyday circular polarization [27], and so $U(1)$ electrodynamics is axiomatically less satisfactory on empirical grounds. The existence of circular polarization, and thus of S_3, is sufficient to demonstrate the existence of $\boldsymbol{B}^{(3)}$ in the vacuum in $O(3)_b$ electrodynamics. In $U(1)$ electrodynamics the third Stokes parameter is axiomatically zero and $U(1)$ electrodynamics is incapable of explaining circular polarization from the first principles of gauge field theory. The reason is that those first principles assert that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in a $U(1)$ gauge field theory is axiomatically zero without auxilliary conditions. The existence of the $\boldsymbol{B}^{(3)}$ field eliminates the radiation reaction problem with $U(1)$ electrodynamics. The field tensor in a gauge theory is in general a commutator of covariant derivatives, which in general is non-linear in $\boldsymbol{A}$. It is this non-linearity that gives rise to the third Stokes parameter and circular polarization. This ties circularly polarized radiation with physical corrections of electrodynamics. If we remove the non-linearity (use $U(1)$ gauge field theory) we remove circular polarization as a source for these corrections.

Equation (2.4.2.14) also produces the electric field part of the tensor in vacuo:

$$\boldsymbol{E}^{(1)} = \frac{\partial \boldsymbol{A}^{(1)}}{\partial t} - igA^{(0)}\boldsymbol{A}^{(1)}$$

$$\boldsymbol{E}^{(2)} = \frac{\partial \boldsymbol{A}^{(2)}}{\partial t} - igA^{(0)}\boldsymbol{A}^{(2)}. \tag{2.4.2.27}$$

The $\boldsymbol{E}^{(3)}$ field is zero [1-8], and there appears to be no empirical evidence for such a field component at low electrodynamic energies. This result is captured by an $SU(2) \times SU(2)$ electroweak theory, and is outlined in chapter 8. There are vacuum polarization terms:

$$\boldsymbol{P}^{(a)} = -igc\epsilon_0 A^{(0)}\boldsymbol{A}^{(a)},\ a = 1,\ or\ 2, \tag{2.4.2.28}$$

which for plane waves self consistently gives the results:

$$\boldsymbol{E}^{(1)} = \frac{\partial \boldsymbol{A}^{(1)}}{\partial t} - ig\omega\boldsymbol{A}^{(1)}$$

$$\boldsymbol{E}^{(2)} = \frac{\partial \boldsymbol{A}^{(2)}}{\partial t} - ig\omega\boldsymbol{A}^{(2)}. \tag{2.4.2.29}$$

Therefore in $O(3)_b$ electrodynamics the field tensor in the vacuum is conveniently summarized as:

$$\boldsymbol{D}^{(2)}{}_{vac} = \epsilon_0 \boldsymbol{E}^{(2)} + \boldsymbol{P}^{(2)}{}_{vac} \tag{2.4.2.30}$$

2.5 REDUCTION TO MAXWELL'S EQUATIONS

The $O(3)_b$ field equations (2.2.1) and (2.2.2) reduce in a straightforward manner to Maxwell's equations in indices (1) and (2) by choosing particular solutions. First consider the homogeneous field equation (2.2.1) and its particular solution:

$$\partial_\mu \tilde{G}^{(a)\mu\nu} = 0 \tag{2.5.31}$$

$$[A_\mu,\ \tilde{G}^{(a)\mu\nu}] = 0. \tag{2.5.32}$$

Equation (2.5.31) gives a complex conjugate pair of homogeneous Maxwell equations:

$$\partial_\mu \tilde{G}^{(a)\mu\nu} = 0, a = 1\ or\ 2, \tag{2.5.33}$$

and the equation:

$$\partial_\mu \tilde{G}^{(3)\mu\nu} = 0. \tag{2.5.34}$$

It can be shown that equation (2.5.32) gives the B Cyclic Theorem self consistently[1-8] if $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is a plane wave (solutions of equation (2.5.33). Eqn. (2.5.34) gives the result in vacuo:

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = 0, \tag{2.5.35}$$

and so there is no "Faraday induction" due to $B^{(3)}$ as observed empirically [28]. The latter is an obvious result because there is no "Faraday induction" due to the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ or third Stokes parameter. The equation of Faraday induction is equation (2.5.33).

The particular solution (2.5.31) gives the Maxwell equations (2.5.33) and therefore the Faraday and Gauss Laws, together with equation (2.5.34) which shows that $\boldsymbol{B}^{(3)}$ is a constant of motion. The solution (2.5.32) gives the B cyclic theorem for plane wave solutions of equation (2.5.33). Therefore Maxwell's electrodynamics are recovered together with equations for the constant of motion $\boldsymbol{B}^{(3)}$. The theoretical structure is a nonabelian gauge field structure, so is Lorentz covariant [12]. As described in section (2.4), the $\boldsymbol{B}^{(3)}$ field is observable through the third Stokes parameter S_3 and the existence of circular polarization. More generally, equation (2,2,1) should be solved numerically making as few assumptions as possible. The Stokes Theorem relevant to this theory is the integral form of equation (2.2.1), the original Feynman Jacobi identity. It is incorrect to apply the $U(1)$ theory of electrodynamics to $\boldsymbol{B}^{(3)}$, and as a result requires considerable auxilliary phenomenology, as attempted recently [29].

The inhomogeneous Maxwell equations in vacuo in indices (1) and (2) are recovered from equation (2.2.2) using the particular solution:

$$\partial_\mu G^{(a)\mu\nu} = 0 \tag{2.5.36}$$

$$J^{(a)\nu} = g\epsilon_0 \epsilon^{abc}[A^{(b)}{}_\mu,\ G^{(c)\mu\nu}] \tag{2.5.37}$$

The inhomogeneous Maxwellian equations are:

$$\partial_\mu G^{(a)\mu\nu} = J^{(a)\nu}, a = 1\ or\ 2, \tag{2.5.38}$$

and the equation for $\boldsymbol{B}^{(3)}$ is:

$$\partial_\mu G^{(3)\mu\nu} = 0, \tag{2.5.39}$$

which in vector form is:

$$\nabla \times \boldsymbol{B}^{(3)} = 0. \tag{2.5.40}$$

This is consistent with the fact that there is no $\boldsymbol{E}^{(3)}$ in the vacuum and with the fact that $\boldsymbol{B}^{(3)}$ is a constant of motion [1-8] whose curl and time derivative both vanish. This is why $\boldsymbol{B}^{(3)}$ is sometimes known as the fundamental spin field of $O(3)_b$ electrodynamics.

The Maxwell equations (2.5.38) give the Coulomb and Ampere-Maxwell laws in the vacuum. The additional cyclic equation (2.5.37) produces an energy term due to $\boldsymbol{B}^{(3)}$ which does not exist in $U(1)$ electrodynamics. We start by considering the current term as described in equation (2.5.38)

$$E = -\int J_\nu^{(a)} A^{(a)}{}_\nu dV, \tag{2.5.41}$$

where V is the volume containing the radiation. This energy term can be developed as follows:

$$E = -\frac{g}{\mu_0}\epsilon^{abc}\int [A^{(b)}{}_\mu,\ G^{(c)\mu\nu}]A^{(a)}{}_\nu dV$$

$$= -\frac{g}{2\mu_0}\epsilon^{abc}\int A^{(b)}{}_\mu [G^{\mu\nu},\ A^{(c)}{}_\nu]dV, \tag{2.5.42a}$$

and since $A^{(3)}{}_\mu = 0$ we see that the this reduces to:

$$E_3 = \frac{g^2}{2\mu_0}\int [A^{(1)\mu},\ A^{(2)\nu}][A^{(1)}{}_\mu,\ A^{(2)}{}_\nu]dV = \frac{1}{2\mu_0}\int \boldsymbol{B}^{(3)}.\boldsymbol{B}^{(3)}dV. \tag{2.5.42b}$$

This is the electromagnetic energy in a volume V of radiation due to the $\boldsymbol{B}^{(3)}$ component of vacuum $O(3)_b$ electrodynamics. The particular solution (2.5.37) therefore gives this energy self consistently with

the inhomogeneous Maxwell equations (2.5.38) and the equation (2.5.39) for the constant of motion $B^{(3)}$.

Inhomogeneous Maxwell equations when there is field matter interaction can always be obtained by writing equation (2.2.2) as:

$$\partial_\mu H^{\mu\nu} = -gA_\mu \times H^{\mu\nu} + J_m{}^\nu + J_1{}^\mu, \tag{2.5.43}$$

and using the particular solutions:

$$\partial_\mu H^{\mu\nu} = J_m{}^\nu \tag{2.5.44}$$

$$J_1{}^\nu = gA_\mu \times H^{\mu\nu} \tag{2.5.45}$$

Equation (2.5.44) gives the Maxwell equations for field matter interaction:

$$\partial_\mu H^{(a)\mu\nu} = J_m{}^{(a)\nu}, \; a = 1 \; or \; 2 \tag{2.5.46}$$

together with:

$$\partial_\mu H^{(3)\mu\nu} = J_m{}^{(3)\nu}, \tag{2.5.47}$$

and the relation (2.5.45) which gives the energy due to the interaction of $B^{(3)}$ with matter, for example an electron or proton:

$$E_3 = -\int J_1{}^\nu A_\nu dV. \tag{2.5.48}$$

Equation (2.5.47) governs the behavior of the magnetization due to $B^{(3)}$, for example in the inverse Faraday effect. If this magnetization is phase free, then the current $J_m{}^{(3)}$ is self-consistently zero because the four-derivative of a phase free magnetization tensor is zero. In general (sect. 2.10) the equation governing field matter interaction is equation (2.2.2) and the self-consistent Maxwellian point of view emerges from a particular solution of this equation together with equations (2.5.47) and (2.5.48).

2.6 THE FUNDAMENTAL LAWS OF $O(3)_b$ ELECTRODYNAMICS

The four fundamental laws of $O(3)_b$ electrodynamics are similar to the Gauss, Faraday, Coulomb and Ampere Maxwell laws of $U(1)$ electrodynamics and as shown in section 2.5 reduce to them under well defined conditions. However, in general, $O(3)_b$ electrodynamics is a nonAbelian gauge field theory which considerably enriches the content of electrodynamics as discussed recently by Barrett [30]. The four basic laws are written out in full in this section in the basis $((1),(2),(3))$ and are also given by Barrett [30] in a slightly different notation.

The $O(3)_b$ Gauss Law

The $O(3)_b$ Gauss law allows for the existence of a topological magnetic monopole and takes the form of three equations:

$$\nabla \cdot \boldsymbol{B}^{(1)*} = ig(\boldsymbol{A}^{([2)} \cdot \boldsymbol{B}^{3])}) \tag{2.6.49a}$$

$$\nabla \cdot \boldsymbol{B}^{(2)*} = ig([\boldsymbol{A}^{([3)} \cdot \boldsymbol{B}^{1])}]) \tag{2.6.49b}$$

$$\nabla \cdot \boldsymbol{B}^{(3)*} = ig(\boldsymbol{A}^{([1)} \cdot \boldsymbol{B}^{2])}) \tag{2.6.49c}$$

The particular solution discussed in Section (2.5) reduces these equations to:

$$\nabla \cdot \boldsymbol{B}^{(a)*} = 0, \; a = 1, \; 2 \tag{2.6.50a}$$

where the square bracket on the internal indices indicates commutation with respect to those indices. We recover the Gauss Law for $a = 1 \; or \; 2$ and for $a = 3$, and obtain that $\boldsymbol{B}^{(3)}$ there is a magnetic flux monopole since the divergence is in general nonzero. In quantum mechanics we have that the electric and magnetic fields are conjugate observables, $[\boldsymbol{E}, \boldsymbol{B}] = i\hbar$ with the commuation rule

$$[E^{(a)}{}_i, \; B^{(b)}{}_j] = i\hbar\delta_{ij}\delta^{ab}. \tag{2.6.50b}$$

This apparently indicates a departure from the purely classical description of $O(3)_b$ electrodynamics. This appears to indicate that

$O(3)_b$ electrodynamics might involve pure vacuum symmetries that are are not determined by the Hamiltonian.

The $O(3)_b$ Faraday Induction Law.

The Faraday Law of induction can be developed as an $O(3)_b$ gauge field theory described by the three equations:

$$\nabla \times \boldsymbol{E}^{(1)} + \frac{\partial \boldsymbol{B}^{(1)}}{\partial t} = -ig(cA_0{}^{([3]}\boldsymbol{B}^{[2])} + [\boldsymbol{A}^{(2)}, \boldsymbol{E}^{(3)}]) \quad (2.6.51a)$$

$$\nabla \times \boldsymbol{E}^{(2)} + \frac{\partial \boldsymbol{B}^{(2)}}{\partial t} = -ig(cA_0{}^{([1]}\boldsymbol{B}^{[3])} + [\boldsymbol{A}^{(3)}, \boldsymbol{E}^{(1)}]) \quad (2.6.51b)$$

$$\nabla \times \boldsymbol{E}^{(3)} + \frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = -ig(cA_0{}^{([2]}\boldsymbol{B}^{[1])} + [\boldsymbol{A}^{(1)}, \boldsymbol{E}^{(2)}]) \quad (2.6.51c)$$

and the particular solution corresponding to the Maxwell theory is:

$$\nabla \times \boldsymbol{E}^{(a)} + \frac{\partial \boldsymbol{B}^{(a)}}{\partial t} = 0, \; for \; a = 1 \; or \; 2,$$

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = 0. \quad (2.6.52)$$

Here commutator notation is used for simplicity in notation. For $a = 1, 2$ these equations are complex valued, and conjugate, Maxwell equations of induction, and the last equation for $a = 3$ is the law for the constancy of $\boldsymbol{B}^{(3)}$.

The $O(3)_b$ Coulomb Law

The $O(3)_b$ Coulomb Law is given by the Poisson equations:

$$\nabla \cdot \boldsymbol{E}^{(1)} = \frac{1}{\epsilon_0}\rho^{(1)} + igA^{([2)} \cdot \boldsymbol{E}^{3])} \quad (2.6.53a)$$

$$\nabla \cdot \boldsymbol{E}^{(2)} = \frac{1}{\epsilon_0}\rho^{(2)} + igA^{([3)} \cdot \boldsymbol{E}^{1])} \quad (2.6.53b)$$

$$\nabla \cdot \boldsymbol{E}^{(3)} = \frac{1}{\epsilon_0}\rho^{(3)} + igA^{([1)} \cdot \boldsymbol{E}^{2])}. \quad (2.6.53c)$$

This leads to the solution of which is:

$$\nabla \cdot \boldsymbol{E}^{(a)*} = \frac{1}{\epsilon_0}\rho^{(a)}, a = 1, 2, \quad (2.6.54)$$

which are complex conjugate pairs corresponding to the original Coulomb law in vacuo for $\rho^{(a)} = 0$.

The $O(3)_b$ Ampere Maxwell Law

Similarly there are three $O(3)_b$ Ampere Maxwell equations:

$$\nabla \times \boldsymbol{B}^{(1)} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(1)*}}{\partial t} =$$

$$\frac{1}{\epsilon_0 c^2}\boldsymbol{J}^{(1)} + i\frac{g}{c}(A_0{}^{([2]}, \boldsymbol{B}^{3])} + c[\boldsymbol{A}^{(2)}, \boldsymbol{B}^{(3)}]) \quad (2.6.55a)$$

$$\nabla \times \boldsymbol{B}^{(2)} - \frac{1}{c^2}\frac{\partial \boldsymbol{B}^{(2)*}}{\partial t} =$$

$$\frac{1}{\epsilon_0 c^2}\boldsymbol{J}^{(2)} + i\frac{g}{c}(A_0{}^{([3]}, \boldsymbol{B}^{1])} + c[\boldsymbol{A}^{(3)}, \boldsymbol{E}^{(1)}]) \quad (2.6.55b)$$

$$\nabla \times \boldsymbol{B}^{(3)} - \frac{1}{c^2}\frac{\partial \boldsymbol{B}^{(3)*}}{\partial t} =$$

$$\frac{1}{\epsilon_0 c^2}\boldsymbol{J}^{(3)} + i\frac{g}{c}(A_0{}^{([1]}, \boldsymbol{B}^{2])} + c[\boldsymbol{A}^{(1)}, \boldsymbol{E}^{(2)}]), \quad (2.6.55c)$$

where the vacuum solution is

$$\nabla \times \boldsymbol{B}^{(a)*} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(a)*}}{\partial t} = 0, \; a = 1, 2 \quad (2.6.56)$$

$$\nabla \times \boldsymbol{B}^{(3)*} = 0$$

For $a = 1, 2$ these are complex conjugate equations corresponding to the original Ampere Maxwell laws and equation for $a = 3$ is the law for the constancy of $\boldsymbol{B}^{(3)}$. It can be seen that in reducing the $O(3)_b$ laws to the $U(1)$ laws plus equations for $\boldsymbol{B}^{(3)}$ we obtain the new energy and angular momentum laws discussed in Section 2.5.

The laws of $O(3)_b$ classical electrodynamics have been given by Barrett [30] in a more condensed notation, and are reproduced as follows:

$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= iq[\boldsymbol{A}, \cdot \boldsymbol{B}] \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= iq[\boldsymbol{A}, \cdot \boldsymbol{E}] \\ \nabla \cdot \boldsymbol{E} &= J_0 - iq[\boldsymbol{A}, \cdot \boldsymbol{E}] \\ \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} + \boldsymbol{J} &= iq[\boldsymbol{A}, \cdot \boldsymbol{B}] \end{aligned} \tag{2.6.57}$$

It is seen that the Barrett Laws and our independently derived Laws are mathematically identical, giving conclusive evidence for the technical correctness of both approaches. This result also demonstrates the fact that recent criticisms [31] aimed exclusively at our equations (2.2.1) to (2.2.2) are wildly erroneous. Significantly, these criticisms are never aimed at the mathematically identical Barrett equations (2.6.57), revealing them to be most unscholarly as well as grossly erroneous [28,29]. The Barrett equations contain the $\boldsymbol{B}^{(3)}$ field by definition and can be developed [30] in a rich variety of variation upon the theme.

2.7 THE LORENTZ FORCE EQUATION IN $O(3)_b$ ELECTRODYNAMICS

The Lorentz force density (force per cubic metre) in $O(3)_b$ electrodynamics is:

$$\mathcal{F}^{\mu} = G^{(a)\mu\nu} J^{(a)}{}_{\nu}, \tag{2.7.58}$$

which is a scalar in the internal $O(3)_b$ symmetry gauge space. Expanding eqn. (2.7.58) we obtain:

$$\mathcal{F}^{\mu} = G^{(1)\mu\nu} J^{(1)}{}_{\nu} + G^{(2)\mu\nu} J^{(2)}{}_{\nu} + G^{(3)\mu\nu} J^{(3)}{}_{\nu}. \tag{2.7.59}$$

The Lorentz force due to $\boldsymbol{B}^{(3)}$ comes from the final term on the right hand side of equation (2.7.59):

$$\mathcal{F} = \boldsymbol{J}^{(3)} \times \boldsymbol{B}^{(3)} = 0. \tag{2.7.60}$$

There is in consequence no Lorentz force due to $\boldsymbol{B}^{(3)}$ as observed empirically [32]. There is however a magnetization (transfer of angular momentum) due to $\boldsymbol{B}^{(3)}$ in the inverse Faraday effect, as observed empirically. This means that the trajectory of an electron in an electromagnetic field is described by the first two terms on the right hand side of eqn. (2.7.59), as in the $U(1)$ Lorentz force equation. There is no $\boldsymbol{E}^{(3)}$ field in $O(3)_b$ electrodynamics so there is obviously no Lorentz force due to it, again as observed empirically[32].

There is however a linear momentum transfer due to $e\boldsymbol{A}^{(3)} = \hbar\boldsymbol{\kappa}$ from the field to an electron in $O(3)_b$ electrodynamics. However, this field is very short ranged and massive, and is present only under very high energies. So the force transfer is due to the transverse components as in $U(1)$ electrodynamics. The Lorentz force density $\mathcal{F}^{\mu}$ is the same in $U(1)$ and $O(3)_b$ electrodynamics, and is defined as [33]:

$$\mathcal{F}^{\mu} = \partial_{\nu} T^{\mu\nu}, \tag{2.7.61}$$

where $T^{\mu\nu}$ is the electromagnetic stress energy momentum tensor. The latter is clearly a scalar in the internal gauge space of $O(3)_b$ electrodynamics because it contains the scalar energy density:

$$U = \frac{1}{\mu_0} \sum_{a=1}^{3} \boldsymbol{B}^{(a)} \cdot \boldsymbol{B}^{(a)}. \tag{2.7.62}$$

The extra energy density due to $\boldsymbol{B}^{(3)}$ is the third term on the right hand side, and so the contribution of $\boldsymbol{B}^{(3)}$ to the stress tensor of the field vanishes:

$$Tr(T^{33}) = \frac{1}{}(\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} - \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)}) = 0 \tag{2.7.63}$$

a result which is consistent with the fact that $\boldsymbol{B}^{(3)}$ makes no contribution to the Lorentz force density. It follows that $\boldsymbol{B}^{(3)}$ makes no contribution to the Poynting vector, which is a part of $T^{\mu\nu}$[32-34]. It contributes however, a constant energy density to the field, as in equation (2.7.62). It is evident that the derivation of the $O(3)_b$ Lorentz force equation is straightforward and in complete agreement with empirical data.

2.8 Continuity Equation and Lorentz Condition in $O(3)_b$ Electrodynamics

The continuity equation of $O(3)_b$ electromagnetism is:

$$\mathcal{D}_\mu J^\mu = 0. \tag{2.8.64}$$

The mathematical form of the continuity equation of $U(1)$ electrodynamics is recovered in the particular solution:

$$\partial_\mu J^\mu = 0 \tag{2.8.65a}$$

$$[A_\mu, J^\mu] = 0. \tag{2.8.65b}$$

Eqn. (2.8.65a) gives the charge and current conservation equations:

$$\partial_\mu J^{(a)\mu} = 0,\ a \in \{1,2,3\}, \tag{2.8.66}$$

and eqn. (2.8.65b) gives the energy conservation law:

$$E = -\int [A_\mu, J^\mu] dV = 0. \tag{2.8.67}$$

Both laws are clearly special cases of the $O(3)_b$ conservation law (2.8.64). If A_μ is assumed to be directly proportional to J_μ[35] equation (2.8.64) implies the Lorentz condition in $O(3)_b$ electrodynamics:

$$\mathcal{D}_\mu A^\mu = 0. \tag{2.8.68}$$

Using the algebra:

$$[A_\mu, A^\mu] = 0, \tag{2.8.69}$$

then this equation implies eqn. (2.8.67) and also:

$$\partial_\mu A^\mu = 0 \tag{2.8.70}$$

from which we recover self-consistently the mathematical form of the $U(1)$ Lorentz condition:

$$\partial_\mu A^{(a)\mu} = 0, a \in \{1,2,3\}, \tag{2.8.71}$$

2.9 Primitive Concepts, Axioms and Constitutive Relations of $O(3)_b$ Electrodynamics

The primitive concepts [36] of $O(3)_b$ electrodynamics include the localization of field energy. As we have just seen, the tensor derivative $\mathcal{F}^\mu$ is the same in $O(3)_b$ and $U(1)$, although the tensor itself is different. The problems encountered with the Poynting vector [36] in $U(1)$ seem to remain in $O(3)_b$. However, in $O(3)_b$, it becomes possible to treat the field momentum classically through an effective $e\boldsymbol{A}^{(3)}$ and to give a classical Compton effect as argued earlier. To accept this representation of the field momentum requires perhaps a slight revision of the primitive concept of charge in electrodynamics. As discussed in ref. [36], charge results from the field in the Maxwell-Poynting interpretation, and charge is an epiphenomenon of the electromagnetic ether. In the microscopic Lorentz-Maxwell theory fields and charges are distinct, the charge on the electron is the source of the field. In $O(3)_b$ electrodynamics there exists the proportionality constant g which leads to the vacuum momentum condition

$$p = eA^{(0)} = \hbar\kappa, \tag{2.9.72}$$

and so the elementary charge, the universal constant e, can exist either in the field or on the electron. This is a primitive (first principles) concept which is consistent with the fact that an electron accelerated to approach c becomes a field [33-37], while e remains the same. The relation (2.9.69) also exists in U(1) electrodynamics as argued and seems to be logically irrefutable.

The fact that $O(3)_b$ electrodynamics is a Yang Mills gauge field theory should not be misinterpreted to mean that the field acts as its own source. The two photons, propagating at c in vacuo, each carry the fluxon $\hbar/e$, and are correlated to each other. The conjugate product $\boldsymbol{A}\times\boldsymbol{A}^*$ is then interpreted as giving rise to a self polarization and magnetization, and to the third Stokes parameter S_3 in the vacuum. Since $p = e\boldsymbol{A}^{(3)}_{eff} = \hbar\kappa$ gives rise to a Compton effect, precisely as observed empirically [39], the Compton effect serves as excellent empirical evidence for $O(3)_b$ electrodynamics and distinguishes it clearly from $U(1)$ electrodynamics. Here $\boldsymbol{A}^{(3)}{}_{eff}$ is an effective potential that defines the $\boldsymbol{B}^{(3)}$ field according to $\boldsymbol{B}^{(3)} = \nabla\times\boldsymbol{B}^{(3)}{}_{eff}$. This potential differs from the more fundamental $\boldsymbol{A}^{(3)}$, which is a massive field that exists only on short lengths and high energy. The exact nature of this potential is discussed in section (2.11). The interpretation of $e\boldsymbol{A}^{(3)}_{eff}$ as a linear momentum may cure some problems with the conventional Poynting vector, which exists unchanged on average in $O(3)_b$ electrodynamics.

The basic axioms of $O(3)_b$ electrodynamics include the use of an $O(3)_b$ symmetry Lagrangian which automatically gives rise to vacuum energy densities due to $\boldsymbol{B}^{(3)}$ As we shall argue, there are profound differences between $O(3)_b$ and $U(1)$ in respect of gauge transformation and the interpretation of the potential as exhibiting products that are classically physical. Therefore there are major advantages of using $O(3)_b$ in unified field theory.

The constitutive relations in $O(3)_b$ electrodynamics [36], arise as in $U(1)$ from the fact that the number of field equations (two) is not enough to determine the four unknowns, $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{H}$ and $\boldsymbol{D}$. So the constitutive relations have to be introduced [36]:

$$\boldsymbol{D}^a = \epsilon(\boldsymbol{E})\boldsymbol{E}^{(a)} \tag{2.9.73}$$

$$\boldsymbol{H}^{(a)} = \frac{1}{\mu(\boldsymbol{B})}\boldsymbol{B}^{(a)}, \tag{2.9.74}$$

for $a \in \{1,2,3\}$. As in $U(1)$ electrodynamics, the scalar permittivity ϵ and scalar permeability μ are complicated tensor functions or functionals [36], and for non-linear media are themselves functions in general of the field and its frequency, as in dielectric spectroscopy [40]. Therefore, $O(3)_b$ electrodynamics gives a description of non-linear optics through the constitutive relations. The symmetry of the gauge theory $O(3)_b$ does not change.

The "ether" relations [36] in $O(3)_b$ electrodynamics are:

$$\boldsymbol{D}^{(a)} = \epsilon_0\boldsymbol{E}^{(a)} \tag{2.9.75}$$

$$\boldsymbol{H}^{(a)} = \frac{1}{\mu_0}\boldsymbol{B}^{(a)}. \tag{2.9.76}$$

As for Maxwell's equations, the $O(3)_b$ field equations are generally covariant, and the invariance of the ether relations restrict the space-time transformations to Lorentz transformations [36]. A Lorentz frame, or covariant ether, is one in which the $O(3)_b$ field equations maintain their form and the speed of light c remains constant. The word "ether" in this context should not be confused with the concept of luminiferous ether [36].

Since $O(3)_b$ electrodynamics reduces to $U(1)$ electrodynamics in particular solution it is not surprising that sometimes the two are closely related. The $O(3)_b$ electrodynamics also reduce straightforwardly to Lehnert's electrodynamics [41] where:

$$\partial_\mu G^{\mu\nu} = J_{vac}{}^{\nu}. \tag{2.9.77}$$

In other words Lehnert assumes a non-vanishing charge-current in the vacuum Coulomb Law, so our equations (2.5.36) and (2.5.37) are modified to:

$$\partial_\mu G^{\mu\nu} = J^{\nu}$$

$$J_1{}^{\nu} = g\epsilon_0[A_\mu, G^{\mu\nu}] \qquad (2.9.78)$$
$$J^{\nu} = J^{\nu}{}_{source} + J_1{}^{\nu}.$$

Eqns. (2.9.74) reduce $O(3)_b$ electrodynamics to Lehnert electrodynamics by generating the vacuum $B^{(3)}$ energy $E = -\int J_1{}^{\nu} A_\nu dV$.

2.10 THE INVERSE FARADAY EFFECT

The inverse Faraday effect [1-8, 26, 42] cannot be described self consistently with $U(1)$ electrodynamics, because the latter denies the existence of the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. A description (Section (1.12)) in terms of the minimal prescription leads back to a non-zero $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, the signature of $O(3)_b$ electrodynamics. The particular, formally Maxwellian, solutions (2.5.46) do not allow for the existence of the longitudinal current $J^{(a)\nu}$, the signature of the inverse Faraday effect, which is empirically observable magnetization in axis (3) by a circularly polarized electromagnetic field. Taking the particular solution leading to the formally Maxwell equations inconsistently removes the inverse Faraday effect, another sign that it cannot be described from first principles in a U(1) gauge theory of electromagnetism. For a consistent first principles description we need the unabridged equation (2.2.2). The latter is expanded in the basis $((1),(2),(3))$ as 1-8:

$$\partial_\mu H^{(a)\mu\nu} = J^{(a)\nu} + ig\epsilon^{abc}[A^{(b)}{}_\mu, H^{(c)\mu\nu}]. \qquad (2.10.79)$$

The constitutive relation:

$$H^{(3)\mu\nu} = \epsilon G^{(3)\mu\nu} \qquad (2.10.80)$$

may be used to demonstrate that:

$$\boldsymbol{H}^{(3)*} = \frac{1}{\mu}\boldsymbol{B}^{(3)*} \qquad (2.10.81)$$

If we define (cf. eqn. (2.4.2.21):

$$\boldsymbol{H}^{(3)} = -i\frac{g}{\mu_0}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \qquad (2.10.82)$$

then we find that

$$g' = \frac{\mu_0}{\mu}g. \qquad (2.10.83)$$

We can write equation (2.10.79a) as:

$$\partial_\mu H^{(1)\mu\nu*} = J^{(1)\nu*} + \Delta J^{(1)\nu*} \qquad (2.10.84)$$

where the transverse current:

$$\Delta J^{(1)\nu*} = ig\epsilon[A^{(2)}{}_\mu, G^{(3)\mu\nu}] \qquad (2.10.85)$$

causes a signal in an induction coil due to the vacuum $\boldsymbol{B}^{(3)}$ field appearing in $G^{(3)\mu\nu}$. This is the inverse Faraday effect as observed empirically [1-8, 26, 42].

The explanation of this effect in $U(1)$ electrodynamics relies [1-8, 26, 42] on the clearly self-inconsistent introduction of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ phenomenologically. This procedure violates the fundamental $U(1)$ gauge condition that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ be everywhere zero under all conditions. As shown in chapter (1), $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in the vacuum is proportional to $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$ and so to the third Stokes parameter. Therefore the $U(1)$ theory cannot describe the third Stokes parameter in a completely consistent manner. In consequence circular polarization defined without reference to $\boldsymbol{A} \times \boldsymbol{A}^*$ and results in an inconsistency that is removed by dropping the term $\boldsymbol{A} \cdot \boldsymbol{A}^*$ term in the Hamiltonian for a particle interacting with the field. In $O(3)_b$ electrodynamics, the inverse Faraday effect and third Stokes parameter are described in a straightforward manner through the vacuum $\boldsymbol{B}^{(3)}$ field, which does not exist in $U(1)$ by definition.

Equation (2.10.79a) can be developed using the result:

$$\Delta J^{(1)\nu*} = \epsilon g^2[A^{(2)}{}_\mu, [A^{(1)\mu}, A^{(2)\nu}]]$$

$$= \epsilon g^2 |A^{(2)}|^2 A^{(1)\nu}. \tag{2.10.86}$$

The coupling constant may be written as

$$g = \frac{\kappa}{A^{(0)}} = \frac{\omega}{cA^{(0)}}, \tag{2.10.87}$$

and may be used to demonstrate that

$$\delta J^{(2)\nu} = -\frac{\epsilon}{c}\omega^2 A^{(2)\nu}. \tag{2.10.88}$$

Now write the four-current for one electron as:

$$\delta J^{(2)\nu} = \frac{e}{mcV} p^{(2)\nu}, \tag{2.10.89}$$

where m is the electronic mass, V the sample volume and $p^{(2)\nu}$. The electronic energy-momentum vector, defined through the $O(3)_b$ minimal prescription:

$$p^{(2)\nu} = eA^{(2)\nu}. \tag{2.10.90}$$

From eqns. (2.10.88) to (2.10.90):

$$\epsilon = -\frac{e^2}{m\omega^2 V}. \tag{2.10.91}$$

By defining the one electron susceptibility as[1-8]:

$$\chi = -\frac{e^2 c^2}{m\omega^2} \tag{2.10.92}$$

then the permattivity is found to be:

$$\epsilon = \frac{\chi}{c^2 V} \tag{2.10.93}$$

This result is self consistent because the inverse Faraday effect can be described through the same susceptibility χ by using the relativistic Hamilton Jacobi equation [1-8] for one electron in the classical electromagnetic field. The current $\Delta J^{(2)\nu}$ is due to the field induced transverse electronic linear momentum as described in section (1.12).

The field equations of $O(3)_b$ electrodynamics give a self consistent and complete description of this process without phenomenology. In $U(1)$ electrodynamics, the inverse Faraday effect can be described only after the phenomenological [1-8, 26] introduction of $\boldsymbol{E} \times \boldsymbol{E}^*$.

Consider next the development of equation (2.10.79c). From equation (2.10.80):

$$\partial_\mu H^{(3)\mu\nu *} = 0, \tag{2.10.94}$$

and so

$$J^{(3)\nu} = -ig[A^{(1)}{}_\mu, H^{(2)\mu\nu}]. \tag{2.10.95}$$

(Equation (2.10.94) follows from the theoretical and empirical finding that $\partial \boldsymbol{B}(3)/\partial t = \nabla \times \boldsymbol{B}^{(3)} = 0$.) In equation (2.10.95) $J^{(3)\nu}$ is an induced current (not a source current), and as we shall see, it is induced in the inverse Faraday effect in a self consistent manner.

Now consider the constitutive relation:

$$H^{(2)\mu\nu} = \epsilon G^{(2)\mu\nu}, \tag{2.10.96}$$

and the definition:

$$\frac{1}{c} G^{(2)\mu\nu} = \partial^\nu A^{(2)\mu} - \partial^\mu A^{(2)\nu} - ig[A^{(3)\mu}, A^{(1)\nu}] \tag{2.10.97}$$

with

$$[A^{(1)}{}_\mu, [A^{(2)\mu}, A^{(1)\nu}]] = 0. \tag{2.10.98}$$

Set $\nu = 3$ in equation (2.11.95), to obtain:

$$J^{(3)3} = 2ig\epsilon \boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{2.10.99}$$

which is a polar current induced by the non-linear cross product $\boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)}$. The $\boldsymbol{B}^{(2)}$ field is

$$\boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)} \tag{2.10.100}$$

the polar current is equal to the axial current of the orbital inverse Faraday effect (cf. Section (1.12)):

$$J^{(3)3} = ig\epsilon\kappa A^{(1)} \times A^{(2)} = \frac{e^2}{m\omega V}. \qquad (2.10.101)$$

Therefore $J^{(3)*}$ is the axial magnetization current due to $B^{(3)*}$ for one electron. This axial current is inversely proportional to the frequency. The fact that the axial and polar currents are equal means that the electron transcribes a chiral (helical) trajectory, involving simultaneous translation and rotation in general[43].

There is no longitudinal source current in equation (2.10.79) for $a = 3$ because the source current of circularly polarized radiation is necessarily transverse, the charge in the source goes around a circle about the (3) axis. The source does not move forward along the (3) axis and there is therefore no polar source current in the (3) axis, the longitudinal axis. As the angular velocity of the charge approaches zero the source stops radiating, and we obtain $O(3)_b$ electrostatics, in which:

$$\boldsymbol{E} = E_x \boldsymbol{i} = \boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} \qquad (2.10.102)$$

so

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = \boldsymbol{E} \times \boldsymbol{E}^* = 0. \qquad (2.10.103)$$

There is no radiated $\boldsymbol{B}^{(3)}$ field in $O(3)_b$ electrostatics because there is no radiation associated with the third component of the gauge algebra. For this reason $O(3)_b$ electrodynamics has really a broken $O(3)$ symmetry. The issue of broken symmetry is discussed in section 8.5. The lack of radiation in the 3-sector of the gauge theory means that the Coulomb Law of $O(3)_b$ electrostatics is:

$$\nabla \cdot \boldsymbol{D} = 0 \qquad (2.10.104)$$

where $\boldsymbol{D}$ is real valued and is defined in the x axis, perpendicular to the $z = e^{(3)}$ axis.

Note carefully from equation (2.11.102) that the gauge symmetry of the Coulomb Law, eqn. (2.11.104), is still $O(3)_b$. It shows that there is a longitudinally, x directed, static electric displacement between one static charge and another.

Similarly, the Ampere Law of $O(3)_b$ magnetostatics is:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}, \qquad (2.10.105)$$

where $\boldsymbol{H}$ is a static magnetic field strength perpendicular to the current density $\boldsymbol{J}$. So $\boldsymbol{H}$ is directed in z if $\boldsymbol{J}$ is directed in x tangentially. Clearly, $\boldsymbol{H}$ is not radiated and should not be confused with radiated $\boldsymbol{B}$ of $O(3)_b$ electrodynamics. The static $\boldsymbol{H}$ is the curl of a vector potential, while $\boldsymbol{B}^{(3)}$ is always defined through $-ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$.

The Gauss and Faraday Laws of $O(3)_b$ magnetostatics and electrostatics are also $O(3)_b$ gauge field equations respectively:

$$\nabla \cdot \boldsymbol{B} = 0 \qquad (2.10.106)$$

and

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad (2.10.107)$$

where

$$\boldsymbol{D} = \epsilon \boldsymbol{D}; \ \boldsymbol{B} = \mu \boldsymbol{H}. \qquad (2.10.108)$$

Note that in eqns. (2.10.104) to (2.10.108) Maxwell's displacement current is absent and there can be no explanation of the inverse Faraday effect, an effect of a radiated field, from a source on an electron in a receiver. These equations are the empirically observable laws of electrostatics and magnetostatics, and are $O(3)_b$ gauge field equations. We can develop equation (2.10.79), with $a = 2$, to show

that it leads to Ohm's Law, which becomes derivable from the first principles of $O(3)_b$ gauge field theory.

From a consideration of:

$$[A^{(3)}{}_{\mu eff},\ [A^{(2)\mu},\ A^{(3)\nu}]] \ = \ 0, \qquad (2.10.109)$$

the induced current from eqn. (2.10.79) for $a \ = \ 2$ is:

$$\Delta J^{(2)\nu *} \ = \ ig\epsilon[A^{(3)}{}_{\mu eff},\ G^{(1)\mu\nu}]$$

$$= \ ig\epsilon[A^{(3)}{}_{\mu eff},\ (\partial^\nu A^{(1)\mu} \ - \ \partial^\mu A^{(1)\nu})] \qquad (2.10.110)$$

where $\nu \ = \ 1$ *or* 2, and where we sum over repeated μ. For $\nu \ = \ 1$, this procedure results in the following relation between scalar components:

$$\Delta J^{(1)1} \ = \ -i\kappa\epsilon E^{(2)1}, \qquad (2.10.111)$$

and since $E^{(2)1}$ is complex, $\Delta J^{(1)1}$ has a real and physical component. This Ohm's Law, i.e.:

$$\boldsymbol{J} \ = \ \sigma \boldsymbol{E}, \qquad (2.10.112)$$

where the conductivity for one electron is:

$$\sigma \ = \ \frac{e^2}{mc\omega V}. \qquad (2.10.113)$$

This is a conductivity resulting from the interaction of radiation with one electron, which is driven in the transverse direction by the transverse field component $\boldsymbol{E}^{(2)1}$. In this situation the conductivity is related to the susceptibility and permittivity:

$$\sigma \ = \ \frac{\omega}{c}\epsilon \ = \ \frac{\chi}{c^3 V}. \qquad (2.20.114)$$

In the static limit $\omega \ \rightarrow \ \infty$, for finite ϵ the static permittivity tends to zero. In the static limit the wavelength becomes infinite, so V should be thought of as becoming infinite. More generally,

dielectric spectroscopy [40] shows that the permittivity is a complex function, related to the power absorption coefficient of spectroscopy, and that the real static permittivity is finite for zero frequency. The imaginary part of the permittivity (dielectric loss) is zero at zero frequency and shows a dispersion, for example the Debye relaxation process[40]. The Drude model of conductivity in metals [14] leads to a result identical with equation (2.11.113) (Jackson's [14] equation (7.89)), except that ω is replaced by the complex valued function $g \ - \ i\omega$, where g is a damping factor. In a tenuous plasma, or in the ionosphere, equation (2.11.113) is identical, within a factor i, with Jackson's [14] equation (7.90). Therefore to have predicted these well known empirical effects from first gauge field principles is something of a triumph for $O(3)_b$ electrodynamics. In $U(1)$ electrodynamics they have to be modeled.

It follows also that Ohm's Law and the inverse Faraday effect for one electron are closely related, and this can be illustrated by considering $\nu \ = \ 2$ in equation (2.11.110) as follows. For $\nu \ = \ 2$:

$$J^{(2)2*} \ = \ -ig\epsilon(A_{0eff}{}^{(3)}E^{(2)2} \ - \ cA_{0eff}{}^{(3)}B^{(2)1}) \qquad (2.10.115)$$

and the first term on the right hand side is ohmic, the second is part of the inverse Faraday effect.

2.11 THE EFFECTIVE $A^{(3)}$ POTENTIAL, PHOTOELECTRIC AND COMPTON EFFECTS AND RADIATION REACTION

The fundamental gauge field theory of $O(3)_b$ electrodynamics leads to the longitudinally directed vector potential $\boldsymbol{A}^{(3)}$, which appears in the vacuum field tensor definition in equation (2.4.2.15). However this potential also does not exist in $O(3)_b$ electrodynamics, due to the vanishing of $\boldsymbol{E}^{(3)} \ = \ \partial \boldsymbol{A}^{(3)}/\partial t$. However, one may define an $\boldsymbol{A}^{(3)}_{eff}$ as an effective potential that defines $\boldsymbol{B}^{(3)} \ = \ \nabla \times \boldsymbol{A}^{(3)}_{eff}$. This effective potential may be defined as:

$$\boldsymbol{A}^{(3)}{}_{eff} \ = \ c(f\boldsymbol{A}^{(1)} \ + \ g\boldsymbol{A}^{(2)}) \qquad (2.11.116)$$

where c is a constant, and $A^{(3)}{}_{eff}$ is a one form, and by default also $A^{(1)}$ and $A^{(2)}$. The application of the differential operator $\mathbf{d}$ to this one form leads to the equation

$$\mathbf{d}A^{(3)}{}_{eff} = c(\mathbf{d}f \wedge A^{(1)} + \mathbf{d}g \wedge A^{(2)})|_{f,g\to 0}$$

$$= (\partial_j f A^{(1)}{}_i + \partial_j g A^{(2)}{}_i)\mathbf{d}x^j \wedge \mathbf{d}x^i. \qquad (2.11.117)$$

Then if f and g are forms of Bogomolny functions that give the gauge potentials $A^{(1)}{}_i = \partial_i g$, $A^{(2)}{}_i = \partial_i f$, we have

$$\mathbf{d}A^{(3)}{}_{eff} = \frac{c}{2}[A^{(1)}{}_i,\ A^{(2)}{}_j]\mathbf{d}x^i \wedge \mathbf{d}x^j. \qquad (2.11.118)$$

Then for $c = 2e/\hbar$ this effective potential defines the $B^{(3)}$ potential. This component $A^{(3)}_{eff}$ does not appear in $U(1)$ electrodynamics but has substantial advantages in particle-field theory as developed in this section.

In $U(1)$ electrodynamics the linear momentum of radiation is the average flux of energy, originally due to Poynting, a well accepted but nevertheless paradoxical concept which sometimes goes astray [38]:

$$\langle \boldsymbol{p} \rangle = \epsilon c \int \boldsymbol{E} \times \boldsymbol{B}^* dV. \qquad (2.11.119)$$

The average momentum as defined above is proportional to beam intensity. The conjugate product $\boldsymbol{E} \times \boldsymbol{B}^*$ removes the frequency and so $\langle p \rangle$ is not linearly proportional to frequency as in the Planck Einstein quantization of radiation [44] into photons with energy and linear and angular momenta. The photoelectric and Compton effects [44] for example lead to the quantum theory, but are paradoxical in classical $U(1)$ electrodynamics. The problem of radiation reaction [14] is paradoxical with the classical setting itself, because it leads to runaway, nonNewtonian, solutions in the Abraham Lorentz equation [14, 36] . It is suggested that $O(3)_b$ electrodynamics removes these paradoxes from field theory.

In $O(3)_b$ electrodynamics there exists a free space (i.e. vacuum) linear momentum:

$$\boldsymbol{p}^{(3)} = e\boldsymbol{A}^{(3)}{}_{eff}, \qquad (2.11.120)$$

which is longitudinally directed as required, i.e. is directed in the propagation axis of the radiation beam. The momentum (2.11.120) is defined through the minimal prescription applied to the beam in the vacuum, and the elementary charge e is a universal, C negative, constant of physics. It is also the charge on the proton. The classical momentum (2.11.120) becomes:

$$\boldsymbol{p}^{(3)} = e\boldsymbol{A}^{(3)}{}_{eff} = \hbar\boldsymbol{\kappa}, \qquad (2.11.121)$$

through Einstein quantization: the photon momentum as proportional to the frequency as observed in the photoelectric and Compton effects. As argued in Section (2.4.1), equation (2.11.121) is also the self consistent result of the $O(3)_b$ symmetry gauge field theory. The result in equation (2.11.121) has a manifestly covariant form:

$$P^{(3)\mu} = eA^{(3)\mu}{}_{eff} = \hbar\kappa^\mu$$

$$A^{(3)\mu}{}_{eff} = \frac{1}{c}(A^0{}_{eff},\ cA^{(3)}{}_{eff}). \qquad (2.11.122)$$

It is well known [14,36] that the energy-momentum of radiation in $U(1)$ electrodynamics is defined through an integral over the tensor $T^{\mu\nu}$ and for this reason is not generally covariant. To make it so requires the use of special hypersurfaces as attempted for example by Fermi and Rohrlich [36]. The root cause of this problem is Poynting's Theorem [14,36]. The $O(3)_b$ energy momentum (2.11.121), in contrast, is generally covariant.

The problem of radiation reaction [14,36] leads in $U(1)$ electrodynamics to the Abraham Lorentz equation in which the radiation reaction force is not Newtonian equation of motion. It has unphysical

runaway solutions [14.36], because the force is proportional to the time derivative of acceleration, a problem addressed by Abraham, Lorentz and Poincare among many others throughout the twentieth century. In $O(3)_b$ electrodynamics the linear momentum $e\boldsymbol{A}^{(3)}$ is classical and constant (conservation of momentum) until the radiation interacts with an electron as in the Compton effect [44]. Upon interaction the frequency of the scattered radiation is found to be different from that of the incoming radiation. In $O(3)_b$ electrodynamics this is explained by the fact that action and reaction are equal and opposite and that force is the rate of change of linear momentum. In this case:

$$F^{(3)\mu} = e\frac{dA^{(3)\mu}{}_{eff}}{dt} = e\frac{dx^\nu}{dt}\partial_\nu A^{(3)\mu}{}_{eff}. \qquad (2.11.123)$$

The derivative can be written according to its symmetric and antsymmetric parts, where the former vanish to obtain that

$$F^{(3)\mu} = \frac{e}{2}V_\nu \epsilon^{\nu\mu\sigma}B^{(3)\sigma}, \qquad (2.11.124)$$

where $B^{(3)\sigma} = \epsilon^{\sigma\nu\mu}(\partial^\nu A^{(3)\mu}_{eff} - \partial^\mu A^{(3)\nu}_{eff})$. This equation is a form of the Lorentz force equation. If this is equated to the frequency through equation (2.11.122) we see that this is equal to $d\kappa/dt$, which means that a change of linear momentum means a change of frequency through Planck Einstein quantization. Further, this equality implies that the total momentum of the electromagnetic wave plus the electron is a constant. There is precise correspondence (the Principle of Correspondence [44]) between linear classical momentum and quantized linear momentum as in equation (2.11.121). Note that both classical and quantized momenta are directed longitudinally as observed in the photoelectric and Compton effects and in the Lebedev effect [1-8]. Note that there is no such correspondence in $U(1)$ electrodynamics because the average momentum $\langle \boldsymbol{p} \rangle$ from Poynting's Theorem can never be proportional to frequency with the classical electrodynamics. The reason for this is that the conjugate product $\boldsymbol{E} \times \boldsymbol{B}$ removes any frequency dependency. For this reason there is no classical explanation [44] of the Compton and photoelectric effects in classical $U(1)$ electrodynamics.

The photoelectric effect is the emission of electrons from metals upon ultraviolet irradiation[44]. Above a threshold frequency the emission is instantaneous and independent of intensity. Below this threshold frequency there is no emission however intense the radiation. This cannot be explained in classical $U(1)$ electrodynamics, because beam energy is proportional to intensity in the Poynting Theorem, and not to frequency as observed. In classical $O(3)_b$ electromagnetism the phenomenon is explained in a straightforward manner by following the information:

$$E = ecA_0{}^{(3)} = \hbar c\kappa \qquad (2.11.125)$$

and in Planck-Einstein quantization the constant of proportionality is $\hbar$, which turns out to be the most fundamental universal constant in physics. The concommitant momentum relation, equation (2.11.121) is shown empirically in the Compton effect as argued already. Equation (2.11.121) means that above a threshold frequency, there is enough energy in the photon to cause electron emission as observed and discussed for example by Atkins[44]. All of the energy and momentum of the photon are transferred to the electron in a collision above a certain threshold frequency because at this point the potential energy responsible for keeping the electron in place is exceeded. Atkins explains the phenomenon simply and decisively in terms of simple Newtonian logic[44]. If we attempt to apply this logic to $\langle \boldsymbol{p} \rangle$ in equation (2.11.119), the Poynting momentum of classical $U(1)$ electrodynamics, there is no threshold frequency possible on the classical level, because $\langle \boldsymbol{p} \rangle$ cannot be proportional to frequency, and we lose correspondence with the quantum theory, in which the momentum of the photon is proportional to the frequency as required empirically in the photoelectric effect. The mean Poynting momentum $\langle \boldsymbol{p} \rangle$ and beam energy are both proportional to beam intensity, but the data in the photoelectric effect show that it is independent of beam intensity. The momentum $\boldsymbol{p}^{(3)} = e\boldsymbol{A}^{(3)}{}_{eff}$ of classical $O(3)_b$ electrodynamics is not proportional to intensity, it is proportional to frequency through the gauge equation (2.4.1.6) which also leads to the B Cyclic Theorem [1-8], the fundamental angular momentum relation of $O(3)_b$ electrodynamics. In the latter, Planck-Einstein

quantization is straightforward.

In the Compton effect in $O(3)_b$ electrodynamics the observable change of wavelength [44] is:

$$\Delta\lambda = 2\frac{e\boldsymbol{A}^{(3)}{}_{eff}}{mc}\lambda_0 sin^2\frac{\theta}{2}, \qquad (2.11.126)$$

where λ_0 is the wavelength of the incident beam, m the electron mass and θ the electron scattering angle. If equation (2.11.121) is applied to this result we recover the quantum description of the Compton effect.

The concept of $\boldsymbol{A}^{(3)}{}_{eff}$ can also be used to suggest a way out of the Dirac paradox [45], in which Dirac maintains that as long as we are dealing with transverse waves we cannot bring in the Coulomb interaction between particles. Dirac suggested the use of longitudinal waves, but in $O(3)_b$ electrodynamics there is a force given by equation (2.11.123) whenever the beam interacts with an electron. This gives a longitudinal force with a change of wavelength as just described in the Compton effect. This is not a Coulomb force however since $\boldsymbol{E}^{(3)}$ is zero in vacuo (there is no electric equivalent of the inverse Faraday effect).

Similarly $\boldsymbol{A}^{(3)}{}_{eff}$ can be used to suggest a way out of the de Broglie paradox [46], which points out that momentum and energy transform differently under Lorentz transformation from frequency, despite Planck-Einstein quantization. This paradox led de Broglie to postulate the existence of empty waves, which have never been found in nature [46]. It is therefore suggested that the Lorentz frequency transform always be applied to $eA_0{}^{(3)} = \hbar\omega/c$ because this momentum is proportional to frequency empirically. For momentum of

a particle traveling at the speed of light, the momentum pertains either to a massless particle, classically indeterminant, or infinite for a massive particle, unless it is always interpreted as being a constant $\hbar$ multiplied by ω/c. This obviously exists empirically at the speed of light. The energy must evidently be interpreted in the same way, i.e. as a constant multiplied by frequency. The Lorentz transform applied to frequency produces the aberration of light as usual in special relativity[14]. In this interpretation there is no de Broglie paradox and no need to postulate the existence of empty waves[46].

2.12 SUMMARY

We have argued that classical $O(3)_b$ electrodynamics produces explanations from first gauge field principles and without paradox of the following: 1) third Stokes parameter in the vacuum, through the modulus of $\boldsymbol{B}^{(3)}$; 2) the inverse Faraday effect, through $\boldsymbol{B}^{(3)}$; 3) Ohm's Law; 4) photoelectric effect, through $\boldsymbol{A}^{(3)}{}_{eff}$; 5) Compton effect, through $\boldsymbol{A}^{(3)}{}_{eff}$; 6) radiation reaction problem, through $d\boldsymbol{A}^{(3)}{}_{eff}/dt$; 7) that the energy momentum four vector $e\boldsymbol{A}^{(3)}{}_{eff}$ of radiation is generally covariant; 8) the Dirac paradox; 9) the de Broglie paradox; 10) the vacuum $\boldsymbol{B}^{(3)}$ field; 11) the vacuum $\boldsymbol{A}^{(3)}{}_{eff}$ potential. .

In the $U(1)$ version of classical electrodynamics we have the following difficulties: 1) the third Stokes parameter contradicts the gauge field principle employed $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = 0$ and is phenomenological and pre Maxwellian (1852); 2) the inverse Faraday effect is phenomenological and contradicts the gauge field principle $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = 0$ in $U(1)$; 3) Ohm's Law is phenomenological; 4) there is no classical photoelectric effect; 5) there is no classical Compton effect; 6) radiation reaction is non-Newtonian and unphysical; 7) the energy momentum vector of radiation is not covariant in general; 8) the Dirac paradox remains; 9) the de Broglie paradox remains; 10) there is no fundamental spin angular momentum, proportional to the $\boldsymbol{B}^{(3)}$ field; 11) there is no

fundamental linear momentum, $e\boldsymbol{A}^{(3)}{}_{eff}$.

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CHAPTER 3
ORIGIN OF ELECTRODYNAMICS IN THE GENERAL THEORY OF GAUGE FIELDS

3.1 CLOSED LOOP IN MINKOWSKI SPACETIME

Electrodynamics can be derived from gauge theory by using a closed loop in Minkowski spacetime 1: a round trip with covariant derivatives. Such a procedure is valid for any internal gauge group symmetry and electrodynamics can be derived in consequence for any internal gauge group. The basics of this process are simple: a general field, such as a quantum wave function, ψ is acted upon by an operator which transports the vector around a closed loop using the theory of infinitesimal generators. The result of the trip around the closed loop is expressed as:

$$\psi' = e^{ix}e^{iy}e^{-ix'}d^{-iy'}\psi, \tag{3.1.1}$$

where ψ' differs from ψ if x, x', y and y' are path variables that noncommuting transport gauge connections as they occur at different regions of spacetime.

The four exponentials in this expression are operators, which can be expanded in a Taylor series. To second order the series takes the form:

$$e^{ix}e^{iy}e^{-ix'}e^{-iy'} = 1 + i(x+y) - i(x'+y') + O((x,y)^2) \tag{3.1.2}$$

If we let $x' = x + \delta x$ and $y' = y + \delta y$ where $\delta x = \delta x^\mu \mathcal{D}_\mu$ and $\delta y = \delta y^\mu \mathcal{D}_\mu$ then the operator on the wave function can be written as

$$e^{ix}e^{iy}e^{-ix'}e^{-iy'} = 1 + i\delta(x^\mu + y^\mu)D_\mu, \tag{3.1.3}$$

where $\mathcal{D}_\mu$ are covariant derivatives and x^μ and y^μ are distinct four vectors in Minkowski (flat) spacetime 1. The covariant derivatives can be defined in any gauge group symmetry and can be expressed in the short-hand notation (cf. Chapters (1) and (2)):

$$\mathcal{D}_\mu = \partial_\mu - igA_\mu, \tag{3.1.4}$$

a notation which is illustrated later in this chapter. Now if we let the variables x^μ and y^μ are variables connected by the parameters τ and λ as $y^{\mu\prime} = y^\mu(1 + \delta\lambda D/d\lambda)$ and with $D/d\lambda = (dx^\nu/d\lambda)D_\nu$, and similarly $x^{\mu\prime} = x^\mu(1 + \delta\tau D/d\tau)$, the phase shift in the wave functions may be written as

$$\psi = \left(1 - (\delta x^\mu \delta y^\nu - \delta x^\nu \delta x^\mu)[D_\mu, D_\nu] + O(dx^4)\right)\psi \tag{3.1.5}$$

The effect of the journey around a small closed spacetime loop that encloses the area element $\delta x^\mu \delta y^\nu - \delta x^\nu \delta x^\mu$ represented by equation (3.1.5) is defined by the way in which the covariant derivatives enter into the expression [1]. If A_μ is the electromagnetic potential, then the electromagnetic field is the commutator:

$$G^{(a)}{}_{\mu\nu} = \partial_\nu A^{(a)}{}_\mu - \partial_\mu A^{(a)}{}_\nu - ig\epsilon^{abc}[A^{(b)}{}_\mu, A^{c)}{}_\nu] \tag{3.1.6}$$

where g is a universal constant, the ratio of e to $\hbar$. The field (3.1.6) is part of the commutator of covariant derivatives $[\mathcal{D}_\mu \mathcal{D}_\nu]$. The Jacobi identity:

$$\sum_{\mathcal{P}(\rho\mu\nu)} [\mathcal{D}_\rho, [\mathcal{D}_\mu, \mathcal{D}_\nu]] = 0 \tag{3.1.7}$$

follows, and is obeyed identically by the commutator $[\mathcal{D}_\mu, \mathcal{D}_\nu]$ for any gauge group. Here $\mathcal{P}$ stands for permutation of the indices. This Jacobi identity is the homogeneous field equation for any gauge group. In general relativity it is the Bianchi identity [1]. These are well known results of modern gauge field theory. Our purpose in this opening section is to show that the gauge group of electromagnetism has to be a non-Abelian group such as $SO(3)$, or here a broken $O(3)$, in order to obtain self-consistently the homogeneous field equation and the first and third Stokes parameters from this general gauge field theory, which we have truncated at second order in the Taylor

series (3.1.2). A theory to higher order in the Taylor series gives non-linear field tensors and equations. These are largely unexplored, but in a simply connected region they correspond to "boundaries of boundaries" that vanish. Most generally, these equations must be solved numerically and this process can be carried out in any gauge group. These remarks show how little we know about electrodynamics, a subject which is currently asserted to be derivable from a U(1) gauge group only, and whose field equations are asserted arbitrarily to be linear in vacuo. In general, they are non-linear to all orders in all internal gauge group symmetries and the process is always Lorentz and gauge covariant. The theory clearly conserves the symmetry CPT [13] on quantization, together with C, P, T, CP, PT and CT for all field types and gauge groups.

Taking the $O(3)_b$, rotation group, symmetry for the internal gauge symmetry of the general theory applied to an electrodynamic vector potential the field tensor becomes equations (2.4.2.15) of chapter 2 [1-12]. One of the components of the tensor is the magnetic field:

$$\boldsymbol{B}^{(3)*} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{3.1.8}$$

which in this gauge symmetry exists in vacuo, where the third Stokes parameter becomes proportional to its modulus, equation (2.4.2.25). Therefore the third Stokes parameter is contained within the definition of the field tensor, and is therefore contained within the commutator of covariant derivatives. The quadratic product of covariant derivatives contains the zero order Stokes parameter:

$$S_0 = \boldsymbol{E}^{(1)} \cdot \boldsymbol{E}^{(2)} = \omega^2 \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)}, \tag{3.1.9}$$

and so an $O(3)_b$ internal gauge group symmetry produces the result:

$$S_0 = \pm S_3 \tag{3.1.10}$$

as required in circular polarization, and as given in equation (1.1.13). The Jacobi identity (3.1.7) becomes the homogeneous field equation

(2.2.1) of $O(3)_b$ electrodynamics. The field tensor is generally covariant [1] as is the theory as a whole. Note that some major features of $O(3)_b$ electrodynamics have been obtained simply and self-consistently by considering a closed loop in Minkowski spacetime, equation (3.1.1).

If we try to apply this method to $U(1)$ electrodynamics, the $\boldsymbol{B}^{(3)}$ field vanishes along with the third Stokes parameter, S_3. The field tensor becomes a scalar in the internal, Abelian, gauge space:

$$G^{U(1)}{}_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu, \tag{3.1.11}$$

and the Jacobi identity (3.1.7) reduces to the homogeneous field equation:

$$\partial_\mu \tilde{G}^{U(1)\mu\nu} = 0, \tag{3.1.12}$$

in which there are components $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ but no $\boldsymbol{B}^{(3)}$. The B Cyclic Theorem is undefined. The quadratic product $\mathcal{D}_\mu \mathcal{D}^\mu$ does not produce the zero order Stokes parameter S_0, unless it is implicitly assumed that the vector potential is a complex quantity. This implicit assumption, however, leads us back to an $O(3)_b$ internal gauge space, because $\boldsymbol{A} := \boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(*)} := \boldsymbol{A}^{(2)}$. The zero and third order Stokes parameters are undefined in $U(1)$. These are attributed to Stokes (1852), and are purely phenomenological parameters. At that point in time the Maxwell equations were unknown, and vector algebra undeveloped. The use of a $U(1)$ gauge group symmetry for electrodynamics leads to an internal inconsistency, because of the occurrence of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$.

3.2 GAUGE TRANSFORMATIONS

The rotation of the general field ψ takes place as follows:

$$\psi' = S\psi, \tag{3.2.13}$$

and is a special case of the closed loop or round trip represented in equation (3.1.1). In special relativity both S and ψ are functions of

x^μ [1], and the derivative:

$$\partial_\mu \psi' = \partial_\mu (S\psi) = S\partial_\mu \psi + \psi \partial_\mu S \tag{3.2.14}$$

in consequence is not covariant because it does not transform under S in the same way as the field itself. There is an extra term $\psi\partial_\mu S$ on the right hand side of equation (3.2.14). The concept of gauge transformation enters into field theory through the use of the covariant derivative as follows:

$$\mathcal{D}'_\mu \psi' = S\mathcal{D}_\mu \psi, \tag{3.2.15}$$

where

$$\mathcal{D}'_\mu = \partial_\mu - igA'_\mu, \tag{3.2.16}$$

with $\partial_\mu S = ig(A'_\mu - A_\mu)$. Equation (3.2.15) is covariant and has the same algebraic form as the original equation (3.2.13). In consequence of the introduction of A'_μ as defined in equation (3.2.16), gauge transformation is a frame transformation in which A_μ changes its value to A'_μ. It can be shown that:

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1}. \tag{3.2.17}$$

If S is constant then A_μ transforms into A'_μ according to:

$$A'_\mu = A_\mu = SA_\mu S^{-1}$$

$$(\partial_\mu S)S^{-1} = 0. \tag{3.2.18}$$

and so the second term on the right hand side of equation (3.2.17), known as the inhomogeneous term, is a direct consequence of our use of a covariant derivative in field theory. The only occasion that equation (3.1.18) will obtain is when the connection is completely "flat" or constant and S corresponds to a global gauge transformation that occurs everywhere.

This procedure is valid for any gauge group in Minkowski spacetime. We illustrate the general equations in this section with reference to

the $U(1)$, $O(3)_b$ and $SU(2)$ gauge groups. The inhomogeneous term, also called the pure gauge term, is responsible for the Aharonov-Bohm effect [1,14] in regions where the original A_μ is regarded as of no physical significance by an experimentalist. This suggests that $(ig)^{-1}(\partial_\mu S)S$ has the physical effect as a phase shift. In the $U(1)$ gauge group the rotation generator S is defined by:

$$S_{U(1)} = e^{i\phi}, \tag{3.2.19}$$

where ϕ is a parameter representing the angle through which the field is rotated by S within the internal space of the complex valued circle. This means that in the U(1) group:

$$A'_\mu = A_\mu + \frac{\hbar}{e}\partial_\mu\phi. \tag{3.2.20}$$

The spacial part of the connection is seen to transform according to:

$$\boldsymbol{A} \rightarrow \boldsymbol{A} - \frac{\hbar}{e}\nabla\phi \tag{3.2.21a}$$

and the temporal component transforms as

$$A^{(0)} \rightarrow A^{(0)} + \frac{\hbar}{e}\frac{\partial\phi}{\partial t} \tag{3.2.21b}$$

The magnetic field in $U(1)$ electrodynamics is defined by:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{3.2.22}$$

and so the function ϕ in the gauge transformation (3.2.20) can be chosen arbitrarily without affecting the value of $\boldsymbol{B}$ by the identity $\nabla\times\nabla\phi = 0$. In classical $U(1)$ electrodynamics the potential is therefore reduced to an arbitrary mathematical quantity, where products of the potentials are not admitted in Lagrangians and Hamiltonians, and for this reason it is difficult to assign to it a physical meaning.

This was Heaviside's original intention [15] in reducing the complicated and numerous quaternionic field equations of Maxwell to the relatively simple vectorial form given in equations (1.4.29) to (1.4.32) of Chapter 1. These became known as "Maxwell's equations". Therefore the potential function, in classical and quantum $U(1)$ electrodynamics, has slightly different interpretations. Usually, it is stated that the potential is physical as a phase shift, due to the parallel transport of a vector potential around a loop, in the quantized theory. The analog of this in the classical theory is absent. This interpretation may be unsatisfactory as argued recently by Barrett [15]: for example he describes the use of a physical potential with the classical setting for the topological phase effect and Aharonov Bohm effect among several others. We encounter a conflict when we use the "traditional" $U(1)$ gauge group, in that the classical gauge potential is unphysical, while within quantum theory the gauge potential is involved with phase shifts of quantum wave functions.

In the $O(3)_b$ gauge group (and other non-Abelian gauge groups such as $SU(3)$) this classical-quantum distinction can be made to disappear[16]. If the internal space of gauge transformations is a physical rotation, where the fields produce products of the gauge potential, then a classical analog exists. However, for the weak and strong forces the short range of the interaction prevents a classical analog from being physically real. These products enter into, ideally within classical field theory, in the field equations. The $O(3)_b$ group is the rotation group in three dimensions and A_μ has physical consequences on both classical and quantum levels. This appears to be more commensurate with the Correspondence Principle between quantum physics and classical physics [17] and as argued also by Barrett [15]. There is an Aharonov-Bohm effect due to the inhomogeneous term, which also leads within nonabelian theories into highly developed areas of field theories, not available when the gauge group is U(1), involving instantons, solitons and vortex theories [1].

3.2.1 GAUGE TRANSFORMATION IN THE $O(3)_b$ GROUP

In this section equation (3.2.17) is worked out in the $O(3)_b$ gauge group, reduced to a rotation about the Z axis. In this case:

$$S = e^{iJ_z\alpha} \tag{3.2.1.23}$$

where J_z is the $O(3)_b$ rotation generator about the z axis by an angle of rotation. In other words a gauge transformation resulting from a rotation of the field ψ is a physical rotation through an angle α, which is not the same described in the $U(1)$ gauge group. That there are similar $U(1)$ transformations of the $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ field indicates that this gauge theory is a broken $O(3)_b$ gauge theory. The $O(3)_b$ algebra is broken due to the vanishment of the $\boldsymbol{A}^{(3)}$ field, but this theory is still generically called the $O(3)_b$ theory of electrodynamics. As a result, $O(3)_b$ electrodynamics involves the addition to the $U(1)$ symmetry the rotation around the z axis defined as:

$$S^{-1} = e^{-iJ_z\alpha} \tag{3.2.1.24}$$

and the Taylor series:

$$S = 1 + iJ_z\alpha - \frac{1}{2}{J_z}^2\alpha^2 + O(\alpha^3)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + i\alpha \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\alpha^2}{2!} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} cos\alpha & \sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2.1.25}$$

These are the 3×3 matrix representation of the $O(3)_b$ group operator S. Its inverse is found through:

$$\alpha \rightarrow -\alpha \tag{3.2.1.26}$$

which results in

$$S^{-1} = \begin{pmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2.1.27}$$

It is then apparent that $SS^{-1} = \mathbf{1}$ and defines a diagonal matrix as required.

The calculation of the inhomogeneous term in the $O(3)_b$ gauge group proceeds simply after noting that $\alpha = \alpha(x^\mu)$. This gives:

$$\partial_\mu cos\alpha = -sin\alpha\partial_\mu\alpha,\ \partial_\mu sin\alpha = cos\alpha\partial_\mu\alpha$$

$$\partial_\mu S = \begin{pmatrix} -sin\alpha & cos\alpha & 0 \\ -cos\alpha & -sin\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \partial_\mu\alpha. \tag{3.2.1.28}$$

It follows that $\partial_\mu S$ vanishes if $\partial_\mu\alpha$ vanishes, which illustrates how a constant phase change that occurs globally has no influence over physics. The potential A_μ is defined through the $O(3)_b$ group generators are [1]:

$$A_\mu = J^a A^a{}_\mu, \tag{3.2.1.29}$$

where repeated indices a are summed over as usual. For a rotation about the z axis:

$$A_\mu = J^z A^z{}_\mu, \tag{3.2.1.30}$$

where the placing of z as an upper or lower index is not significant. Thus, for a z axis rotation:

$$A_\mu = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A^z{}_\mu, \tag{3.2.1.31}$$

where the components of $A^z{}_\mu$ are interpreted as scalars in the $O(3)_b$ internal space of the gauge theory. The inhomogeneous term is therefore:

$$-\frac{i}{g}(\partial_\mu S)S^{-1} = \frac{J_z}{g}\partial_\mu\alpha, \tag{3.2.1.32}$$

that vanishes if $\alpha = const$ for a global gauge transformation. Since the Euler angle α always depends on x and y in general, and since by definition:

$$x^\mu = (ct,\ x,\ y,\ z),\ \partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t},\ -\frac{\partial}{\partial x},\ -\frac{\partial}{\partial y},\ -\frac{\partial}{\partial z}) \tag{3.2.1.33}$$

the object $\partial_\mu \alpha$ is in general non-zero in flat Minkowski spacetime. To illustrate this we write:

$$\partial_\mu S S^{-1} = i J_z \partial_\mu \alpha. \qquad (3.2.1.34)$$

The Aharonov-Bohm effect in $O(3)_b$ electrodynamics can be interpreted as a physical rotation, treated classically, corresponding to the usual quantum interpretation [1]. The scalar g is the same in both $U(1)$ and $O(3)_b$ gauge theories and was discussed in Section (2.4.1).

Using the matrix method it follows from elementary methods that:

$$S A_\mu S^{-1} = A_\mu \qquad (3.2.1.35)$$

so the overall result of gauge transformation in $O(3)_b$ electrodynamics is:

$$A^z{}_\mu J_z \rightarrow \left(A^z{}_\mu + \frac{1}{g}\partial_\mu \alpha\right) J_z, \qquad (3.2.1.36)$$

which corresponds to the transformation of the gauge connection by,

$$A^z{}_\mu \rightarrow \left(A^z{}_\mu + \frac{1}{g}\partial_\mu \alpha\right). \qquad (3.2.1.37)$$

3.3 THE SAGNAC EFFECT

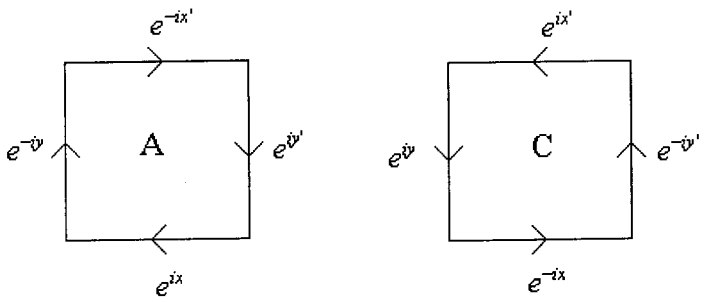
The Sagnac effect is an interferogram that is created by an apparatus that recombines two beams sent around a closed physical loop (optical fiber) in opposite directions, anticlockwise (A) and clockwise(C). An interferogram is an intensity pattern [18-20], where the intensity at each point is proportional to the phase difference between two interfering (i.e. combining) beams of light. If there is no electromagnetic phase difference in the two beams there is no interferogram. In $U(1)$ electrodynamics the electromagnetic phase is arbitrary [21],

there is gauge invariance, the field is described completely by the field tensor. In the Sagnac interferometer, there is no phase difference between beams A and C, see figure 3.1, no intensity or polarization difference if there is no net gauge transformation due to parallel transport of the field vectors around the loop. Therefore, $U(1)$ electrodynamics cannot describe the Sagnac interferogram without local gauge transformations. When the platform on which the interferometer is built is under a rotation, the interferogram changes. In this section it is shown that $O(3)_b$ electrodynamics explain the Sagnac effect when the platform is at rest, and explain the Sagnac effect (motion of the platform) through equation (3.2.1.37) of $O(3)_b$ gauge transformation, a physical rotation. The Sagnac effect with platform in motion is a Doppler effect given in $O(3)_b$ electrodynamics by a change of frequency of the electromagnetic radiation:

$$\omega \rightarrow \omega \pm \Omega, \qquad (3.3.38)$$

where Ω is the angular frequency $\partial\alpha/\partial t$, or equivalently by the change of the Euler angle with time.

The explanation of the Sagnac effect in general is given by a comparison of two closed loops, in opposite directions in Minkowski spacetime.



These loops are illustrations of equation (3.1.1). The two loops are described as follows:

$$\psi_A = e^{ix}e^{-iy}e^{-ix}e^{iy}\psi \tag{3.3.39}$$

$$\psi_C = e^{-iy}e^{ix}e^{iy}e^{-ix}\psi \tag{3.3.40}$$

where ψ is the quantum wave function that describes the phase $e^{-(\boldsymbol{k}\cdot\boldsymbol{r}\,-\,\omega t)}$ as the field is transported around the loop in the two directions within the Sagnac interferometer. The displacement of the field along the x and y directions is given by:

$$\delta x = \delta x^{\mu}\mathcal{D}_{\mu},\ \delta y = \delta y^{\mu}\mathcal{D}_{\mu}. \tag{3.3.41}$$

For simplicity let us define $\delta\mathcal{A}^{\mu\nu}{}_A = \delta x^{\mu}\delta y^{\nu} - \delta x^{\nu}\delta x^{\mu}$ as a small area, with the orientation in of the A loop, within the Sagnac interferometer. A summation over all these areas produces the total area $\mathcal{A}^{\mu\nu}{}_A$. The opposite orientations of the A and C loops results in $\mathcal{A}^{\mu\nu}{}_A = -\mathcal{A}^{\mu\nu}{}_C = \mathcal{A}^{\nu\mu}{}_C$. Using equation (3.1.2) it follows that the phase shifts between the two loops are equal in magnitude but opposite direction:

$$\mathcal{A}^{\mu\nu}{}_A[\mathcal{D}_{\mu},\ \mathcal{D}_{\nu}] = -\mathcal{A}^{\mu\nu}{}_C[\mathcal{D}_{\mu},\ \mathcal{D}_{\nu}]. \tag{3.3.42}$$

The covariant derivatives for the A and C loops may be defined in either $U(1)$ or $O(3)_b$ electrodynamics. In $U(1)$ electrodynamics, the potential A_{μ} is a determined by a scalar in the internal gauge space, and in consequence the covariant derivatives result in potentials that commute. The commutator of potentials vanishes in $U(1)$ and from equation (3.3.42) we have that the two commutators are restricted to the case that $[A_{\mu},\ A_{\nu}] = 0$:

$$G_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{3.3.43}$$

At this point we need to compute the interference from $U(1)$ electrodynamics. This is given by the modulus square of the total wave function

$$\Psi = (1/2)(\psi_A + \psi_C). \tag{3.3.44}$$

Further in general we exponentiate the phase shift according to

$$\psi_A \rightarrow e^{i\mathcal{A}^{\mu\nu}G_{\mu\nu}}\psi_A, \psi_C \rightarrow e^{-i\mathcal{A}^{\mu\nu}G_{\mu\nu}}\psi_C, \tag{3.3.45}$$

where the difference in the sign is due to the different area orientations. We write $\mathcal{A}^{\mu\nu}G_{\mu\nu} = AG$ and we have a more compact notation. From here the modulus square of the wave function is then

$$\begin{aligned}|\Psi|^2 &= (1/4)(|\psi_A|^2 + |\psi_C|^2 + \psi_A{}^*\psi_C + \psi_C{}^*\psi_A)\\ &= \frac{1}{4}(2 + e^{2iAG} + e^{-2iAG})|\psi|^2 \\ &= \frac{1}{2}(1 + cos(2AG))|\psi|^2 = cos^2(AG)|\psi|^2,\end{aligned} \tag{3.3.46}$$

and this is the interference effect in a rather compact notation. This interference occurs in the $U(1)$ case as well as the $O(3)_b$ case. In the $U(1)$ case this is the standard Michelson-Morley type of result. However, commutators of potentials in the $O(3)_b$ case will result in departures from the $U(1)$ case. Further, if the apparatus is rotating the Euler angle and J_z will induce further angular dependent terms not captured by $U(1)$ electrodynamics.

Within $O(3)_b$ symmetry internal gauge space. When we move from x to $x + dx$ on loop A the axes of the internal space's frame are rotated [1,14] this gives rise to:

$$d\psi_A = igJ^aA^a{}_{\mu}dx^{\mu}\psi = id\theta\psi. \tag{3.3.47}$$

When we move from x to $x - dx$ in loop C the rotation is in the opposite sense, giving rise to:

$$d\psi_C = -igJ^aA^a{}_{\mu}dx^{\mu}\psi = -id\theta\psi. \tag{3.3.48}$$

Then with $d\psi/\psi = id\theta$ we then have that

$$\psi_A = e^{i\oint d\theta},\ \psi_C = e^{-i\oint d\theta} \tag{3.3.49}$$

as evaluated around the loop in both directions. The Stokes law lets us to evaluate the line integral as

$$-i \oint d\theta \;=\; \int d\mathcal{A}^{\mu\nu} F_{\mu\nu} \;+\; ig \int d\mathcal{A}^{\mu\nu} [A_\mu,\; A_\nu], \qquad (3.3.50)$$

where $F_{\mu\nu}$ is the abelian field tensor that results from ${A^{(1)}}_\mu$ and ${A^{(2)}}_\mu$ and the commutator is the $O(3)_b$ contribution that results from commutation of these two potential fields. The departure due to $O(3)_b$ electromagnetism is the commutator part of the field tensor generated by loops A and C.

Let $\psi_{A:U(1)}$ and $\psi_{C:U(1)}$ be the shifts in the wave function due to the abelian portion of the field strength tensor evaluated around the A and C loops respectively. We then have that the wave functions are phase shifted around the two paths by

$$\psi_A \;=\; e^{i\phi} \psi_{A:U(1)},\; \psi_C \;=\; e^{-i\phi} \psi_{C:U(1)} \qquad (3.2.51)$$

where $\phi \;=\; \int d\mathcal{A}^{\mu\nu} [A_\mu,\; A_\nu]$. Then the evaluation of the interference of the wave function around the interferometer, in the same manner as arrived at in equation (3.3.46), gives the $O(3)_b$ departure from the Michelson-Morely result

$$|\Psi|^2 = \frac{1}{2}(1 \;+\; cos(2\phi))|\psi'|^2 \;=\; cos^2\phi |\psi'|^2, \qquad (3.3.52)$$

where $\Psi \;=\; 1/2(\psi_A \;+\; \psi_C)$ and $|\psi'|^2$ is the interference due to the contribution from strictly $U(1)$ terms. From equation (3.3.47) when $J^{(3)} \;=\; 0$ the nonabelian contribution to the phase shift vanishes, as $[J^a,\; J^b] \;=\; \epsilon^{abc} J^c$. This is an additional term to the interferometric shift that is predicted by $O(3)_b$ electrodynamics.

When the platform is rotated about the z axis perpendicular to the plane containing loops A and C, a shift in the fringes of the interferogram is observed [18-20]. In $O(3)_b$ electromagnetism this phenomenon is explained through an application of equation [3.3.52],

i.e. is explained by a nonabelian gauge transformation, i.e. a physical rotation of the de Broglie wavefunction. Taking the index $\mu \;=\; 0$ in the gauge transformation of the potentials

$${A^2}_0 \;\rightarrow\; {A^2}_0 \;+\; \frac{1}{g}\partial_0 \alpha, \qquad (3.3.53)$$

and the connection to the momentum

$$e{A^2}_0 \;=\; \hbar\omega/c \qquad (3.3.54)$$

indicates that a gauge transformation induced by $\Omega \;=\; \partial\alpha/\partial t$ that there is a Doppler shift due to the rotation of the platform depending on the direction of the platform rotation (clockwise or anticlockwise):

$$\omega \;\rightarrow\; \omega \;\pm\; \Omega. \qquad (3.3.55)$$

The effect is also an optical Aharonov Bohm effect, where the Ahronov Bohm phase shift exhibits a time dependency. Using $k \;=\; \omega/c$ we find that the elecromagnetic phase is shifted by an amount equation (3.3.51):

$$\Delta\phi \;\propto\; \frac{2\Omega}{c} \oint dl. \qquad (3.3.56)$$

The standard equation for the Sagnac effect is obtained from this general result by using the identity [21]:

$$\oint \Omega \times \boldsymbol{r} \cdot d\boldsymbol{l} \;=\; 2\Omega\mathcal{A} \qquad (3.3.57)$$

where $\mathcal{A}$ is the area enclosed by the loops A and C. We obtain:

$$\mathcal{A} \;=\; \frac{1}{2} \oint \boldsymbol{k} \times \boldsymbol{r} \cdot d\boldsymbol{l}. \qquad (3.3.58)$$

If we use the smallest possible unit for r, the wavelength of the radiation $\boldsymbol{r} \;=\; \lambda\hat{i}$, we arrive at:

$$\oint d\boldsymbol{l} \;=\; \frac{2\mathcal{A}}{\lambda}. \qquad (3.3.59)$$

Using this result in equation (3.3.56) gives the standard expression [21] for the Sagnac effect with a platform rotating at an angular frequency Ω:

$$\Delta\phi = \frac{4\Omega\mathcal{A}}{c\lambda}. \tag{3.3.60}$$

It is worth pointing out that this effect is dependent upon the angular velocity of the platform and the frequency of light as

$$\Delta\phi \propto \omega\Omega. \tag{3.3.61}$$

Therefore the Sagnac effect with rotating platform is recovered in $O(3)_b$ electrodynamics as a Doppler effect of the rotating platform caused by a gauge transformation, or optical Aharonov Bohm effect. In $U(1)$ electrodynamics the products of the four-vector potential are disallowed and the derivation of the Sagnac effect with rotating platform is obscure [21] unless concepts are used which fall outside the usual domain of definition of special relativity. In $O(3)_b$ electrodynamics we are working within standard special relativity and the role of time on and off the platform is always defined by the structure of generally covariant gauge field theory. The speed of light in this derivation is the universal constant c, and we do not use the concept of time dilation, for the tangential velocity $v << c$, and the effect is described by a gauge transformation.

3.4 OBSERVATION OF $B^{(3)}$ IN THE TOPOLOGICAL PHASE

Simon [22] has shown that the topological phases [23-27] are equivalent to a gauge potential in parameter or momentum space, and in general are due to parallel transport in the presence of a gauge field. They are therefore features specific to a nonAbelian gauge field theory. The Pancharatnam phase [23] arises from the cycling in the polarization state of light while keeping the direction of the beam of light fixed. It has been observed [28-31] in different types of interferometry, and has several distinguishing features when compared with the dynamical phase [32]. The Berry phase [24] arises from a cycling in the direction of the beam of light so that the tip of the

spin vector of a photon in the beam traces out a closed curve on the sphere of spin directions. For example, there is a change in the plane of polarization of a beam of light sent through an optical fiber wound on a cylinder.

Pancharatnam demonstrated that a phase difference between two differently polarized beams of light arises when they interfere. Light waves do not interfere when they have orthogonal polarizations, and interfere to a maximum extent when they have the same polarization. The intermediate case gives rise to Pancharatnam's phase, which is related [32] to Berry's phase. The Pancharatnam (P) phase depends only on polarization while the usual dynamical phase has a wave length dependence. The P phase does not depend on the change of the optical path, unlike the dynamical phase, and there are several other differences which are empirically verifiable [32].

In this section it is shown that the topological phase is proportional in free space to the $\boldsymbol{B}^{(3)}$ field through the non Abelian Stokes Theorem associated with the Wu Yang phase. The starting point of the demonstration is the following relation between the P phase and the Wu Yang phase [15]:

$$\gamma(P) = -\frac{e}{\hbar}\oint_C A_\mu dx^\mu, \tag{3.4.62}$$

in which the de Broglie equivalence states the wave particle duality:

$$\frac{e}{\hbar} = \frac{\kappa}{A^{(0)}}. \tag{3.4.63}$$

Using the non-Abelian Stokes Theorem [15] produces the topological phase difference:

$$\delta\gamma(P) = \frac{e}{\hbar}\int\int_{\mathcal{A}}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + i\frac{e}{\hbar}[A_\mu, A_\nu]\right)d\sigma^{\mu\nu}, \tag{3.4.64}$$

and this contains an area integral over a commutator of potentials (which is of course zero in $U(1)$ electrodynamics). In equation 3.4.64) the internal indices are implied. When considering the cycling motion of the electromagnetic field vector as it propagates in free space, this commutator is proportional [33] to the $\boldsymbol{B}^{(3)}$ field in the required $O(3)_b$ gauge theory of electrodynamics. Therefore the P phase contains a term directly proportional to the area integral over $\boldsymbol{B}^{(3)}$, which is the magnetic flux. The P phase is then:

$$\Delta\gamma(P) \;=\; \frac{e}{\hbar}(\boldsymbol{B} \;+\; \boldsymbol{B}^{(3)}) \cdot \mathcal{A} \qquad (3.4.65)$$

where $\mathcal{A}$ is the beam area. Here there is the contribution to the phase by the ordinary magnetic field plus the $\boldsymbol{B}^{(3)}$ field. For two beams of opposite circular polarization the phase contribution from $\boldsymbol{B}^{(3)}$ to P phase is zero, because $\boldsymbol{B}^{(3)}$ is equal and opposite for opposite circular polarizations [34]. For a light beam that travels through the loop the contribution by the standard magnetic field $\boldsymbol{B}$ can be set to zero if the length of the loop is an integer multiple of the wavelength of the light. In this way the average contribution from the oscillating field is zero. Consider a beam that orbits the loop a multiple number of times. Even if the length of the loop is not an integer multiple, then it may be arranged so there will be some n traverses of the loop where effectively the average contribution of the standard magnetic field will average to zero. For two beams of the same circular polarization the $\boldsymbol{B}^{(3)}$ contribution to the phase is maximized, as is observed; and for intermediate cases the vector sum of the two $\boldsymbol{B}^{(3)}$ fields is used.

If we choose a beam made up of one photon, the magnetic flux carried by it is $\frac{\hbar}{e}$, and so:

$$\Delta\gamma(P) \;=\; \pm 1, \qquad (3.4.66)$$

the P phase is 1 for one sense of circular polarization, and -1 for the other, and is proportional to the photon helicity [15]. Without the use of $O(3)_b$ electrodynamics, (parallel transport in an $O(3)_b$ internal gauge space) this topological phase cannot be related to $\boldsymbol{B}^{(3)}$, and is

undefined. This is a conclusive demonstration of the limits of $U(1)$ electrodynamics.

When the phase change equivalent to Berry's phase [35] is observed by winding a fiber on a cylinder, there is a topological phase due to the $\boldsymbol{B}^{(3)}$ field generated in the direction of the axis of the fiber by longitudinal vector potentials (cf. Section 3.3). The angle of rotation in the plane of polarization of light is:

$$\Delta\theta \;\propto\; (\frac{e^2}{\hbar})^2 \int\int [A_\mu,\; A_\nu] d\sigma^{\mu\nu} \;=\; V|\boldsymbol{B}^{(3)}|, \qquad (3.4.67)$$

where V is a Verdet constant. The Berry phase is then a topological number that is induces by the $\boldsymbol{B}^{(3)}$ field. These results are straightforward deductions from the Wu-Yang phase applied to the photon using the wave particle equivalence (3.4.63). Using this method the dynamical phase can also be deduced, as in Section (3.3). The topological and dynamical phases therefore have their origins in the Wu-Yang phase and in the de Broglie equivalence (3.4.63) for the photon and wave in free space. They are expressions of the wave and particle dualism of light and all matter. In explaining the Sagnac effect in Section 3.3 the non Abelian dynamical phase was deduced from the Wu- Yang phase and shown to be the cause of the interferogram with this particular interferometer at rest. This deduction can be reinforced as follows, using the fact that the non Abelian dynamical and topological interpretations of the Wu-Yang phase are expressions of de Broglie's wave particle dualism, and therefore have a common origin.

The B Cyclic Theorem of $O(3)_b$ electrodynamics states that:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} \;=\; iB^{(0)}\boldsymbol{B}^{(3)*}, \qquad (3.4.68)$$

and using this result in equation (3.4.63) it follows that:

$$Icos\delta \;=\; \frac{c}{\mu_0} B^{(0)} B^{(3)*}, \qquad (3.4.69)$$

where we have used the definition of the vector cross product:

$$B^{(1)} \times B^{(2)} = {B^{(0)}}^2 cos(\delta \pm \pi/2) e^{(3)*} \tag{3.4.70}$$

More generally, Pancharatnam [23] deduced that:

$$Icos\delta = \sqrt{I_1 I_2} \hat{P_1 P_2} cos\delta \tag{3.4.71}$$

from a consideration of the intensity pattern of two interfering beams of intensities I_1 and I_2 and elliptical polarizations P_1 and P_2. The object $\hat{P_1 P_2}$ is the angular distance [32] between P_1 and P_2 on the Poincaré sphere and δ their phase difference. The P phase was then shown to be proportional to a solid angle on the Poincare sphere through Pancharatnam's Theorem [23,32]. A simplifirf of the geometry is where we consider two co-axial circularly polarized beams. The P phase is then seen through equation (3.4.69) to be determined by the $\boldsymbol{B}^{(3)}$ fields of the two beams. If for example the circular polarization of the two beams is opposite, the P phase is zero. If the circular polarizations are the same, the P phase is maximized, and intermediate elliptical polarizations give intermediate P phases.

The existence of the P phase has been demonstrated empirically on the one photon level [36,37] and therefore these recent experiments serve as a direct demonstration of the existence of $\boldsymbol{B}^{(3)}$ for one photon - the photomagneton operator introduced in 1992 [38]. Experiments demonstrate [32] that the P phase exists in all types of interferometry, and that a non Abelian phase change due to a change in optical path difference (for example in Michelson interferometry [32] is the same precisely as that induced topologically. This is a straightforward consequence of the non-Abelian nature of the Wu-Yang phase and of the de Broglie duality, equation (3.4.62). The P phase is deduced from the area integral of the non-Abelian Stokes

Theorem (equation (3.4.63)):

$$\gamma(P) = \Omega_0 = i \int \int [D_\mu, D_\mu] d\sigma^{\mu\nu}, \tag{3.4.72}$$

and the dynamical phase is deduced from the line integral of the same Theorem:

$$\gamma(\underline{D}) = \omega_0 = i \oint D_\mu dx^\mu. \tag{3.4.73}$$

For example, the dynamical phase difference of the well known Young interferometer (two slit experiment) is deduced from the Wu-Yang phase by replacing $\frac{e}{\hbar}$ in equation (3.4.65) by $\kappa/A^{(0)}$. Straight forward algebra gives the well known Young interferogram:

$$\mathcal{R}e(\Delta\phi) = Pcos\Big(2\pi\Big(\frac{\Delta r}{\lambda} \pm n\Big)\Big), \tag{3.4.74}$$

and this result is applicable to the photon, electron, neutron, atoms and all particulate matter. This possibly suggests that the photon is also a particle with material properties similar to massive particles, as originally suggested by deBroglie. The empirically observable [32] topological P phase, equation (3.4.72), in the same experiment (Young interferometry) is deduced from the nonAbelian area integral of $O(3)_b$ electrodynamics. This integral is directly proportional to a topological magnetic monopole [15]. The recent empirical observation of the topological phase in Young interferometry [15] is therefore an observation of the topological magnetic monopole [15], or magnetic flux due to the $\boldsymbol{B}^{(3)}$ field.

Note carefully that the topological P phase cannot be defined without the area integral over $\boldsymbol{B}^{(3)}$, and this area integral does not exist in $U(1)$ electrodynamics. As shown by Simon [22] the topological phase is generated by a round trip with nonAbelian, for example $O(3)_b$ covariant derivatives. In $U(1)$ electrodynamics we obtain the dynamical phase:

$$\phi_0 = \boldsymbol{\kappa} \cdot \boldsymbol{r} \tag{3.4.75}$$

from the line integral of the Wu-Yang phase, but as argued in section 3.3, this phase is insufficient to describe the Sagnac interferogram with the interferometric platform at rest. For a complete description of Young interferometry, $O(3)_b$ electrodynamics are essential.

Dultz and Klein [32] have recently given a useful summary of the differences between the topological (P) phase and dynamical (D) phase. The P phase is achromatic, and depends only on polarization. The D phase is chromatic and path and frequency dependent. The P phase is non-additive in the optical path, $\boldsymbol{r}$; unlike the D phase. The P phase is unbounded, and is changed by the state of polarization. The D phase is bounded in a finite range of $\boldsymbol{r}$. Beam coding in the P phase occurs in unseparated beams (for example the Sagnac interferometer), while separate beams are always required for the D phase. Both phases occur in the one photon case [36,37] and have been observed empirically [32] in the Sagnac, Mach, Zehnder, Michelson and Young interferometers.

Therefore interferometry, and all physical optics, are more generally branches of $O(3)_b$ electrodynamics. This conclusion can be illustrated by considering as in Section 3.3 the Sagnac effect as a phase difference generated by equal and opposite Wu-Yang phase differences between loops A and C. Replace $\frac{e}{\hbar}$ by $\frac{\kappa}{A^{(0)}}$ using equation (3.4.63) and integrate around the A and C loops with the $\boldsymbol{B}^{(3)}$ field projected on the loops:

$$\begin{aligned} \boldsymbol{B}^{(3)} \cdot d\boldsymbol{r}_A &= \kappa A^{(0)} |\boldsymbol{k}| \\ \boldsymbol{B}^{(3)} \cdot d\boldsymbol{r}_C &= -\kappa A^{(0)} |\boldsymbol{k}|. \end{aligned} \tag{3.4.76}$$

These are equal and opposite in$O(3)_b$ electrodynamics, but do not exist in $U(1)$ electrodynamics. These integrals generate equal and

opposite nonAbelian dynamical and topological phases in loops A and C: phases which do not exist in $U(1)$ electrodynamics. The phase difference observed with the Sagnac platform at rest can be shown simply from this analysis to be:

$$\mathcal{R}e\phi = cos(2\omega^2 \mathcal{A}/c^2 \pm 2\pi n), \tag{3.4.77}$$

where $\mathcal{A}$ is the area enclosed by loops A or C and ω is the angular frequency of the light beam from the Sagnac beamsplitter. When the platform is rotated at an angular frequency Ω, there is a Doppler shift as shown in Section (3.3), so ω in equation (3.4.77) is replaced by:

$$\omega \rightarrow \omega \pm \Omega. \tag{3.4.78}$$

For $\omega >> \Omega$ the difference in phase interference generated in the Sagnac interferometer with platform at rest and in motion is therefore the well known classical result:

$$\Delta(\mathcal{R}e) \simeq cos(4\omega\Omega\mathcal{A}/c^2 \pm 2\pi n), \tag{3.4.79}$$

which is an approximation of:

$$\Delta(\mathcal{R}e) \simeq cos(2(2\omega\Omega + \Omega^2)\mathcal{A}/c^2 \pm 2\pi n). \tag{3.4.80}$$

When the platform is spun very quickly, the result (3.4.80) may produce observable departures. In this explanation of the Sagnac effect the equal and opposite topological phase due to $\boldsymbol{B}^{(3)}$ fields appear in an axis perpendicular to the plane of the loops A and C, together with a topological magnetic monopole and a topological phase, all stemming from the commutator $[\mathcal{D}_\mu, \mathcal{D}_\nu]$ in the integrand of the area integral of the non-Abelian Stokes Theorem (3.4.72). The commutator is zero by definition in $U(1)$ electrodynamics. All these effects are inter-related and the $\boldsymbol{B}^{(3)}$ field is associated with a topological magnetic monopole whose origins are similar to the well known 't Hooft Polyakov monopole in non Abelian gauge field theory 1. As argued in Section 3.3, there is no phase difference in the Sagnac interferometer in $U(1)$ electrodynamics, and so there is no explanation for the observed interferogram with platform at rest. In $O(3)_b$ there is an explanation which can be developed simultaneously according

to an $O(3)_b$ dynamical phase or an $O(3)_b$ topological phase induced on a quantum wave function as represented by equation (3.4.63). When the platform is rotated, an $O(3)_b$ gauge transformation occurs as argued in Section (3.3), leading to a Doppler effect and to the classical result (3.4.79), an approximation of equation (3.4.80).

The Michelson interferogram can be explained in $O(3)_b$ electrodynamics in the wave interpretation (dynamical phase) by noting that the phase differences generated in arms X and Y by a round trip from beamsplitter to mirror and back are:

$$\begin{aligned} \Delta\phi_X &= exp\Big(k\frac{\kappa}{A^{(0)}}\oint A_X dX\Big) \\ \Delta\phi_Y &= exp\Big(k\frac{\kappa}{A^{(0)}}\oint A_Y dY\Big). \end{aligned} \tag{3.4.81}$$

In general these phase differences are not the same, and so the Michelson interferogram is observed by moving one of the mirrors. The corresponding topological phase, observed recently in the Michelson interferometer [32] is given by the difference of $\boldsymbol{B}^{(3)}$ fields in arms X and Y, a difference generated in the particle interpretation from the difference of area integrals:

$$\Delta(\Delta\phi) = cos\Big(\frac{e}{\hbar}(\boldsymbol{B}^{(3)}{}_X - \boldsymbol{B}^{(3)}{}_Y) \pm 2\pi n\Big). \tag{3.4.82}$$

In order to observe this phase difference, which is equal [32] to the dynamical phase difference, the polarizations in axes X and Y must be adjusted without moving either mirror of the interferometer. This is accomplished using polarizers as reviewed by Dultz and Klein [32].

In $U(1)$ electrodynamics it is not possible to obtain a Michelson interferogram without moving one mirror. In $O(3)_b$, if the solid angle

in Pancharatnam's theorem is denoted Ω_0, the P phase difference is:

$$\begin{aligned} \Delta\phi &= exp(i\Omega_0), \\ \Omega_0 &= i\oint \mathcal{D}_\mu dx^\mu = i\int\int [\mathcal{D}_\mu,\ \mathcal{D}_\nu] d\sigma^{\mu\nu}, \end{aligned} \tag{3.4.83}$$

and is given by a difference in $\boldsymbol{B}^{(3)}$ fields in X and Y induced by a polarizer with fixed mirrors. It is emphasized that in $U(1)$ electrodynamics, there does not exists this topological phase difference in Michelson interferometry. This is because the beams in X and Y always exhibit no phase difference due to $\boldsymbol{B}^{(3)}$ at the beamsplitter. Specifically, the $U(1)$ phases at the recombination point at the beamsplitter is sum to e^{2iFA} for all beamsplitter to mirror path lengths in X and Y. Here we are using the compact notation of equation (3.3.44). The reason for this is that the contribution to the phase in X and Y always vanish as follows:

$$\begin{aligned} \Delta\phi_X &= e^{i(FA + gJ^3\int A^3\cdot dx)} = e^{i\phi} \\ \Delta\phi_X &= e^{i(-FA + gJ^3\int A^3\cdot dy)} = e^{i\phi'}, \\ J^x &= J^y = 0. \end{aligned} \tag{3.4.84}$$

In conclusion therefore the Pancharatnam phase is related to the $\boldsymbol{B}^{(3)}$ field of $O(3)_b$ electrodynamics through the non-Abelian Stokes Theorem:

$$\Omega_0 = i\oint \mathcal{D}_\mu dx^\mu = i\int\int [\mathcal{D}_\mu,\ \mathcal{D}_\nu] d\sigma^{\mu\nu}, \tag{3.4.85}$$

The observable phase change is then determined by de Moivre's Theorem:

$$\Delta\phi = e^{i\Omega_0} = cos\Omega_0 + isin\Omega_0, \tag{3.4.86}$$

where upon it becomes clear that Ω_0 is indeed an angle - an angle in Minkowski spacetime, or in other words a solid angle on the Poincare sphere. The most important point to bear in mind is: the empirically observed features of the P phase (and the Berry phase) cannot be deduced theoretically unless the Stokes Theorem (3.4.85)

is non-Abelian, with covariant derivatives that give rise to the $\boldsymbol{B}^{(3)}$ field in the special case of circular polarization:

$$\Omega_0 = \frac{e}{\hbar}\boldsymbol{B}^{(3)} \cdot \mathcal{A} = \frac{e}{\hbar}\Phi = \frac{\Phi}{\Phi_0}. \tag{3.4.87}$$

The P phase is therefore directly proportional to $\boldsymbol{B}^{(3)}$ and is the ratio of the magnetic flux Φ to the elementary fluxon, $\phi_0 = \frac{\hbar}{e}$.

Interferometry and physical optics depend on the existence in vacuo of the $\boldsymbol{B}^{(3)}$ field for one photon. The $\boldsymbol{B}^{(3)}$ field is observed empirically as the topological phase.

3.5. THE NONABELIAN STOKES THEOREM AND THE ELECTROMAGNETIC PHASE

In $O(3)_b$ electrodynamics the nonlinear electromagnetic phase is defined through the non-Abelian Stokes theorem:

$$\gamma = i\oint \mathcal{D}_\mu dx^\mu = i\int\int [\mathcal{D}_\mu, \mathcal{D}_\nu] d\sigma^{\mu\nu}, \tag{3.5.88}$$

where

$$\mathcal{D}_\mu = \partial_\mu - igA_\mu \tag{3.5.89}$$

s a covariant derivative. The quantum operator correspondence is:

$$i\partial_\mu = P_\mu/\hbar = K_\mu \tag{3.5.90}$$

nd the electromagnetic field is:

$$G_{\mu\nu} = \frac{i}{g}[\mathcal{D}_\nu, \mathcal{D}_\mu] = \partial_\nu A_\mu - \partial_\mu A_\nu - ig[A_\nu, A_\mu]. \tag{3.5.91}$$

The phase is therefore:

$$\gamma = \oint (K_\mu + gA_\mu) dx^\mu = g\int\int G_{\mu\nu} d\sigma^{\mu\nu}. \tag{3.5.92}$$

The left hand side is a line integral and the right hand side an area integral over the electromagnetic field tensor: they give the dynamical and topological phases respectively [34]. The line integral is equal and opposite for loops A and C by definition: over any closed curve, the line integral clockwise (or in one sense of direction) is always the negative of the line integral anticlockwise. As we have argued, this is the basis of the Sagnac effect and of interferometry in general, leading us to reject $U(1)$ electrodynamics in favor of a non-Abelian electrodynamics such as $O(3)_b$.

The limitations of $U(1)$ (Heaviside Maxwell) electrodynamics are abundantly exemplified when we trace the electromagnetic phase around the A and C loops of the Sagnac effect with platform at rest. The $U(1)$ phase contains a dynamical phase plus a topological phase due to the magnetic field $\boldsymbol{B}^{(1,2)}$ that project through the solenoidal area:

$$\gamma_{U(1)} = K_\mu x^\mu + i\int\int (\partial_\nu A_\mu - \partial_\mu A_\nu) d\sigma^{\mu\nu}. \tag{3.3.93}$$

If the Sagnac loops A and C are in a plane perpendicular to the Z axis, the wave-vector is in general:

$$\boldsymbol{K} = K_X\boldsymbol{i} + K_Y\boldsymbol{j}. \tag{3.5.94a}$$

$$\boldsymbol{r} = A\boldsymbol{i} + Y\boldsymbol{j}. \tag{3.5.94b}$$

The phases in the A and C loops, if there is no net parallel translation of the magnetic fields, are respectively:

$$\gamma_A = \omega t$$

$$\gamma_C = \omega t \tag{3.5.95}$$

and are identical. There is no phase difference for a rotating platform, contrary to observation [32,36]. The only way of addressing this problem within $U(1)$ symmetry is the ad hoc introduction of a random phase difference, an essentially unsatisfactory procedure

which follows the phenomenon and is therefore not based on first principles. By definition of the $U(1)$ gauge:

$$[A_\mu,\ A_\nu] \ = \ 0, \tag{3.5.96}$$

and both sides of equation (3.5.88) vanish after averaging over many cycles. The $U(1)$ covariant derivative cannot be used to give a satisfactory definition of this electromagnetic phase associated with the Sagnac effect.

In the $O(3)_b$ gauge theory it is of key importance to realize that the potential contributes, as a product, to physical observables on the classical level. It is defined through the rotation generators of $O(3)_b$ [39]:

$$A_\mu \ = \ J^a A^a{}_\mu, \tag{3.5.97}$$

where:

$$[J^1,\ J^2] \ = \ iJ^3 \tag{3.5.98}$$

is an angular momentum commutator relation within $\hbar$. It follows that the area integral of equation (3.5.88) is:

$$\gamma_A \ = \ \gamma_C \ = \ -ig^2 \int\int [A^1{}_\mu,\ A^2{}_\nu] d\sigma^{\mu\nu}, \tag{3.5.99}$$

and if the loops A and C are circular we can write:

$$\boldsymbol{A} \ = \ \boldsymbol{A}^* \ = \ \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} \ + \ \boldsymbol{j})e^{i\omega t}. \tag{3.5.100}$$

Laser beams sent around optical fibers A or C therefore generate equal and opposite projections of

$$\boldsymbol{B}^{(3)} \ = \ -ig\boldsymbol{A} \times \boldsymbol{A}^*. \tag{3.5.101}$$

onto the area that encloses the loop.

This is a topological magnetic field with a monopole defined by Stokes law and the divergence of the field. It is essential to understand that this magnetic field does not exist in Heaviside Maxwell electrodynamics, and is therefore fundamentally different from a static magnetic field. Analogously, a topological phase is fundamentally different from a dynamical phase [34], and it contains a term that is numerically computed through the non-Abelian Stokes Theorem. (A line integral is equal to an area integral but the integrals are local and global in nature.) An additional feature in the non-Abelian Stokes Theorem is that there is an internal vector space, so if the area integral points in a given direction, so must the line integral. Some practice is needed to become accustomed to this property, in the usual Abelian Stokes Theorem, as applied in $U(1)$ electrodynamics, the internal vector space is missing. In the Sagnac effect, for example, the area integral reverses sign for loops A and C so must be an axial vector perpendicular to the plane of the loops, i.e. an axial vector in Z. Therefore the line integral is also an axial vector in Z. The magnitude of this vector is the same for left and right hand sides (line and area integrals). If the constant g is $\kappa/A^{(0)}$ the difference in the area integrals for A and C is:

$$\Delta\gamma \ = \ 2\kappa^2\mathcal{A} \ = \ 2\frac{\omega^2}{c^2}\mathcal{A} \tag{3.5.102}$$

and the observable phase shift of the Sagnac effect with platform at rest is:

$$\Delta\phi \ = \ cos(\Delta\gamma \ \pm \ 2\pi n), \tag{3.5.103}$$

a topological phase difference. It is numerically equal to the dynamical phase difference

$$\Delta\phi \ = \ cos(\oint \kappa \cdot d\boldsymbol{r} \ \pm \ 2\pi n) \tag{3.5.104}$$

obtained from the line integral on the left side of the non-Abelian Stokes Theorem. This line integral is zero in U(1) electrodynamics for connection coefficients that have zero parallel transport around the loop. This is a vector in the axis perpendicular to the Sagnac plane. Such a concept does not exist in Heaviside Maxwell electrodynamics and is the root cause of the Sagnac effect. Some care must be taken in the interpretation of equation (3.5.99).

When the platform is set in motion there is an $O(3)_b$ gauge transformation, a physical rotation about Z which produces the result:

$$K_\mu \rightarrow K_\mu + \partial_\mu \alpha \tag{3.5.105}$$

as we have seen. This feature is specific to $O(3)_b$ electrodynamics. Taking the $\mu = 0$ index of equation (3.5.105) produces the observable phase shift of the Sagnac effect when the platform is set in motion:

$$\Delta\Delta\gamma = \frac{2}{c^2}((\omega + \Omega)^2 - (\omega - \Omega)^2)\mathcal{A} = \frac{4}{c^2}\omega\Omega\mathcal{A}. \tag{3.5.106}$$

This phase shift has been observed to a precision of one part in 10^{20} [38]. It cannot be explained with $U(1)$ electrodynamics [39].

The Sagnac effect with platform in motion is the same to an observer on and off the platform [38], so is a Lorentz invariant phenomenon to a very good approximation. The observed phase shift does not depend on the shape of the loops, or on the way in which the platform is rotated - offset rotation produces the same result [38].

The explanation for these observations in $O(3)_b$ electrodynamics is that if we take the difference of the Lorentz invariants:

$$L_1 = (K_\mu + \partial_\mu \alpha)(K^\mu + \partial_\mu \alpha) \tag{3.5.107}$$

$$L_2 = (K_\mu - \partial_\mu \alpha)(K^\mu - \partial_\mu \alpha), \tag{3.5.108}$$

we find that

$$\Delta = L_1 - L_2 = 2(\partial_\mu \alpha K^\mu + K^\mu \partial^\mu \alpha), \tag{3.5.109}$$

another invariant. If we assume that the space part of $\partial_\mu \alpha$ is very small, we find that:

$$\Delta = \frac{4\omega\Omega}{c^2} \tag{3.5.110}$$

is a Lorentz invariant as required. The Sagnac effect is the same on and off the platform as observed with great precision [38]. This is a direct result of $O(3)_b$ electrodynamics in Minkowski spacetime.

In order to demonstrate the topological origin of the Sagnac effect it can be proven as follows that it is a special case of the Tomita Chiao effect [40]. First write equation (3.5.106) as:

$$\Delta\Delta\gamma = 4\pi\frac{\omega}{\Omega}\left(\frac{v}{c}\right), \tag{3.5.111}$$

using $\Omega = v/R$ and $\mathcal{A} = \pi R^2$. As $\Omega \rightarrow \omega$ and $\Omega \rightarrow \omega$:

$$\Delta\Delta\gamma \rightarrow 4\pi. \tag{3.5.112}$$

Now rewrite equation (3.5.99) using $\mathcal{A} = \pi R^2$; $\omega = v/R$; $\Omega \rightarrow \omega$; $v \rightarrow c$ to obtain:

$$\Delta\gamma_{A-C} \rightarrow 2\pi. \tag{3.5.113}$$

The limits (3.5.113) describes a platform so that if it is regarded as massless then this describes a laser beam sent around an optical fiber loop A. (The limit (3.5.112) is the difference in $\Delta\gamma$ for lasers in loops A and C.) The topological phase of the Tomita Chiao effect, regarded as a Pancharatnam phase, is given by 40:

$$\gamma = \mp\frac{1}{2}\int_0^{2\pi}\left(1 - \frac{p}{\sqrt{p^2 + (2\pi R)^2}}\right)d\phi, \tag{3.5.114}$$

and so

$$\gamma_{p\rightarrow 0} \rightarrow \mp\pi \; and \; \Delta\gamma_{A-C} \rightarrow 2\pi \tag{3.5.115}$$

when $p \rightarrow 0$. Here p is the length of a helically wound optical fiber of radius R. The limit (3.5.115) is that in which the helix reduces to the circle of the Sagnac effect. The latter is therefore a topological phase measured by interferometry. The Tomita Chiao effect is the same (global) topological phase picked up by rotation of the phase of polarization of linearly polarized light propagating through the fiber.

Both equations (3.5.113) and (3.5.115) can be written as

$$\Delta = 2\pi\frac{e}{\hbar}B^{(3)}\cdot\mathcal{A}, \tag{3.5.116}$$

where

$$\Phi = B^{(3)}\cdot\mathcal{A} \tag{3.5.117}$$

is the magnetic flux due to the topological magnetic flux density $B^{(3)}$. For one photon [2-12]:

$$\Phi_0 = \frac{\hbar}{e}. \tag{3.5.118}$$

Equation (3.5.111) illustrates the fact that the phase shift in the Sagnac effect is area independent, and this is observed in the Michelson Gale experiment, where the area enclosed by the loop is much smaller than the Earth's area. The Michelson-Gale effect is due to the Earth's diurnal rotation, v being the Earth's surface velocity.

The only difference between the well known Michelson Gale and Michelson Morley experiments is that the laser beam in the former encloses an area on the earth's surface. In the Michelson Morley experiment the beam is sent in opposite directions in one arm of a Michelson interferometer. In one direction the Earth's surface moves at v and in the other at $-v$. This movement can be thought of as that of a rotating platform and is therefore an $O(3)_b$ gauge transform. The relevant phase difference with platform at rest in the Michelson interferometer is generated by a journey from the beam-splitter (O) to a mirror (A) and back to the beamsplitter:

$$\Delta\gamma = \int_O^A \boldsymbol{k}\cdot\boldsymbol{r} + \int_A^O \boldsymbol{k}\cdot\boldsymbol{r} = \oint \boldsymbol{k}\cdot\boldsymbol{r}, \tag{3.5.119}$$

where

$$\int_O^A \boldsymbol{k}\cdot\boldsymbol{r} = -\int_A^O \boldsymbol{k}\cdot\boldsymbol{r} \tag{3.5.120}$$

and on reflection at A:

$$\boldsymbol{k} \rightarrow -\boldsymbol{k}. \tag{3.5.121}$$

This then results in:

$$\Delta\gamma = \oint \boldsymbol{k}\cdot\boldsymbol{r}. \tag{3.5.122}$$

The journey of the wave-vector around the Earth's diameter can be described as:

$$\boldsymbol{k}^{(1)} = \boldsymbol{k}^{(2)*} = \frac{k^{(0)}}{\sqrt{2}}(\boldsymbol{i}\boldsymbol{i} + \boldsymbol{j})e^{i\omega t}, \tag{3.5.123}$$

where $k^{(0)} = \omega$. So if $\omega \rightarrow \omega \pm \Omega$ as in the Michelson Gale or Sagnac effect then the relative change in $K^{(0)}$ is $\simeq \Omega/\omega = 10^{-22}$. This seems to allow for the possibility of an extremely small effect in the Michelson Morley experiment.

3.6 LINK BETWEEN B CYCLIC THEOREM AND THE NONABELIAN STOKES THEOREM

In this final section of chapter three a link is proposed between the B cyclic theorem and the non-Abelian Stokes theorem. Start with the B cyclic theorem in the form:

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)} \tag{3.6.124}$$

and use the relation:

$$B^{(0)} = \kappa A^{(0)}, \tag{3.6.125}$$

so that

$$\kappa A^{(0)}\boldsymbol{k} = -i\frac{\kappa}{A^{(0)}}A^{(1)}\times A^{(2)}\pi R^2. \tag{3.6.126}$$

Without loss of generality it is possible to multiply both sides of this equation by the area of the space and so that:

$$\kappa A^{(0)}\sigma R^2\boldsymbol{k} = i\frac{\kappa}{A^{(0)}}\pi R^2. \tag{3.6.127}$$

Therefore it is deduced that

$$\pi R\kappa A^{(0)}(\boldsymbol{k}\cdot\boldsymbol{R})\boldsymbol{k} = \boldsymbol{B}^{(3)}\cdot\mathcal{A}\boldsymbol{k}. \tag{3.6.128}$$

This expression can be integrated to give the non-Abelian Stokes theorem in the form:

$$2\pi\kappa A^{(0)} \oint \boldsymbol{R} \cdot d\boldsymbol{R} \; = \; \int\int \boldsymbol{B}^{(3)} \cdot d\mathcal{A}. \qquad (3.6.129)$$

Finally, let:

$$R \; = \; \frac{1}{\kappa} \; = \; \frac{\lambda}{2\pi}, \qquad (3.6.130)$$

and multiply both sides by $g \; = \; \frac{\kappa}{A^{(0)}}$ to define the phase:

$$\gamma \; = \; 2\pi \oint \kappa \cdot d\boldsymbol{R} \; = \; \frac{\kappa}{A^{(0)}} \int\int \boldsymbol{B}^{(3)} \cdot d\mathcal{A}. \qquad (3.6.131)$$

The line integral must be evaluated along a closed path as indicated in figure 3.1. A closed curve is therefore a helix along the propagation axis (Z axis). The circumference of the helix remains $2\pi R$. The line integral starts at the origin O and proceeds around the helix back along the propagation axis from A to O. Off the Z axis the line integral is zero. Integration around a circle for example gives:

$$\oint dl \; = \; -\int_0^{2\pi} sin(t)dt \; + \; \int_0^{2\pi} cos(t)dt. \qquad (3.6.132)$$

The line integral along the off axial helix is formed from:

$$\mathcal{I} \; = \; -i \int_0^{2\pi} sin(t)dt \; + \; j \int_0^{2\pi} cos(t)dt \qquad (3.6.133)$$

which is zero for all ϕ.

The line integrals for all r (along the Z axis) are defined therefore by:

$$\gamma \; = \; 2\pi \oint_{AO} \kappa \cdot d\boldsymbol{R} \; = \; -2\pi \oint_{OA} \kappa \cdot d\boldsymbol{R}, \qquad (3.6.134)$$

and it is these line integrals that give the phenomenon of Michelson interferometry. As soon as the concept of line integral is introduced,

the area integral follows, and the non-Abelian Stokes theorem links together the dynamical and topological phases. In terms of the vector potential:

$$\gamma \; = \; 2\pi g \oint \boldsymbol{A}^{(3)} \cdot d\boldsymbol{R} \; = \; g \int\int \boldsymbol{B}^{(3)} \cdot d\mathcal{A}. \qquad (3.6.135)$$

As the helical path closes to a circle:

$$R \; \rightarrow \; 0; \; \kappa \; \rightarrow \; \infty, \qquad (3.6.136)$$

and the line integral becomes indeterminate. The situation is then that of the Sagnac interferometer.

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CHAPTER 4
NONLINEAR PROPAGATION IN $O(3)_b$ ELECTRODYNAMICS: SOLITONS AND INSTANTONS

4.1 LIMITATIONS OF THE U(1) THEORY

Barrett [1] has discussed the limitations of the $U(1)$ theory when it comes to the calculation of propagation velocities. Group velocities in $U(1)$ are almost always larger than the speed of light for radio frequency transmissions through the atmosphere, and under certain conditions derivations imply a transmission rate of information equal to zero. Barrett also points out that the $U(1)$ theory cannot always calculate the propagation velocity of signals with bandwidth propagating in a lossy medium. Finally, sinusoidal or linear signals are always used in $U(1)$. It is demonstrated [1] that the Harmuth ansatz 2-5 allows these calculations to be made. The Harmuth ansatz described by Barrett as occurs in three stages. 1) A mapping of the Heaviside Maxwell $U(1)$ equations into a higher symmetry SU(2) form. 2) Solutions for the propagation velocities. 3) Mapping the solved equations back into $U(1)$ symmetry, removing thereby the magnetic monopole.

In this chapter the $O(3)_b$ field equations are used to accomplish the calculation of propagation velocities, but without mapping in and out of $SU(2)$ symmetry. The $O(3)_b$ gauge symmetry is used in all situations, and allows the propagation velocities to be calculated satisfactorily, thus demonstrating another major advantage over the $U(1)$ form of electrodynamics. This can be accomplished by identifying the internal gauge space with the $((1),(2),(3))$ representa-

tion of space, a representation based on circular polarization itself. (All other polarizations are constructed from circular polarization, so the basis is fundamental.) The quantities appearing in the Harmuth field equations can therefore be identified with their equivalents in the $O(3)_b$ field equations, and the $O(3)_b$ equations can therefore be used straightforwardly to calculate propagation velocities using Harmuth's method. The net result is a clear advantage over $U(1)$ electrodynamics.

4.2 IDENTIFICATION OF THE HARMUTH AND $O(3)_b$FIELD EQUATIONS

We identify terms in Harmuth and $O(3)_b$ equation by equation, starting with the Gauss law, which is equation (2.6.49) in $O(3)_b$. In the Harmuth equations this has the representation 1:

$$\nabla \cdot \boldsymbol{B} = \rho_m, \tag{4.2.1}$$

where ρ_m is a phenomenological magnetic charge density which is assumed to vanish [1] in the vacuum (source free condition). In the $O(3)_b$ Gauss Law, equation (3.6.49), component (3) is perpendicular to (1) and (2) by definition, so the Law becomes:

$$\nabla \cdot \boldsymbol{B}^{(1)} = 0, \ \nabla \cdot \boldsymbol{B}^{(2)} = 0 \tag{4.2.2a}$$

$$\nabla \cdot \boldsymbol{B}^{(3)} = ig\boldsymbol{A}^{([1)} \cdot \boldsymbol{B}^{(2])}. \tag{4.2.2b}$$

The topological magnetic monopole or instanton construct exists if and only if the current term proportional to $[\boldsymbol{A}^{(1)}, \boldsymbol{B}^{(2)}]$ is non-zero. In the vacuum however, $\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)}$ and if $\boldsymbol{A}^{(1)}$ is a plane wave this current term vanishes, leaving the $O(3)_b$ Gauss Law:

$$\nabla \cdot \boldsymbol{B}^{(1)} = \nabla \cdot \boldsymbol{B}^{(2)} = \nabla \cdot \boldsymbol{B}^{(3)} = 0. \tag{4.2.3}$$

Therefore there is automatic agreement with the Harmuth equation in the vacuum, in that the topological magnetic monopole does not

exist in the vacuum. Under other conceivable circumstances a magnetic monopole of this type may exist in $O(3)_b$ electrodynamics. Equation (4.2.3) is consistent with the B Cyclic Theorem:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)}. \tag{4.2.4}$$

There is a clear difference, however, between the philosophy of Harmuth and the systematic development in this book of $O(3)_b$ electrodynamics, in that an embedding of $U(1)$ in $SU(2)$ and back again is not used. This means that Harmuth still appears to accept $U(1)$ electrodynamics as the fundamental electrodynamics, whereas the development in this book takes $O(3)_b$ electrodynamics as being fundamental.

The Faraday Law of induction extended phenomenologically by Harmuth by using a magnetic current density proportional to

$$\boldsymbol{J}_m{}^{(1)*} = cA_0{}^{([3)}\boldsymbol{B}^{(2])} + [\boldsymbol{A}^{(2)}, \boldsymbol{E}^{(3)}], \tag{4.2.5}$$

in equation (2.6.51) is demonstrated. This is the Faraday law of induction in $O(3)_b$ electrodynamics. Harmuth assumes that this magnetic current density obeys a magnetic Ohm's law through a magnetic conductivity which is assumed to be non-zero in free space as described by Barrett [1], pp. 56 ff. Therefore we must investigate the circumstances under which such a term can exist in $O(3)_b$ electrodynamics in order to calculate propagation velocities using Harmuth's method.

The Faraday law of induction is regained from $O(3)_b$ electrodynamics in the form:

$$\nabla \times \boldsymbol{E}^{(1)} = \frac{\partial \boldsymbol{B}^{(1)*}}{\partial t} = 0 \tag{4.2.6}$$

if the magnetic current density (4.2.5) vanishes. Otherwise the Law is in Harmuth's form, which allows a much richer variety of solutions

[1] and allows a calculation to be made of propagation velocities where the $U(1)$ symmetry field equations fail [1-5]. These are good reasons to accept equation (2.6.51) as the Faraday law of induction without attempting to map it back into $U(1)$ form. The Faraday law in the familiar form (4.2.6)is retrieved from equation (2.6.51) by using particular solutions as follows. The potentials must be of the form:

$$A^{(1)\mu} = (0, \boldsymbol{A}^{(1)}), \; A^{\mu(2)} = (0, \boldsymbol{A}^{(2)}), \; A^{(3)\mu} = (0, 0), \quad (4.2.7)$$

so that ${A_0}^{(2)} = 0$. The $\boldsymbol{E}^{(3)}$ field is zero [6], so in order for the current (4.2.5) to vanish:

$$c{A_0}^{(3)}\boldsymbol{B}^{(1)} = \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(2)}; \quad (4.2.8)$$

a result which is consistent with

$$c\boldsymbol{B}^{(2)} = \boldsymbol{k} \times \boldsymbol{E}^{(2)}, \quad (4.2.9)$$

which is consistent with equation (4.2.6). This procedure loses the advantage developed by Harmuth [1-5] of being able to calculate propagation velocities as described already. Only the unabridged form (2.6.5) retains this important advantage. This is precisely the point being illustrated; $O(3)_b$ electrodynamics has advantages over $U(1)$ electrodynamics and it serves no purpose to map the $O(3)_b$ structure back to the $U(1)$ structure. This mapping loses information in general. The difficulty of course is that those long used to the Faraday Law in its $U(1)$ form will find the $O(3)_b$ form to be at first unfamiliar. It is nevertheless rigorously founded in gauge theory, illustrated in earlier chapters. The most general solutions of equation (2.6.51) and (4.2.5) must be numerically solved.

4.3 STRUCTURE OF THE $O(3)_b$ AND HARMUTH-BARRETT FIELD EQUATIONS

The Harmuth-Barrett field equations were originally developed by Harmuth and shown by Barrett [1] to be Yang Mills equations. The

method developed by both Harmuth and Barrett consists of mapping, or embedding, the original $U(1)$ field equations into an $SU(2)$ symmetry and back again into $U(1)$ symmetry at the end of the calculation. The $O(3)_b$ field equations developed in this book apply in all situations and are regarded as the fundamental field equations of electrodynamics and electrostatics. This is a philosophical step further than Harmuth and more in the spirit of the work of Barrett [1]. The structure of the $O(3)_b$ field equations is given as follows in a vector form. The homogeneous equations include:

$$\nabla \cdot \boldsymbol{B}^{(3)*} = \rho_m \quad (4.3.10)$$

$$\nabla \times \boldsymbol{E}^{(1)*} + \frac{\partial \boldsymbol{B}^{(1)*}}{\partial t} + {\boldsymbol{J}_0}^{(1)*} = 0, \quad (4.3.11)$$

where ρ_m has the units of magnetic charge density ($JsC^{-1}m^{-3}$). Those of magnetic current density are ($JC^{-1}m^{-2}$). There are two other equations with the structure of equation(4.3.11) for indices (2) and (3). By comparison with Barrett's equation (5.3.6) it is seen that the homogeneous $O(3)_b$ and Harmuth-Barrett field equations have the same algebraic form. The magnetic charge density ρ_m is given by:

$$\rho_m = ig\boldsymbol{A}^{([1)} \cdot \boldsymbol{B}^{(2])}, \quad (4.3.12)$$

and the magnetic current density by:

$${\boldsymbol{J}_m}^{(1)*} = ig(c{A_0}^{([3)}\boldsymbol{E}^{(2])} + [\boldsymbol{A}^{(2)}, \boldsymbol{B}^{(3)}]). \quad (4.3.13)$$

In general these quantities are not zero, but they vanish in special cases. For example when $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is a plane wave and $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is defined in the transverse gauge. In this case the homogeneous field equations (4.3.10) and (4.3.11) decouple into equations in indices (1) and (2) which look like the usual Faraday Law of induction, and into $\boldsymbol{B}^{(3)}$ equations as discussed in chapter (2). In electrostatics the magnetic charge and current densities are zero and the $\boldsymbol{B}^{(3)}$ field also vanishes.

More generally however, it is important to note that the non-linear equations (4.3.10) and (4.3.11) are regarded as the equations that

must be solved for the field quantities appearing in them. They control the form of the electromagnetic field in the vacuum, showing that there are three components, in indices ((1), (2) and (3)).

In Harmuth's method, on the other hand, as described by Barrett [1], the philosophy is different. A magnetic current density is added to the Gauss Law phenomenologically, but this is essentially a calculational device, putting the field equations into a transitional $SU(2)$ symmetry. After completion of a calculation the $SU(2)$ symmetry is removed. This is a needless complication if it is accepted that the $O(3)_b$ equations are more self consistent than their $U(1)$ counterparts, as argued in this monograph in many ways. Nevertheless, the work of both Harmuth and Barrett in this context is pioneering, and of great merit. The $O(3)_b$ electrodynamics allows for the existence of ρ_m and $\boldsymbol{J}_m$ under well defined circumstances. These may perhaps include those reviewed by Barrett [1], Tellegren's gyrator, the experiments by Mikhailov on the Ehrenhaft effect. The Heaviside Maxwell, or $U(1)$, electrodynamics does not allow these effects to exist, yet they are observed repeatably and reproducibly [1]. The importance of the work of Harmuth and Barrrett has been to draw attention to their existence, and to attempt to address the problem within $U(1)$ gauge theory. It seems almost certain that electrodynamics, aas with the case of gauge theory in general, is far richer than supposed in the nineteenth century when Heaviside devised the equations known in all the textbooks as the Maxwell equations (1.4.29). The $O(3)_b$ symmetry electrodynamics is just one example of this likelihood using a particular basis ((1), (2), (3)) based on circular polarization as it manifests itself in the third Stokes parameter and the inverse Faraday effect. In other words the internal gauge space in $O(3)_b$ electrodynamics is a an internal space that carries physical consequences, in complex representation ((1), (2), (3)).

In the Harmuth method there is a covering space and a physical

space, and this idea is carried over into Barrett's work [1]. What all three approaches have in common (Harmuth, Barrett and $O(3)_b$) is the existence of products between physical potentials that can contribute observable physics. In the $U(1)$ electrodynamics of Heaviside, the potential does not contribute such physical observables. By using physical content derived from potentials in $O(3)_b$ electrodynamics, topological explanations can be found for phase effects, as discussed in chapter three, and that alone is strong support for an $O(3)_b$ symmetry theory on a fundamental level in classical and also in quantum electrodynamics (see later chapters of this monograph).

For those highly mentored in the Heaviside electrodynamics this is a very difficult step to take, but nevertheless it seems reasonable and fruitful because it leads to more self-consistency and a great deal more information. It brings instanton theory into electrodynamics as just described.

The structure of the inhomogeneous field equations in $O(3)_b$ electrodynamics includes:

$$\nabla \cdot \boldsymbol{D}^{(3)*} = \rho_e + \rho_T, \tag{4.3.14}$$

$$\nabla \times \boldsymbol{H}^{(1)*} = \frac{\partial \boldsymbol{D}^{(1)*}}{\partial t} + \boldsymbol{J}^{(1)*} + \boldsymbol{J}_T{}^{(1)*}, \tag{4.3.15}$$

where ρ_T is an extra electric charge density and $\boldsymbol{J}_T$ an extra electric current density. Again, this structure is regarded as the fundamental one to be solved for all the field quantities appearing in the equations. The two new quantities are defined (chapter 2) by:

$$\rho_T = ig\boldsymbol{A}^{([2)} \cdot \boldsymbol{D}^{(3])} \tag{4.3.16}$$

$$\boldsymbol{J}_T{}^{(1)*} = ig[\boldsymbol{A}^{(2)}, \boldsymbol{H}^{(3)}], \tag{4.3.17}$$

and the equations must be solved in general without approximation. In the special cases developed in chapter (2) they reduce to more

familiar forms. However, they must be in their unabridged form to describe the inverse Faraday effect, as described in section (2.10). If they are artificially reduced to a form resembling Heaviside's electrodynamics they lose the ability to describe this well known magneto-optical effect. By reference to Barrett's equation (5.3.6) [1] it becomes clear that the inhomogeneous $O(3)_b$ equations (4.3.14) and (4.3.15) have the same form as the Harmuth-Barrett equations when these are mapped into this higher symmetry.

On this classical level constitutive relations are needed in $O(3)_b$ electrodynamics as explained in Chapter (2), where Ohm's Law was derived from first principles. Harmuth also introduces a magnetic Ohm's Law in his transitional symmetry. In $O(3)_b$ electrodynamics the electric and magnetic Ohm Laws are:

$$\sigma_E \boldsymbol{E}^{(1)*} = \boldsymbol{J}_E{}^{(1)*} + ig[\boldsymbol{A}^{(2)}, \boldsymbol{H}^{(3)}]$$
$$\sigma_m \boldsymbol{H}^{(1)*} = \boldsymbol{J}_m{}^{(1)*} = ig[\boldsymbol{A}^{(2)}, \boldsymbol{E}^{(3)}] \qquad (4.3.18)$$

where the last term on the right hand side of the first of equation (4.3.18) we identify as $\boldsymbol{J}_T{}^{(1)*}$. These are direct consequences of the existence of the non-linear currents.

4.4 LINK BETWEEN THE $O(3)_b$ EQUATIONS AND THE SINE GORDON EQUATION

The procedure developed by Harmuth [2-5] and reviewed and extended by Barrett [1] can be used to reduce the $O(3)_b$ equations to a set of sine Gordon equations. This is an important link because the sine Gordon equation has soliton solutions and topological currents which do not exist in $U(1)$. We start with the vacuum $O(3)_b$ equations in their general vector form [1-5]:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{J}_m = 0, \qquad (4.4.19)$$

$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial} - \boldsymbol{J} - \boldsymbol{J}_T = 0. \qquad (4.4.20)$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}, \ \boldsymbol{D} = \epsilon_0 \boldsymbol{E}, \ \boldsymbol{J}_m = \sigma_m \boldsymbol{H}, \ \boldsymbol{J} + \boldsymbol{J}_T = \sigma_m \boldsymbol{E}, \qquad (4.4.21)$$

and generalize to an $O(3)_b$ field component such as:

$$\nabla \times \boldsymbol{H}^{(1)} - \epsilon_0 \frac{\partial \boldsymbol{E}^{(1)}}{\partial t} - \sigma_E \boldsymbol{E}^{(1)} = 0 \qquad (4.4.22)$$

$$\nabla \times \boldsymbol{E}^{(1)} + \epsilon_0 \frac{\partial \boldsymbol{H}^{(1)}}{\partial t} + \sigma_E \boldsymbol{H}^{(1)} = 0 \qquad (4.4.23)$$

These equations can be put into the form of a sine Gordon equation by using components of the curl of the fields for a photon propagating in the z direction [1-5]:

$$(\nabla \times \boldsymbol{H}^{(1)})_x = -\frac{\partial \boldsymbol{H}_y{}^{(1)}}{\partial z}, \ (\nabla \times \boldsymbol{E}^{(1)})_x = \frac{\partial \boldsymbol{E}_y{}^{(1)}}{\partial z}. \qquad (4.4.24)$$

The difference in sign between the two vector curls in equation (4.4.25) is needed to obtain the correct d'Alembert equation from a limit of the sine Gordon equation [1-5]. Using equation (4.4.24) in equations (4.4.22) and (4.4.23) we arrive at:

$$\frac{\partial H_y{}^{(1)}}{\partial z} + \epsilon_0 \frac{\partial E_y{}^{(1)}}{\partial t} + \sigma_E E_y{}^{(1)} = 0, \qquad (4.4.25)$$

$$\frac{\partial E_y{}^{(1)}}{\partial z} + \mu_0 \frac{\partial H_y{}^{(1)}}{\partial t} + \sigma_m H_y{}^{(1)} = 0, \qquad (4.4.26)$$

a result attributed to Harmuth [2-5]. Now differentiate (4.4.25) with respect to Z:

$$\frac{\partial^2 H_y{}^{(1)}}{\partial z^2} + \epsilon_0 \frac{\partial}{\partial z}\left(\frac{\partial E_y{}^{(1)}}{\partial t}\right) + \sigma_E \frac{\partial E_y{}^{(1)}}{\partial z} = 0. \qquad (4.4.27)$$

Similarly now differentiate (4.4.26) with respect to t:

$$\mu_0 \frac{\partial^2 H_y{}^{(1)}}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{\partial E_y{}^{(1)}}{\partial z}\right) + \sigma_m \frac{\partial H_y{}^{(1)}}{\partial t} = 0. \qquad (4.4.28)$$

From equation (4.4.26) in equation (4.4.27):

$$\frac{\partial^2 H_y{}^{(1)}}{\partial z^2} + \epsilon \frac{\partial}{\partial z}\left(\frac{\partial E_y{}^{(1)}}{\partial t}\right) - \sigma_E\left(\mu_0 \frac{\partial H_y{}^{(1)}}{\partial t} + \sigma_m H_y{}^{(1)}\right) = 0. \qquad (4.4.29)$$

Now assume a solution such that:

$$\frac{\partial}{\partial t}\Big(\frac{\partial {E_y}^{(1)}}{\partial z}\Big) = \frac{\partial}{\partial z}\Big(\frac{\partial {E_y}^{(1)}}{\partial t}\Big) \tag{4.4.30}$$

then from equation (4.4.28) in equation (4.4.29):

$$\frac{\partial^2 {H_y}^{(1)}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 {H_y}^{(1)}}{\partial t^2} = (\epsilon_0\sigma_m + \mu_0\sigma_E)\frac{\partial {H_y}^{(1)}}{\partial t} + \sigma_E\sigma_m {H_y}^{(1)}. \tag{4.4.31}$$

If $\sigma_m = \sigma_E = 0$ we recover the d'Alembert wave equation in vacuo and a plane wave for the component ${H_y}^{(1)}$. This result is consistent with the fact that σ_m and σ_E vanish for plane waves [6], and if a transverse gauge is assumed for the potential $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$.

As shown by Harmuth [1-5] equation (4.4.31) is a sine Gordon equation. This is seen if we write $\sigma_m = \sigma_m({H^{(1)}}_y)$ and $\sigma_E = \sigma({H^{(1)}}_y)$ as a power series in the field ${H^{(1)}}_y$ then the right hand side is a power series, which can be a sum of $cos(\beta {H_y}^{(1)})$ and $sin(\beta {H_y}^{(1)})$. A choice of the sine solution leads to a Sine-Gordon equation

$$\frac{\partial^2 {H_y}^{(1)}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 {H_y}^{(1)}}{\partial t^2} = \alpha sin(\beta {H_y}^{(1)}), \tag{4.4.32}$$

which is conveniently normalized using $H_N = {H_y}^{(1)}/H^{(0)}$ to obtain:

$$\frac{\partial^2 H_N}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 H_N}{\partial t^2} = \alpha sin(\beta H_N), \tag{4.4.33}$$

This equation is one of a set of sine Gordon equations for different components of the $O(3)_b$ field. The reduction has been illustrated for one component ${H_y}^{(1)}$.

The properties of the sine Gordon equation are well known [1-5,7] and in general if electrodynamics is described by such an equation

it becomes a much richer subject. Since $\boldsymbol{B}^{(3)}$ is a solution of the $O(3)_b$ field equations it is also associated with an infinite number of conservation laws, or symmetries, of the sine Gordon equation. Soliton wave equations have an infinite number of symmetries, and the topological index associated with the $\boldsymbol{B}^{(3)}$ field is then associated with these symmetries. Apart from the solutions of the latter discussed by Harmuth [2-5] and Barrett [1] there is an infinite number of constant solutions of the form:

$$H_N = \frac{2\pi n}{\beta};\ n = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{4.4.34}$$

as discussed by Ryder [7]. If we accept the hypothesis of $O(3)_b$ electrodynamics these solutions are electrodynamic solutions which do not exist in the Heaviside Maxwell model. If the vacuum is defined as a classical field configuration of zero energy the solutions (4.4.34) show that the vacuum is degenerate [7]. The sine Gordon equation also gives solitons, which is a solitary wave in electrodynamics which moves along the propagation axis without changing size or shape, and therefore without dissipation [7]. The solitons appear if and only if the parameters σ_E and σ_m are non-zero.

Most real world physics involves approximations of exact theories. With this in mind we now Expand the function $y = sin(\beta H_N)$ with a two variable Taylor series about the point $z_0 = (x_0,\ t_0)$, we find that:

$$H_N(z,\ t) = \frac{1}{\beta}sin^{-1}y$$

$$\simeq H_N(z_0,\ t_0) + (z - z_0)\frac{\partial H_N}{\partial z}(z_0) + \frac{1}{2!}(z - z_0)^2\frac{\partial^2 H_N}{\partial z}(z_0) + \ldots, \tag{4.4.35}$$

or conversely,

$$y \simeq \sum_{z=x}^{t}\Big(\beta(z - z_0)\frac{\partial H_N}{\partial z}(z_0) - \frac{1}{3!}\beta^3(z - z_0)^3\frac{\partial^3 H_N}{\partial z^3}(z_0) + \ldots\Big), \tag{4.4.36}$$

where the sum includes the z and t components. Therefore, when $z = z_0 + \delta z$:

$$y = \alpha\beta(z - z_0)\partial_z H_y, \tag{4.4.37}$$

and with $\partial_z H_y \simeq H_y/(z - z_0)$ we see that $\alpha\beta \simeq \sigma_E\sigma_m$. This result implies that $\alpha\beta$ in the sine-Gordon model is proportional to a mass term as discussed by Ryder [7].

Start with the sine Gordon equation for the scalar component of the electromagnetic field in the form given by Ryder's eqn. (10.1):

$$\frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \alpha sin(\beta\phi), \tag{4.4.38}$$

in naturalized units with $c = 1$. The soliton solution is of the form

$$\phi(x.\, t) = \frac{4}{b}tan^{-1}(e^{\pm\gamma\xi}), \tag{4.4.39}$$

which is a solitary wave which moves without changing shape or size. The Lagrangian in Ryder's reduced units is

$$\mathcal{L} = \frac{1}{2}\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right) - V, \tag{4.4.40}$$

with a potential energy chosen so that the constant solutions

$$\phi = \frac{2\phi n}{b},\ n = 0,\ \pm 1,\ \pm 2,\ \ldots, \tag{4.4.41}$$

have $V = 0$. The potential energy may be written as:

$$V(\phi) = \frac{a}{b}(1 - cos(b\phi)), \tag{4.4.42}$$

where there is a particle mass $m = ab$ and a self-interaction coupling parameter. Therefore it follows that in $O(3)_b$ electrodynamics there is a "particle mass" proportional to $\sqrt{\sigma_E\sigma_m}$. It follows from the soliton solution of the sine Gordon equation that there is a conserved topological charge and current. Examples of topological charges and currents in $O(3)_b$ electrodynamics are given in the basic equations (2.6.49) and (2.6.51). Since each sine Gordon equation of the form (4.4.33) is also an equation of $O(3)_b$ electrodynamics the properties deduced from the former are also properties of the latter. The stability of the soliton solutions is a consequence of topology in each case because the equations are non-linear.

The $B^{(3)}$ solutions of the $O(3)_b$ equations can be considered to be a vortex line [8], and $O(3)_b$ gauge theory does indeed support one non-trivial vortex. The $O(3)_b$ field equations also give a non-trivial topological magnetic monopole as given on the right hand side of equation (2.6.49). This is clearly not a point magnetic monopole.

4.5 INSTANTONS

Instantons are minimum action solutions of the self-dual Yang Mills equations [1,7]. In order to explain these abstractions in terms of electrodynamics this section is initiated with a discussion of duality in $U(1)$ and $O(3)_b$ electrodynamics. We start with the familiar field tensor of $U(1)$ electrodynamics. The dual tensor is [7]:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho} \tag{4.5.43}$$

where

$$\epsilon_{\mu\nu\sigma\rho} = -\epsilon_{\mu\nu\rho\sigma} \tag{4.5.44}$$

Therefore, in contravariant-covariant notation with naturalized units [7]:

$$F^{ij} = \epsilon^{ijk}B^k,\ F^{0i} = \epsilon^{0ij}E^j;$$
$$\tilde{F}^{ij} = \epsilon^{ijk}E^k,\ \tilde{F}^{0i} = \epsilon^{0ij}B^j. \tag{4.5.45}$$

There is a field duality in $U(1)$ symmetry which can be expressed as [9-10]:

$$\boldsymbol{E} \rightarrow \boldsymbol{B},\ -\boldsymbol{B} \rightarrow \boldsymbol{E}. \tag{4.5.46}$$

Instanton theory, based upon field dualism, considered in Euclidean spacetime results in computational simplicity. Here the coordinate system is:

$$x_{\mu\ eucl} = (x_1,\ x_2,\ x_3,\ x_4). \tag{4.5.47}$$

The dual tensor in euclidean spacetime is:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F_{\sigma\rho}, \tag{4.5.48}$$

with

$$\epsilon_{1234} = 1. \tag{4.5.49}$$

The first step towards the construction of an instanton theory for electrodynamics is therefor to construct the field tensor and its dual, and it is helpful to illustrate this procedure firstly by reference to U(1) electrodynamics. The coordinates are such that:

$$A_\mu = (A_1,\ A_2,\ A_3,\ A_4);\ \partial_\mu = (\partial_1,\ \partial_2,\ \partial_3,\ \partial_4), \tag{4.5.50}$$

and in the abelian case:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.5.51}$$

For convenience of illustration denote:

$$A_\mu = (A_0,\ A_1,\ A_2,\ A_3);\ \partial_\mu = (\partial_0,\ \partial_1, \partial_2, \partial_3), \tag{4.5.52}$$

so that the euclidean field tensor is:

$$F^{ij} = \epsilon^{ijk}B^k,\ F^{0i} = \epsilon^{0ij}E^j \tag{4.5.53}$$

The euclidean dual tensor is:

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F_{\sigma\rho}, \tag{4.5.54}$$

and using the rule:

$$\epsilon_{0123} = 1,\ \epsilon_{0132} = -1, \tag{4.5.55}$$

and further permutation symmetry, the dual of the Euclidean field tensor is the dual of the Euclidean field tensor is:

$$\tilde{F}^{ij} = \epsilon^{ijk}E^k,\ F^{0i} = \epsilon^{0ij}B^j. \tag{4.5.56}$$

So the Euclidean field duality is:

$$\boldsymbol{E} \rightarrow \boldsymbol{B},\ \boldsymbol{B} \rightarrow \boldsymbol{E}. \tag{4.5.57}$$

In Euclidean spacetime the electric-magnetic field duality amounts to an equality.

These considerations can be extended to $O(3)_b$ electrodynamics by defining the field tensor as:

$$F^{\mu\nu} = \sum_{a=1}^{3} F^{a\mu\nu}e^a \tag{4.5.58}$$

so the duality in Minkowski spacetime for $O(3)_b$ electrodynamics is:

$$\boldsymbol{E}^a \rightarrow \boldsymbol{B}^a,\ \boldsymbol{B}^a \rightarrow -\boldsymbol{E}^a. \tag{4.5.59}$$

The nonzero field tensors for the (3) components are the antidiagonal elements:

$$F^{3\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E^{(3)}{}_z \\ 0 & 0 & -B^{(3)}{}_z & 0 \\ 0 & B^{(3)}{}_z & 0 & 0 \\ E^{(3)}{}_z & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{F}^{3\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -B^{(3)}{}_z \\ 0 & 0 & E^{(3)}{}_z & 0 \\ 0 & -E^{(3)}{}_z & 0 & 0 \\ B^{(3)}{}_z & 0 & 0 & 0 \end{pmatrix}, \tag{4.5.60}$$

The field components appearing in these tensors are defined in $O(3)_b$ gauge field theory by

$$G^{(3)\mu\nu} = \partial^\nu A^{(3)\mu *} - \partial^\mu A^{(3)\nu *} - ig[A^{(1)\mu},\ A^{(2)\nu}], \tag{4.5.61}$$

and if we assume a particular solution [12]:

$$A^{(1,2)\mu} = (0,\ \boldsymbol{A}^{(1,2)}),\ A^{(3)\mu} = (cA^{(0)},\ \boldsymbol{A}^{(3)}), \tag{4.5.62}$$

it follows that:

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)},\ \boldsymbol{E}^{(3)*} = 0. \tag{4.5.63}$$

Therefore, the magnetic component is real and physical and the electric component has no real or imaginary part. This result is consistent with the fact that there is an observable empirical magnetization due to the $\boldsymbol{B}^{(3)}$ component, but apparently no known effect of $\boldsymbol{E}^{(3)}$. However, a zero $\boldsymbol{E}^{(3)}$ is the result of choosing the vector potentials according to equation (4.5.57). This choice of vector potentials leads to a duality of the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B^{(3)}{}_z & 0 \\ 0 & B^{(3)}{}_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & -B^{(3)}{}_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B^{(3)}{}_z & 0 & 0 & 0 \end{pmatrix}, \tag{4.5.64}$$

which can be summarized by [9-12]:

$$\nabla \times \boldsymbol{B}^{(3)} + \frac{1}{c}\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = 0. \tag{4.5.65}$$

This equation is unchanged if

$$\boldsymbol{B}^{(3)} \rightarrow \boldsymbol{B}^{(3)} \tag{4.5.66}$$

and so $\boldsymbol{B}^{(3)}$ is self-dual. There is never any "Faraday induction" due to $\boldsymbol{B}^{(3)}$.

However, the use of a rigorously zero $\boldsymbol{E}^{(3)}$ is not compatible with the existence of instantons in electrodynamics, for instantons to exist requires a pure imaginary $\boldsymbol{E}^{(3)}$ [9-12]. To prove this result the $O(3)_b$ field tensors must be set up in the required Euclidean spacetime:

$$[F^{(a)}{}_{\mu\nu}] = \begin{pmatrix} 0 & -E_1^a & -E_2^a & -E_3^a \\ E_1^a & 0 & B_3^a/c & -B_2^a/c \\ E_2^a & -B_3^a/c & 0 & B_1^a/c \\ E_3^a & B_2^a/c & -B_1^a/c & 0 \end{pmatrix} \tag{4.5.67a}$$

$$[\tilde{F}^{(a)}_{\mu\nu}] = \begin{pmatrix} 0 & -B_1^a & -B_2^a & -B_3^a \\ B_1^a & 0 & E_3^a/c & -E_2^a/c \\ B_2^a & -E_3^a/c & 0 & E_1^a/c \\ B_3^a & E_2^a/c & -E_1^a/c & 0 \end{pmatrix} \tag{4.4.67b}$$

In order to obtain instantons [7] we must use the self dual condition:

$$F^{(a)}{}_{\mu\nu eucl} \rightarrow \tilde{F}^{(a)}_{\mu\nu eucl} \tag{4.5.68}$$

which means that in Euclidean spacetime:

$$\begin{aligned} E^{(a)}{}_1 &\rightarrow B^{(a)}{}_1 \\ E^{(a)}{}_2 &\rightarrow B^{(a)}{}_2 \\ E^{(a)}{}_3 &\rightarrow B^{(a)}{}_3, \end{aligned} \tag{4.5.69}$$

for $a = 1$, 2,and 3. Translating this result into Minkowski spacetime produces:

$$\begin{gathered} iE^{(1)}{}_x \rightarrow B^{(1)}{}_x :\ iE^{(1)}{}_y \rightarrow B^{(1)}{}_y \\ -iE^{(2)}{}_x \rightarrow B^{(2)}{}_x,\ -iE^{(2)}{}_y \rightarrow B^{(a)}{}_y \\ iE^{(3)}{}_z \rightarrow B^{(3)}{}_z, \end{gathered} \tag{4.5.70}$$

which is consistent with transverse plane wave for indices 1 and 2:

$$\begin{aligned} \boldsymbol{E}^{(1)} &= \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}; \\ \boldsymbol{B}^{(1)} &= \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{i\phi}, \end{aligned} \tag{4.5.71}$$

and for index 3:

$$B^{(3)}{}_z = B^{(0)};\ E^{(3)}{}_z = -iE^{(0)}. \tag{4.5.72}$$

Therefore, the real part of $\boldsymbol{E}^{(3)}$ is zero, and the imaginary part of $\boldsymbol{B}^{(3)}$ is zero. The observation of electrodynamic instantons will therefore be an indication of the existence of a real $\boldsymbol{B}^{(3)}$ and an imaginary $\boldsymbol{E}^{(3)}$:

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)} = B^{(0)},\ \boldsymbol{E}^{(3)} = -iE^{(0)}\boldsymbol{e}^{(3)} = -iE^{(0)}. \tag{4.5.73}$$

It would be wrong to conclude that because $\boldsymbol{E}^{(3)}$ is imaginary it is unphysical, because it is needed for the instanton, a physical pseudoparticle obtained from the self dual Yang Mills equations. Therefore, the potentials in equation (4.4.42) must be chosen so that they give a pure imaginary $\boldsymbol{E}^{(3)}$, rather than a rigorously zero $\boldsymbol{E}^{(3)}$. This ad hoc method might exist in much more rigorous form. If however we adopt the usual rule in electrodynamics that a pure imaginary field component is in itself unphysical, the self-duality of $\boldsymbol{B}^{(3)}$ still follows because equation (4.5.45) is still correct. These concepts arise from the $O(3)_b$ symmetry adopted for electrodynamics and concomitant Yang Mills equations.

4.6 HIGHER ORDER SOLITON EQUATONS

The following is an illustration of how the $B^{(3)}$ field results in the existence of pseudoparticles that are magnetic monopoles. Physically these particles are similar to electron holes in solids and mathematically are instantons.

We start with an $SU(2)$, $SO(3)$, or the broken $SO(3)$ as $O(3)_b$ field theory in R^4, where the metric is regarded as Euclidean. Later a discussion of this solution in Minkowski space with the metric signature $[1, -1, -1, -1]$ will be given. The field strength tensor $F^a{}_{\mu\nu}$ carries the Euclideanized spacetime indices μ, ν that can be decomposed into the temporal parts and the spacial parts 0, $i \in \{1, 2, 3\}$. In addition there is the Lie algebraic index $a \in \{1, 2, 3\}$. These indices give vector components in the internal 3-sphere isomorphic to the Lie algebra $SU(2)$. The field tensors $F^a{}_{\mu\nu}$ are then nonabelian and obey Yang-Mills theory. This definition of the field tensor is composed of terms that are purely spacial and those that mix temporal and spacial parts:

$$F^a{}_{ij} = \partial_j A^a{}_i - \partial_i A^a{}_j + ig\epsilon^{abc}[A^b{}_i, A^c{}_j]$$
$$F^a{}_{0i} = \partial_i A^a{}_0 - \partial_0 A^a{}_i + ig\epsilon^{abc}[A^b{}_0, A^c{}_i]. \qquad (4.6.74)$$

We consider the source free Yang-Mills equations

$$\partial^\nu F^a{}_{\mu\nu} + \epsilon^{abc}[A^{b\nu}, F^c{}_{\mu\nu}] = 0 \qquad (4.6.75)$$

in its spacial and temporal decomposition

$$\partial^j F^a{}_{ij} + \epsilon^{abc}([A^{bj}, F^c{}_{ij}] + [A^{b0}, F^c{}_{0i}]) = 0, \qquad (4.6.76)$$

and

$$\partial^i F^a{}_{0i} + \epsilon^{abc}[A^{bi}, F^c{}_{0i}] = 0. \qquad (4.6.77)$$

These field strength tensors define a topological charge by the action

$$n = \frac{1}{8\pi^2}\int d^4x F^{a\mu\nu}F^a{}_{\mu\nu}. \qquad (4.6.78)$$

The field strength tensor contains self-dual and antiself-dual components. The dual field strength tensor is $*F^a{}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{a\sigma\rho}$. The self-dual and antiself-dual tensors may then be written as $G^{\pm a}{}_{\mu\nu} = F^a{}_{\mu\nu} \pm *F^a{}_{\mu\nu}$ and the action is

$$n = \frac{1}{8\pi^2}\int d^4x G^{\pm a\mu\nu}G^{\pm a}{}_{\mu\nu} + \frac{1}{4\pi^2}\int d^4x G^{+a\mu\nu}G^{-a}{}_{\mu\nu}, \qquad (4.6.79)$$

where the bound is obtained for self duality.

Now we consider the stress energy to be smooth and vanishing at infinity. This means that the gauge potential at infinity is a pure gauge transformation, where if g is a map between different charts $g_{\alpha\beta} : \mathcal{U}_\alpha \to \mathcal{U}_\beta$ then the gauge potential is

$$A^a{}_\mu = i\epsilon^{abc}g^b\partial_\mu g^{c-1}. \qquad (4.6.80)$$

This gauge potential then defines a map between the three sphere and the group $SU(2)$. This mapping between the three sphere and $SU(2)$ can be shown to be given by,

$$m(x) = \frac{\boldsymbol{I}^4 x_4 + i\boldsymbol{\sigma}\cdot\boldsymbol{x}}{r},\ r^2 = x_4^2 + x^2, \qquad (4.6.81)$$

which means that the gauge potentials are of the form

$$A^a{}_\mu = ia(r)\epsilon^{abc}m^b(x)\partial_\mu m^{c-1}(x). \quad (4.6.82)$$

This form of the potential is similar to the form presented by Uhlenbeck and further gives a correspondence between the Stokes indices and the internal group on the base space. By using the Yang-Mills equations this gauge potential can be shown to have the explicit form

$$A_\mu = i\frac{r^2}{r^2 + r_0^2}m(x)\partial_\mu m^1(x). \quad (4.6.83)$$

where the components of the gauge potential on the spacial subsurface in R^4 determine the nature of the Lie indices according to our map. The form of these gauge potentials is then suggests that the logarithm of a function defines them in the manner of a Bogomolny theory [13]

$$A^a{}_\mu = \partial_\mu log\phi^a. \quad (4.6.84)$$

By using the Pauli matrices explicitly we find that the gauge potentials are then

$$\begin{aligned} A_1 &= (\phi_4, \phi_3, -\phi_2)\phi^{-1}, \\ A_2 &= (-\phi_3, \phi_4, \phi_1)\phi^{-1}, \\ A_3 &= (\phi_2, -\phi_1, \phi_4)\phi^{-1}, \\ A_4 &= -(\phi_1, \phi_2, -\phi_3)\phi^{-1}. \end{aligned} \quad (4.6.85)$$

A substitution into the field equations illustrates that these give a solution only if they satisfy the Klein-Gordon equation

$$\partial^\mu\partial_\mu\phi = 0. \quad (4.6.86)$$

A solution that only depends upon the 4 distance reduces this equation to the form

$$\frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr} = 0, \quad (4.6.87)$$

which has a solution that is a charge 1 instanton. An examination of the equations for the vector potentials reveal that $O(3)_b$ electromagnetic theory obtains when $\phi_1 = \phi_2 = 0$ in this self dual case.

This is since in the case of the $O(3)_b$ electrodynamics $A^3{}_\mu = 0$ on the physical vacuum.

Now we take the field equations for $O(3)_b$ where we transform to polar coordinates that mix A_1 with A_2 so that

$$A_1 = A_r cos\theta + r^{-1}A_\phi sin\theta, \; A_2 = A_r sin\theta - r^{-1}A_\phi cos\theta. \quad (4.6.88)$$

The gauge fields then assume a form in cylindrical coordinates

$$\begin{aligned} A_\phi &= (0, \xi_1, \xi_2), \\ A_z &= (\zeta_1, 0, 0), \\ A_r &= (\zeta_2, 0, 0), \\ \Phi &= (0, \Phi_1, \Phi_2). \end{aligned} \quad (4.6.89)$$

With an analysis with the $O(3)_b$ field equations we find that if define the variables ψ and f that we have the rather simple looking equations

$$\begin{aligned} \Phi_1 &= f^{-1}\partial_z\psi = -\zeta_1, \\ \Phi_2 &= -r^{-1}f^{-1}\partial_z\psi, \\ \xi_1 &= -rf^{-1}\partial_r\psi = r\zeta_2, \\ \xi_2 &= rf^{-1}\partial_r f, \end{aligned} \quad (4.6.90)$$

with some further calculations these equations can be reduced to the differential equation

$$E(\partial_r^2 E + r^{-1}\partial_r E + \partial_z{}^2 E) - ((\partial_z E)^2 + (\partial_r E)^2) = 0, \quad (4.6.91)$$

where $E = f + i\psi$. This differential equation is the Ernst equation for an axial-symmetric spacetime. This differential equation is has solutions that are vortex and antivortex solutions for solitons. This equation is also associated with problems for axially symmetric spacetimes in general relativity. This differential equation is solved through prolongation techniques and Bäcklund transformations [14].

As this book is not a mathematical methods text, these techniques are not discussed.

If the potential function ϕ describes a potential of the form $V(|\phi|) = a(|\phi|^2 - 1)$ then this leads to a broken $SU(2)$ type of physical model. This is in chapter 8 in connection to the unification of electromagnetism and weak interactions.

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CHAPTER 5
PHYSICAL PHASE EFFECTS IN $O(3)_b$ ELECTRODYNAMICS

5.1 PHASE EFFECTS

In chapter three topological phase effects were developed in terms of $O(3)_b$ electrodynamics. In this chapter the theory is extended to physical phase effects, namely the radiatively induced Aharonov-Bohm; Josephson and quantum Hall effects. The key to the existence of all these effects is the radiated object $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ through which the $\boldsymbol{B}^{(3)}$ field is defined. This object, known in nonlinear optics [1-3] as the "conjugate product", originates in foundational gauge theory. It is the from the covariant derivative of the potential. This product is demonstrated empirically by the existence of circular polarization; magneto-optical effects; topological phase effects; and the optically induced physical phase effects such as that of Aharanov and Bohm, first observed with a static magnetic field. Therefore the fundamental nature in gauge theory of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is key to the development of the whole of $O(3)_b$ electrodynamics. There are two other fundamental issues raised by this realization, the first is under what circumstances is the potential in electrodynamics a physical object, having a measurable effect; and the second is the reduction of $O(3)_b$ electrodynamics to electrostatics, whose laws are well verified empirically, and where there are no radiative effects by definition, so $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ disappears. These two questions must be answered satisfactorily before we can discuss the optically induced phase effects, which are the topic of this chapter.

In the received view of non-linear optics [1-3] the object $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is introduced empirically, as first suggested by Pershan [4] in his de-

scription of the inverse Faraday effect. The introduction is made within the confines of abelian gauge theory, whose gauge symmetry is considered to be $U(1)$, homomorphic with $SO(2)$. This abelian and linear gauge theory is the linear Heaviside-Maxwell theory of electrodynamics as we have seen in earlier chapters. The non-linear conjugate product $A \times A^*$ is considered within the current $U(1)$ theory to be an operator, but an operator which is not proportional to a field component. Without the phenomenological introduction of $A^{(1)} \times A^{(2)}$, the Heaviside-Maxwell theory cannot describe the inverse Faraday effect. The third Stokes parameter (chapter one), which describes circular polarization, and predates the original Maxwell theory, is also described phenomenologically because by definition, $A^{(1)} \times A^{(2)}$ is zero in any abelian gauge theory [5]. In $O(3)_b$ gauge theory on the other hand $A^{(1)} \times A^{(2)}$ is an intrinsic part of the field tensor and defines the physical and gauge invariant field component $B^{(3)}$ an observable quantity. Therefore, in $O(3)_b$ electrodynamics $B^{(3)}$ and $A^{(1)} \times A^{(2)}$ become parts of gauge theory at a fundamental level. In $U(1)$ electrodynamics $A^{(1)} \times A^{(2)}$ is phenomenological and $B^{(3)}$ is undefined.

The fundamentally important object in $O(3)_b$ electrodynamics is therefore $A^{(1)} \times A^{(2)}$, which arises from a covariant change in potential by transport around a closed loop in Minkowski space-time (chapter three). The observable $A^{(1)} \times A^{(2)}$ signals the existence of an $O(3)_b$ symmetry covariant derivative in this closed loop, or round trip [5] by parallel transport. The use of a $U(1)$ covariant derivative would result in $A^{(1)} \times A^{(2)} = 0$ by definition, contradicting empirical data. There must be recourse either to a different gauge symmetry such as $O(3)_b$ or to phenomenology. The introduction of $A^{(1)} \times A^{(2)}$, however, without the introduction of $B^{(3)}$ is a clear violation of gauge theory. The origin of $A^{(1)} \times A^{(2)}$ is to be found therefore in a holonomy of $O(3)_b$ symmetry gauge theory, a covariant derivative, or a type of difference, in potentials at different points in spacetime. We shall argue in this chapter that under an $O(3)_b$ gauge transformation $A^{(1)} \times A^{(2)}$ induces an optical Aharonov-Bohm effect

[6-10]. It is seen in chapter three that this type of gauge transformation suggests a topological explanation for the extra phase shift seen in the Sagnac effect when the platform is rotated. So the Sagnac effect was shown to be an optical Aharonov-Bohm effect. Changes in potential, through the object $A^{(1)} \times A^{(2)}$, are responsible for the Sagnac effect. If this is accepted, $O(3)_b$ electrodynamics will have been shown to be able to explain an effect whose origin in $U(1)$ is a mystery [11]. The fact that it is differences in potential that are physically significant is of key importance, because of the analogy with the well known tenet on dynamics that difference in potential energy is the only measurable.

Ryder [5] emphasizes the physical effect of the potential in electrodynamics in a slightly different way by defining it as the difference in value of a field vector after it has been transported around a closed loop with covariant derivatives. The commutator of covariant derivatives defines the field. This again means that it is the $O(3)_b$ symmetry covariant derivative of a potential that gives rise to $A^{(1)} \times A^{(2)}$. The latter is non-zero if and only if the gauge symmetry is non Abelian, for example $O(3)_b$. The major oversight in contemporary gauge theory is the incorrect assertion within gauge theory that $A^{(1)} \times A^{(2)}$ is zero in the electromagnetic sector. This is why gauge theory mistakenly attributes a $U(1)$ or $SO(2)$ symmetry to the electromagnetic sector. In consequence a lot of information is lost despite the fact that a $U(1)$ sector is regarded almost as axiomatic. In later chapters of this book we will begin the task of developing unified field theory with a non-Abelian electromagnetic sector, of symmetry $SU(2)$ homomorphic with $SO(3)$. This will be seen to have immediate observable consequences in the first subject area addressed, electro-weak theory of symmetry $SU(2) \times SU(2)$ rather than the standard symmetry $U(1) \times SU(2)$.

In $O(3)_b$ electrodynamics $A^{(1)} \times A^{(2)}$ is gauge covariant, and is a

physical observable. The reason is with the fundamental nature of the vacuum itself. The object $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ transforms homogeneously in general as can be seen once the differential is made $O(3)_b$ covariant. Since $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is by definition proportional to a field component that transforms homogeneously, the $\boldsymbol{B}^{(3)}$ component, an additional constraint equation appears, one which allows the existence of the optical Aharonov-Bohm effect. These properties of $O(3)_b$ electrodynamics are developed in Section 5.1 and depend on the fact that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is a physical object. The $O(3)_b$ gauge transform rules give rise to the infinitely degenerate Yang Mills vacuum, a highly structured vacuum, and to the existence of instantons in electrodynamics, as argued in chapter four. The vacuum in contemporary gauge theory is therefore a structured entity that depends on the symmetry adopted for the gauge theory. The Aharonov-Bohm effect is supported [5] by $U(1)$ or $O(3)_b$ gauge field symmetry, but not by $SU(2)$. The reason is to be found in topology.

Since $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is well defined within gauge theory of $O(3)_b$ symmetry it is an object that is well defined within standard special relativity. Therefore $\boldsymbol{B}^{(3)}$ is also so defined, and propagates at c in the vacuum along with $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$. Within this framework, $vec B^{(3)}$ is an object not well defined in instantaneous action at a distance theory. If the photon is massless for the sake of argument, $\boldsymbol{B}^{(3)}$ propagates at c with $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and with the third Stokes parameter that defines circular polarization. Since other polarizations of light can be built from circular polarization $\boldsymbol{B}^{(3)}$ is a foundational field component that exists in a one photon beam [6-10]. This is true if and only if the covariant derivative in electrodynamics is an $O(3)_b$ symmetry covariant derivative, originating in this view in the existence of circular polarization itself. The latter is used to define [6-10] the $((1),(2),(3))$ basis.

It should be realized, however, that in contemporary gauge field theory [5] the covariant derivative plays a role for all gauge symmetries, including $U(1)$ [10]. The general gauge field theory for any gauge group defines the field through the commutator of covariant derivatives, giving [10]:

$$G_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] \tag{5.1.1}$$

where the commutator is non-zero in general. The connection, or potential, A_μ is defined in general through the gauge group symmetry. The field tensor is covariant within some gauge group, and so the general gauge theory is compatible with special relativity for all gauge group symmetries. The field tensor $G_\mu\nu$ is also covariant with respect to the Poincare symmetries of special relativity. In the general theory [5], the homogeneous field equations are:

$$\mathcal{D}_\mu \tilde{G}^{\mu\nu} = 0 \tag{5.1.2}$$

for all gauge group symmetries, including $U(1)$, with their particular structure constants. The covariant derivatives are defined as:

$$\mathcal{D}_\mu = \partial_\mu + i\frac{e}{\hbar}A_\mu. \tag{5.1.3}$$

Therefore the homogeneous field equation in general is:

$$(\partial_\mu + i\frac{e}{\hbar}A_\mu)\tilde{G}^{\mu\nu} = 0, \tag{5.1.4}$$

and reduces to the corresponding $U(1)$ Heaviside-Maxwell field equation if and only if we adopt the particular solutions:

$$\partial_\mu \tilde{G}^{\mu\nu} = 0 \tag{5.1.5a}$$

$$A_\mu \tilde{G}^{\mu\nu} = 0. \tag{5.1.5b}$$

In vector notation equation (5.1.4b) implies the Abelian relations:

$$\boldsymbol{A} \cdot \boldsymbol{B} = 0 \tag{5.1.6a}$$

$$\boldsymbol{A} \times \boldsymbol{E} = 0 \tag{5.1.6b}$$

which are compatible [10] with transverse plane wave solutions in the vacuum. Therefore the same interpretation must pertain to the

vector potential in all gauge field symmetries, differences between potentials are observable, and these differences are always generated by covariant derivatives. Otherwise the gauge theory is not a local gauge field theory [5] and so is not a theory of special relativity. This is a more precise way of discussing the role of the potential in classical field theory than simply abandoning it as a mathematical artifice. In the textbooks, $U(1)$ gauge transformation is developed by adding a gradient function to the original vector potential, and then taking the curl to give a magnetic field. It is concluded that since the curl of a gradient function is always zero, the gradient function itself can be arbitrary. More precisely, in $U(1)$ gauge theory the gradient function is the four derivative of an angle which depends on x^μ through special relativity. This four-derivative is responsible for the Aharonov-Bohm effect due to a static magnetic field, and so it its difference around a path is not an arbitrary function. It determines a physical observable in regions where the original static magnetic field is absent [5].

The reduction of $O(3)_b$ electrodynamics to electrostatics must therefore be considered carefully before proceeding to the optically induced equivalent effect due to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The laws of electrostatics and magnetostatics, and the original Aharonov-Bohm effect are well verified empirically within the context of a $U(1)$ symmetry gauge field theory which must therefore be a well-defined limit of the $O(3)_b$ symmetry electrodynamic theory. The difference is that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ exists in electrodynamics, and vanishes in electro-statics.

These consequences obtain by writing equation (2.2.1) in its component form:

$$\begin{aligned} \partial_\mu \tilde{G}^{(1)\mu\nu *} &= ig[A^{(2)}{}_\mu,\ \tilde{G}^{(3)\mu\nu}] \\ \partial_\mu \tilde{G}^{(2)\mu\nu *} &= ig[A^{(3)}{}_\mu,\ \tilde{G}^{(1)\mu\nu}] \\ \partial_\mu \tilde{G}^{(3)\mu\nu *} &= ig[A^{(1)}{}_\mu,\ \tilde{G}^{(2)\mu\nu}]. \end{aligned} \tag{5.1.7}$$

When the (3) component is zero these equations reduce to:

$$\partial_\mu \tilde{G}^{(1)\mu\nu} = \partial_\mu \tilde{G}^{(2)\mu\nu} = 0 \tag{5.1.8}$$

and

$$[A^{(1)}{}_\mu,\ \tilde{G}^{(2)\mu\nu}] = 0. \tag{5.1.9}$$

A solution of equation (5.1.8) is:

$$\tilde{G}^{(1)\mu\nu} = \tilde{G}^{(2)\mu\nu} \tag{5.1.10}$$

which means that the field tensor is real valued. Equation (5.1.9) gives two vector equations:

$$\boldsymbol{A}^{(1)} \cdot \boldsymbol{B}^{(2)} = \boldsymbol{A}^{(2)} \cdot \boldsymbol{B}^{(1)} \tag{5.1.11}$$

and

$$\boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(2)} = \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(1)} \tag{5.1.12}$$

which are satisfied by plane waves.

The absence of a (3) component means that this rotational field component of the electromagnetic field about the Z axis does not exist. This leaves only $X(t)$ and $Y(t)$ components of the vector potential. These are the real valued scalars A_x and A_y. The third Stokes parameter, expressed as proportional to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, has no nonlinear magneto-optical effects. The Maxwell displacement current is abelian and there are no nonlinear effects from radiation. In this limit:

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)}, \tag{5.1.13}$$

and the vector potential is a real function. This is a consequence of the fact that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero and that there is no $\boldsymbol{B}^{(3)}$ field. In the static limit the potential are written in general as:

$$\boldsymbol{A} = A_x \boldsymbol{i} + A_y \boldsymbol{j}, \tag{5.1.14}$$

i.e. as a vector in a two dimensional space (a plane) with orthonormal basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. Rotation of the vector in this plane through

an angle Λ are characterized by the group $SO(2)$, homomorphic with the group $U(1)$.

Gauge transformations act as unitary transformations of wave functions and are therefore [5]:

$$\psi \;=\; e^{-i\Lambda}\psi,\; \psi^* \;=\; \psi^* e^{i\Lambda}. \tag{5.1.15}$$

In the situation where the gauge transformation is local, or dependent upon coordinates the overlap of a wave function defined at different points in space gives:

$$\psi^*(x)\psi(x+\delta x) \;=\; \psi^*(x)e^{-i\delta\lambda}\psi(x) \;\simeq\; \psi^*(x)(1 \;-\; i\delta x^\mu \partial_\mu \lambda)\psi(x). \tag{5.1.16}$$

Now invoke the rule:

$$P_\mu \;=\; i\hbar\partial_\mu \;+\; eA_\mu \tag{5.1.17}$$

in equation (5.1.16) requires the usual gauge transformation [5] in order for momentum to be a gauge dependent quantity:

$$A_\mu \;\rightarrow\; A_\mu \;+\; \frac{\hbar}{e}\partial_\mu \Lambda \tag{5.1.18}$$

that describes the Aharonov-Bohm effect due to a static magnetic field. This procedure gives a conserved charge [5] and current through the Noether Theorem and the action is invariant under the $U(1)$ symmetry gauge transform (5.1.15). In the case where there are local gauge transformations the angle Λ is a function of x^μ [5]. This is the root cause of the Aharonov-Bohm effect due to a static magnetic field, because the derivative $\partial_\mu \Lambda$ must be non-zero. As we shall see, this is also the case in the optical Aharonov-Bohm effect, which needs an $O(3)_b$ symmetry transform for its description (Section 5.1) because it is a dynamical effect. The gauge transform with Λ a function of x^μ is a local gauge transform. If Λ were not a function of the rotation represented in operator form by equations

(5.1.15) and (5.1.18) would take place at the same time at all points [5]. This is a global gauge transformation.

Therefore electro-statics can be considered as a special case of $O(3)_b$ electrodynamics for global gauge transformations when quantities with index (3) are zero, and quantities with index (1) are equal to those with index (2). These conditions restrict the symmetry to a sub-symmetry of $O(3)_b$, a subsymmetry which can be identified as $U(1)$, homomorphic with $SO(2)$. The laws that appear under this sub-symmetry are the usual Coulomb, Faraday, Gauss and Ampere Laws. The Maxwell displacement current is zero, for global gauge transformations, because there is no radiation, and vice versa. The four laws of electro-statics and magneto-statics are thus determined by the equations:

$$\partial_i \tilde{G}^{ij} \;=\; 0,\; \partial_i H^{ij} \;=\; J^j \tag{5.1.19}$$

in the static limit. These equations represent a special case of $O(3)_b$ symmetry, a special case which can be characterized as having a $U(1)$ sub-symmetry because $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero.

Therefore electrodynamics and electro- or magneto-statics can be considered as having an overall $O(3)_b$ symmetry which depends on the existence nonlinear radiation of circular polarization. The latter defines the complex basis $((1),(2),(3))$ [6-10] which is used as the basis for the internal space of the $O(3)_b$ symmetry gauge field theory that $O(3)_b$ electrodynamics represents. There are no point magnetic monopoles because of the structure of equation (5.1.19). There are topological magnetic monopoles as discussed in chapter three, and there are solutions to the field equations of $O(3)_b$ electrodynamics which do not exist in the Heaviside-Maxwell theory. Some of these are summarized in chapter four. Higher order non-linear optical effects can be described within $O(3)_b$ electrodynamics with

constitutive relations, or more rigorously by extending to higher order terms the Taylor series in closed loop in Minkowski space-time that gives rise to the $O(3)_b$ field tensor. The potentials in $O(3)_b$ electrodynamics will, however, always be defined in terms of a sum of $O(3)_b$ rotation generators. Higher order non-linear optical effects will arise from tensor products of these potential components.

5.1.1. THE OPTICAL AHARONOV-BOHM EFFECTS

The existence of $O(3)_b$ electrodynamics, implied by the existence of the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in radiation in the vacuum (the third Stokes parameter) means that there are Aharonov-Bohm effects due to radiation rather than to a static magnetic field. As described in chapter three, these are closely related to the topological phase effects [11], which can be traced to an integral over the radiated $\boldsymbol{B}^{(3)}$ field. In general there are radiative Aharonov Bohm effects due to $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$. These are analogues of the equivalent effect due to a static magnetic field [5], analogues which exist because of the rules of gauge transformation in $O(3)_b$ rather than in $U(1)$. These radiated or optical Aharonov-Bohm effects can be detected by phase shifts, and as developed in the following sections, there are similar effects of radiation which are also physical phase effects, the theme of this chapter. One of these phase shifts is observed when the platform of a Sagnac interferometer is rotated, the $O(3)_b$ gauge transformation (chapter three) is identified with this rotation.

The difference between an $O(3)_b$ and a $U(1)$ gauge transformation is that in $O(3)_b$ there are three components of the vector potential that must be transformed, and each transformation is accompanied by a distinct inhomogeneous term [5,10]. The gauge transformations for $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ can be represented by:

$$\boldsymbol{A}^{(1)} \rightarrow \boldsymbol{A}^{(1)} + \boldsymbol{a}^{(1)}, \; \boldsymbol{A}^{(2)} \rightarrow \boldsymbol{A}^{(2)} + \boldsymbol{a}^{(2)} \qquad (5.1.1.20)$$

where $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ denote inhomoge

tion of the cross product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ therefore generates four terms in general:

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \rightarrow \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} + \boldsymbol{a}^{(1)} \times \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(1)} \times \boldsymbol{a}^{(2)} + \boldsymbol{a}^{(1)} \times \boldsymbol{a}^{(2)}. \qquad (5.1.1.21)$$

However, the rules of gauge theory [5] show that the same transformation applied to $\boldsymbol{B}^{(3)}$ produces $\boldsymbol{B}^{(3)}$ itself, without any additional terms. This result is demonstrated later in this chapter with matrices. Therefore $\boldsymbol{B}^{(3)}$ is gauge invariant, i.e. does not change when the gauge transform is applied to it. The process of gauge transformation can therefore be thought of as a rotation about the axis in which $\boldsymbol{B}^{(3)}$ is defined. Since $\boldsymbol{B}^{(3)}$ is unchanged, it continues to be equal to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and it can be demonstrated that there is the additional constraint equation:

$$\boldsymbol{a}^{(1)} \times \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(1)} \times \boldsymbol{a}^{(2)} + \boldsymbol{a}^{(1)} \times \boldsymbol{a}^{(2)} = 0. \qquad (5.1.1.22)$$

In regions where $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)}$ is zero the term $\boldsymbol{a}^{(1)} = \boldsymbol{a}^{(2)}$ is non-zero, and so there is a new Aharonov-Bohm effect due to radiation. There is a phase shift [5] in interfering electron beams due to a laser placed between the cavities of a Young interferometer. The laser generates an analog to the static magnetic field of the original Aharonov-Bohm effect. Later in this section the order of magnitude expected from such an effect will be estimated.

In order to understand the nature of gauge transformation in non-Abelian gauge theory we can for example consider a rotation of a field ψ in $SU(2)$ symmetry:

$$\psi' = exp(i\sigma_z \alpha/2)\psi, \qquad (5.1.1.23)$$

where σ_z is a Pauli matrix component and α an Euler angle. The potential four vector transforms under the field rotation (5.1.1.23) as:

$$A_\mu{}' = SA_\mu A^{-1} - \frac{i}{q}(\partial_\mu)SS^{-1} \qquad (5.1.1.24)$$

and is defined in $SU(2)$ as:

$$A_\mu = A^a{}_\mu \sigma^a/2 = A^z{}_\mu \sigma^z/2. \tag{5.1.1.25}$$

Therefore, a gauge transformation in $SU(2)$ is the direct consequence of rotating the field geometrically. Equation (5.1.1.24) is the basis of the Aharonov-Bohm effect and instanton theory. In $SU(2)$, A_μ is put into the form of a Pauli matrix:

$$A_\mu = \begin{pmatrix} A^z{}_\mu/2 & 0 \\ 0 & -A^z{}_\mu/2 \end{pmatrix} \tag{5.1.1.26}$$

and the gauge transformed $A^z{}_\mu$ can be evaluated as follows:

$$A_\mu{}' = A_\mu - \frac{\sigma}{2g}\partial_\mu \alpha \tag{5.1.1.27}$$

giving the result:

$$A^z{}_\mu = A^z{}_\mu + \frac{1}{g}\partial_\mu \alpha^z. \tag{5.1.1.28}$$

which explains the Sagnac effect as in chapter three.

Therefore, in non-Abelian field theory gauge transformation is a geometrical process. We illustrate such gauge shifts in the small angle limit. In the $O(3)_b$ basis the gauge rotation can be represented as the action of the matrix:

$$\mathbf{1} + i\boldsymbol{J}\cdot\boldsymbol{\Lambda} = \begin{pmatrix} 1 & i\Lambda_3 & 0 \\ -i\Lambda_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.1.1.29}$$

in vector notation:

$$\psi' = (\mathbf{1} + i\boldsymbol{J}\cdot\boldsymbol{\Lambda})\psi, \rightarrow \psi' = e^{i\boldsymbol{J}\cdot\boldsymbol{\Lambda}}\psi, \tag{5.1.1.30}$$

Where ψ is a column vector of ψ_1, ψ_2, ψ_3. Now apply formula (5.1.1.24) with:

$$SA_\mu = exp(i\boldsymbol{J}\cdot\boldsymbol{\Lambda})A_\mu \simeq A_\mu - \Lambda \times A_\mu, \tag{5.1.1.31}$$

to obtain:

$$A_\mu{}' = (A_\mu - \boldsymbol{J}\cdot\boldsymbol{\Lambda}\times A_\mu + \frac{1}{g}\partial_\mu(\boldsymbol{J}\cdot\boldsymbol{\Lambda}))S^{-1}, \tag{5.1.1.32}$$

where

$$S = e^{i\boldsymbol{J}\cdot\boldsymbol{\Lambda}} = \mathbf{1} + i\boldsymbol{J}\cdot\boldsymbol{\Lambda} + O(\boldsymbol{\Lambda}^2). \tag{5.1.1.33}$$

Therefore we have the gauge transformation in its standard form:

$$A_\mu{}' = A_\mu - \boldsymbol{\Lambda}\times\boldsymbol{A}_\mu + \frac{1}{g}\partial_\mu\boldsymbol{\Lambda} + \ldots, \tag{5.1.1.34}$$

and the gauge transform differs from the $U(1)$ counterpart [5]. For Z axis rotation

$$\boldsymbol{\Lambda}\times\boldsymbol{A}_\mu = \Lambda_3 A_{\mu 1}\boldsymbol{j} - \Lambda_3 A_{\mu 2}\boldsymbol{i} \tag{5.1.1.35}$$

and

$$\begin{aligned} A'_{\mu 1} &= A_{\mu 1} + \Lambda_3 A_{\mu 2} \\ A'_{\mu 2} &= A_{\mu 2} + \Lambda_3 A_{\mu 1} \\ A'_{\mu 3} &= A_{\mu 3} + \frac{1}{g}\partial_\mu \Lambda_3, \end{aligned} \tag{5.1.1.36}$$

which is a geometrical result emphasizing the fact that there are three gauge transforms possible, of $A_{\mu 1}$, $A_{\mu 2}$, and $A_{\mu 3}$.

The same general equations of non-Abelian gauge field theory apply in $O(3)_b$ symmetry. The vector potential transforms inhomogeneously as in equation (5.1.1.24) and the field transforms homogeneously:

$$G_{\mu\nu} = SG_{\mu\nu}S^{-1}. \tag{5.1.1.37}$$

The potential four vector is defined as a sum over rotation generators:

$$A_\mu = J^a A^a{}_\mu, \tag{5.1.1.38}$$

which is a sum that is equivalent in the basis $((1),(2),(3))$ to:

$$A_\mu = \sum_{a=1}^{3} A^{(a)}{}_\mu e^{(a)} \qquad (5.1.1.39)$$

The field tensor is similarly defined in this basis as:

$$G_{\mu\nu} = \sum_{a=1}^{3} G^{(a)}{}_{\mu\nu}. \qquad (5.1.1.40)$$

The inhomogeneous terms $(\partial_\mu S)S^{-1}$ are responsible for the non-abelian Aharonov-Bohm effects and if J_z is the infinitesimal rotation generator about Z this term gives the (3) component :

$$\frac{i}{g}(\partial_\mu S)S^{-1} = \frac{J_z}{g}\partial_\mu \alpha. \qquad (5.1.1.41)$$

The transformation of the gauge potential $A^z{}_\mu$ is therefore:

$$A^z{}_\mu \rightarrow A^z{}_\mu + \frac{1}{g}\partial_\mu \alpha, \qquad (5.1.1.42)$$

as used in chapter three.

Therefore in non-Abelian gauge field theory the term $a^{(1)} \times a^{(2)}$ in equation (5.1.1.21) is accompanied by the term $(1/g)\partial_\mu \alpha$ in equation (5.1.1.42).

Finally, in this Section we illustrate with matrices the basic properties that the field tensor transforms homogeneously and the vector potential inhomogeneously. The field tensor, in general gauge field theory, is defined using a Taylor series, one term of which (chapter three) is a commutator of covariant derivatives. The field tensor transforms homogeneously according to equation (5.1.1.37) and if

$\boldsymbol{B}^{(3)}$ is defined in the Z axis the process of gauge transformation about Z leaves $\boldsymbol{B}^{(3)}$ unchanged. This deduction is illustrated as follows using matrices in $O(3)_b$ symmetry. The $\boldsymbol{B}^{(3)}$ field itself is defined from the basic $G_{\mu\nu}$ as follows [5]:

$$G_{\mu\nu} = \frac{i}{g}[\mathcal{D}_\nu, \mathcal{D}_\mu]. \qquad (5.1.1.43)$$

Define the component matrices of A_μ in the basis $((1),(2),(3))$:

$$G^{(3)}{}_{\mu\nu} = \partial_\nu A^{(3)}{}_\mu - \partial_\mu A^{(3)}{}_\nu - ig[A^{(1)}{}_\mu, A^{(2)}{}_\nu] \qquad (5.1.1.44)$$

so

$$G^{(3)}{}_{xy} = ig[A^{(1)}{}_x, A^{(2)}{}_y]. \qquad (5.1.1.45)$$

This is equivalent to the vector form:

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}. \qquad (5.1.1.46)$$

It is easily demonstrated that rotation about the Z axis leaves $\boldsymbol{B}^{(3)}$ unchanged. Given the rotational matrix

$$\boldsymbol{O} = \begin{pmatrix} cos\alpha & sin\alpha & 0 \\ -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (5.1.1.47)$$

so that the field is unchanged by rotation $\boldsymbol{B}' = \boldsymbol{OBO}^T$ about the Z axis. The same process for the potential results in the characteristic inhomogeneous term, and results in a change in the potential components due to the rotation. For example, if:

$$\boldsymbol{A}^{(1)}{}_x = A^{(1)}{}_x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad (5.1.1.48)$$

then

$$S\boldsymbol{A}^{(1)}{}_x S^{-1} = A^{(1)}{}_x \begin{pmatrix} 0 & 0 & -isin\alpha \\ 0 & 0 & -icos\alpha \\ isin\alpha & icos\alpha & 0 \end{pmatrix}. \qquad (5.1.1.49)$$

It is easily checked that the commutator $[SA^{(1)}{}_xS^{-1},\ SA^{(1)}{}_yS^{-1}]$ is a Z axis rotation, thus confirming the geometrical nature of the gauge transformation. The latter changes $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, but leaves $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ invariant, confirming that $\boldsymbol{B}^{(3)}$ remains equal to the conjugate product, and confirming the existence of the subsidiary condition (5.1.1.21).

The inhomogeneous terms for a Z axis rotation of $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ can be evaluated through:

$$A^{(1)}{}_\mu = SA^{(1)}{}_\mu S^{-1} - \frac{i}{g}((\partial_\mu S)S^{-1})^{(1)}, \tag{5.1.1.50}$$

$$A^{(2)}{}_\mu = SA^{(2)}{}_\mu S^{-1} - \frac{i}{g}((\partial_\mu S)S^{-1})^{(2)}. \tag{5.1.1.51}$$

However, for a Z axis rotation it is known that:

$$-\frac{i}{g}((\partial_\mu S)S^{-1})^{(1)} = \frac{i}{g}((\partial_\mu S)S^{-1})^{(2)*} = \frac{J_z}{g}\partial_\mu \alpha, \tag{5.1.1.52}$$

and so the commutator of the inhomogeneous terms, labeled $a^{(1)} \times a^{(2)}$ in equation (5.1.1.42), is zero. The only optical Aharonov-Bohm effect for this type of gauge transformation therefore contains:

$$A^z{}_\mu \rightarrow A^z{}_\mu + \frac{J_z}{g}\partial_\mu \alpha, \tag{5.1.1.53}$$

which as argued in chapter three is responsible for the Sagnac effect.

5.1.2 PHASE SHIFT OF THE INVERSE FARADAY EFFECT

The order of magnitude of any non-zero optical Aharonov-Bohm effect can be estimated from the phase shift expected in the inverse

Faraday effect due to the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The magnetic field induced in a bulk sample in the inverse Faraday effect is given for a gas of electrons [12] by:

$$\boldsymbol{B}^{(3)*} = = ig'\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{5.1.2.54}$$

where the factor g' is given by:

$$g' = \frac{N}{V}\frac{\mu_0 e^3}{2m^2\omega}. \tag{5.1.2.55}$$

Here N/V is the number of electrons in a sample volume V; μ_0 the permeability in vacuo; e the charge on the electron; m the electron mass and ω the angular frequency. Using the nonAbelian Stokes Theorem, the change in phase in the electronic wave function expected in this form of the inverse Faraday effect is:

$$\Delta\delta = g'\int\int \boldsymbol{B}^{(3)} \cdot \mathcal{A}. \tag{5.1.2.56}$$

An order of magnitude estimate of the magnitude of the phase shift is then obtained. For a sample That contains $N/V = 10^{26}$ electrons per cubic meter, and for an intensity and angular frequency corresponding to a pulsed Nd YaG laser, the factor g' is $2.5 \times 10^7 CJ^{-1}s^{-1}$; which compares with $e/\hbar = 1.5 \times 10^{15} CJ^{-1}s^{-1}$ as used in the interaction of a magnetic field with an electron. Therefore, this type of optically induced phase shift is eight orders of magnitude smaller than the magnetically induced equivalent, even when using an intense laser pulse. Therefore, if geometry allows an optical Aharonov-Bohm effect from equation (5.1.1.21) due to $\boldsymbol{a}^{(1)} \times \boldsymbol{a}^{(2)}$, it is much smaller in magnitude than the equivalent effect due to a static magnetic field.

In a simplified view of the standard $U(1)$ based Aharonov-Bohm effect [5] the observable shift in a fringe pattern due to interfering fermion matter waves is attributed to a change in phase:

$$\Delta\delta = \frac{1}{\hbar}\boldsymbol{p} \cdot \boldsymbol{r}, \tag{5.1.2.57}$$

where $\boldsymbol{p}$ is the non-relativistic limit of the fermion linear momentum at a point $\boldsymbol{r}$ in three dimensional space. The phase $\Delta\delta$ is an action or angular momentum divided by the quantized unit of action or angular momentum, $\hbar$. The intrinsic angular momentum of the fermion does not appear [12] because $\boldsymbol{S}$ is a relativistic concept. Reinstating, $\boldsymbol{A}$ produces the spin phase of the fermion:

$$\delta \;=\; \frac{1}{\hbar}\boldsymbol{S}^{(3)}\cdot\boldsymbol{\sigma}^{(3)}, \qquad (5.1.2.58)$$

which exists due to topology. There therefore exists a phase shift in the inverse Faraday effect, which is due to the interaction of the conjugate product with the intrinsic spin of the fermion. Following Talin et al. [13] we can write the field fermion interaction energy E in the form:

$$E \;=\; \omega J \;=\; \omega S, \qquad (5.1.2.59)$$

which originates in the conservation laws of angular momentum $\boldsymbol{J}$ and energy. Here $\boldsymbol{S}$ has the units of action [13]. The energy of the photon is $\hbar\omega$ by definition, and this is the limit in which J has been replaced by $\hbar$, the quantum of action or angular momentum of an electromagnetic or matter wave. In the presence of an electromagnetic field the spin phase of one fermion shifts by:

$$\Delta\delta \;=\; \frac{E}{\hbar\omega}, \qquad (5.1.2.60)$$

which is the action E/ω generated by the field-fermion interaction divided by $\hbar$. This indicates that $\Delta\delta$ will exist in discrete units. This is the change in energy E caused by fermion-field interaction divided by the quantum of energy, $\hbar\omega$. Alternatively, the phase change can be viewed [12] as $J/\hbar$, i.e. as the change in angular momentum J generated by the interaction of the fermion and the field divided by $\hbar$. If there is no interaction energy or interaction angular momentum, no phase shift exists, and vice versa. For one fermion the change in phase is,

$$\Delta\delta \;=\; i\frac{e^2}{2m\hbar\omega}\sigma^{(3)}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}, \qquad (5.1.2.61)$$

and is proportional to the conjugate product. The phase change is expressible as:

$$\Delta\delta \;=\; \mp\Big(\frac{e^2c\mu_0}{2m\hbar}\frac{I}{\omega^3}\Big). \qquad (5.1.2.62)$$

These considerations can be developed for the gauge transformed $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$, giving the optical Aharonov Bohm effect [12], assuming that there are circumstances under which the product $\boldsymbol{a}^{(1)}\times\boldsymbol{a}^{(2)}$ in equation (5.1.21) is non-zero. In both types of inverse Faraday effect we use the non-Abelian Stokes Theorem to show that for a flat connection:

$$\Delta\delta \;=\; g'\oint A_\mu dx^\mu \;=\; g'^2\int\int A^{(0)2}d\mathcal{A} \qquad (5.1.2.63)$$

5.2 PHASE SHIFT OF THE OPTICAL JOSEPHSON EFFECT

The optical Josephson effect is a physical influence by means of a phase change 14 producing a circulating current

$$J \;=\; sin\Delta\delta, \qquad (5.2.64)$$

in a superconductor junction. The phase change for these flat connections is:

$$\Delta\delta \;=\; \frac{\Phi}{\Phi_0} \;=\; \frac{e}{\hbar}\int\int \boldsymbol{B}^{(3)}\cdot\mathcal{A} \qquad (5.2.65)$$

from the non-Abelian Stokes Theorem and is another example of a physical influence due to $\boldsymbol{B}^{(3)}$ that is not present in abelian electrodynamics. The optical Josephson effect is therefore the inverse Faraday effect. It occurs in a superconductor, due to a phase factor in the junction gap of a Josephson junction. The origin of the phase factor is the conjugate product $\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}$. It is another example

of the effect of the conjugate product, which sets up a magnetic flux in a superconducting ring.

5.3 PHASE FACTOR OF THE OPTICAL HALL EFFECT

Finally, in this chapter we discuss the optical Hall effect. Here the magnetic field of the ordinary Hall effect is replaced by a circularly polarized electromagnetic field generating the phase factor due to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in non-Abelian gauge field theory. The optical Hall effect can be observed in principle in a modified Wien filter, which is used [15] to observe phase shifts in the original Aharonov Bohm effect to high precision. The electric force on the electron is balanced by the magnetic force when the Wien filter is compensated [12]. Hasselbach et al. [15] have discussed the role played by the electric and magnetic Aharonov Bohm phase shifts in such as device. The magnetic field in the conventional Wien filter causes an Aharonov-Bohm phase shift due to the difference in magnetic flux enclosed by the two coherent electron beams. Therefore, when the Wien filter is modified to replace the static magnetic field by a circularly polarized electromagnetic field, there will be an optically induced Aharonov-Bohm effect if the term $\boldsymbol{a}^{(1)} \times \boldsymbol{a}^{(2)}$ is non-zero. There might also be a quantum analogue of this effect.

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CHAPTER 6
QUANTUM ELECTRODYNAMICS AND THE $B^{(3)}$ FIELD

This chapter is a discussion of the nonrelativistic $O(3)_b$ quantum electrodynamics. This discussion covers the basic physics of $U(1)$ electrodynamics and leads into a discussion of nonrelativistic $O(3)_b$ quantum electrodynamics. This discussion will introduce the quantum picture of the interaction between a fermion and the electromagnetic field with the $\boldsymbol{B}^{(3)}$ magnetic field. Here it is demonstrated that the existence of the $\boldsymbol{B}^{(3)}$ field implies photon-photon interactions. In nonrelativistic quantum electrodynamics this leads to nonlinear wave equations. Some presentation is given on relativistic quantum electrodynamics and the occurrence of Feynman diagrams that emerge from the $\boldsymbol{B}^{(3)}$ are demonstrated to lead to new subtle corrections. Numerical results with the interaction of a fermion, identical in form to a 2-state atom, with photons in a cavity are discussed. This concludes with a demonstration of the Lamb shift and renormalizability.

6.1 INTRODUCTION TO QUANTUM ELECTRODYNAMICS

One of the oldest subjects of physical science is electrodynamics. The study has its early origins in the study of optics by Willebrord Snellius and the studies of magnetism by William Gilbert in the sixteenth century[1]. It took nearly three centuries for the theory of classical electromagnetism to reach fruition with Maxwell[2]. This grand synthesis at first appeared to solve the most fundamental questions of the day, but an historical retrospective shows that it posed as many questions as it solved. The resolutions to these problems were found in the theory of special relativity and in quantum theory. The first of these was an answer to the problem of what is the speed of an elec-

tromagnetic wave on any given reference frame and the second was a resolution to the blackbody radiation problem. The latter solution advanced by Planck assumed that light existed in discrete packets of energy that were emitted and absorbed[3]. This initiated the study of the interaction between quantized electromagnetic waves and matter with discrete quantized energy levels. This theory is called quantum electrodynamics.

The formalism of quantum electrodynamics may appear arcane to the uninitiated, but in reality it is based on rather simple concepts. The first of these is that the radiation field is described by a set of harmonic oscillators. The harmonic oscillator is essentially a spring loaded with a mass or a pendulum that swings through a small angle. The pendulum has an old history with physics that began with Galileo. Early in the formalism of quantum mechanics this was a system examined and quantized. An analysis with the Schrödinger wave equation leads to some complexities with recurrence relations and Hermite polynomials. However, with the Heisenberg formalism the quantum theory of the harmonics oscillator reduces to a simple model with evenly spaced states that have an associated energy $(n + 1/2)\hbar\omega$. Here the number n corresponds to the number of photons with angular frequency $\omega = ck$ in the system. For $n = 0$ we see that the absence of photons predicts that there is still an energy associated with the vacuum. This nonzero value for the ground state of the harmonic oscillator has been a source of controversy as well as profound physical insight. A second assumption that is often made is that these photons exist within a cavity. This allows for a simplification of the meaning to counting modes. The third concept is that atoms that interact with these photons also have energy levels. The simplest example would be atoms with two states. Here an atom that absorbs a photon can only do so by changing its internal state from the lower state to the excited state, and an atom can emit a photon only by changing its internal state from the excited state to the lower state. These atomic interactions with the electromagnetic field will change the photon number by ± 1.

How does one proceed to take the classical theory of electromagnetism, or Maxwell's equations, and cast them in a quantum mechanical context? It is best to start with the definitions of the electric and magnetic fields:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{6.1.1}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{6.1.2}$$

The quantity $\mathbf{A}$ appears in these equations and is the vector potential of electromagnetic theory. In a very elementary discussion of the static electric field we are introduced to the theory of Coulomb. It is demonstrated that the electric field can be written as the gradient of a scalar potential $\mathbf{E} = -\nabla\phi$, $\phi = \kappa q/r$. It is also demonstrated that the addition of a constant term to this potential leaves the electric field invariant. Where you choose to set the potential to zero is purely arbitrary. In order to describe a time varying electric field a time dependent vector potential must be introduced $\mathbf{A}$. If one takes any scalar function χ and uses it in the substitutions

$$\mathbf{A}' = \mathbf{A} - \kappa\nabla\chi, \ \kappa = constant, \tag{6.1.3}$$

$$\phi' = \phi + \gamma\frac{\partial \chi}{\partial t}, \ \gamma = constant, \tag{6.1.4}$$

it is easy to demonstrate that the electric and magnetic fields are left invariant. This means that the analyst can choose the form of the vector potential in an arbitrary fashion. This is defined as a choice of gauge that is described by either writing an explicit form for the vector potential or by writing an auxiliary differential equation. As an example we may then choose

$$\mathbf{A}(\mathbf{r},t) = A_0\mathbf{e}\ exp(i(\mathbf{k}\cdot\mathbf{r} - \omega t)), \tag{6.1.5}$$

$$\phi = 0,$$

which is equivalent to stating that $\nabla \cdot \mathbf{A} = 0$. It is then fitting that the Maxwell's equations are presented as they are invariant under all possible gauge transformations

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}, \ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

$$\nabla \cdot \boldsymbol{D} = \rho, \ \nabla \cdot \boldsymbol{B} = 0, \tag{6.1.6}$$
$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon \boldsymbol{E}, \ \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu \boldsymbol{H}.$$

The connection to quantum theory is made with the recognition that this transformation changes the phase of a wave function of a particle that interacts with the electromagnetic field:

$$\psi \rightarrow e^{-i\chi}\psi. \tag{6.1.7}$$

The equation that describes the interaction of a nonrelativistic electron with the electromagnetic field is the Pauli equation,

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\Big(\boldsymbol{\sigma}\cdot\Big(\mathbf{p} - \frac{e}{c}\mathbf{A}\Big)\Big)\Big(\boldsymbol{\sigma}\cdot\Big(\mathbf{p} - \frac{e}{c}\mathbf{A}\Big)\Big)\psi + e\phi\psi \tag{6.1.8}$$

where the last potential term is dropped in $U(1)$ electrodynamics. Now consider this equation under the phase shift $\psi \rightarrow e^{-i\chi}\psi$:

$$\nabla\psi = \nabla(e^{-i\chi}\psi) = e^{-i\chi}\nabla\psi - i\nabla\chi e^{i\chi}\psi,$$
$$= e^{-i\chi}(\nabla - i\nabla\chi)\psi. \tag{6.1.9}$$

This means that the generalized momentum operator is

$$\Big(\boldsymbol{\sigma}\cdot(\mathbf{p} - \frac{e}{c}\mathbf{A})\Big)\psi' \rightarrow e^{i\chi}\Big(\boldsymbol{\sigma}\cdot(\mathbf{p} - \hbar\nabla\chi - \frac{e}{c}\boldsymbol{A})\Big)\psi, \tag{6.1.10}$$

which recovers the above gauge transformations for $\boldsymbol{A}$ as $\boldsymbol{A} \rightarrow \boldsymbol{A} - (e/c)\nabla\chi$ so the quantity in equation (6.1.10) is gauge invariant.

We have our first connection between quantum mechanics and electromagnetism: a local phase shift in a wave function is coexistent with a local gauge transformation in the vector and scalar potential for the electromagnetic field. So far nothing has been changed

with the formal description of the electric and magnetic field. This is good news, for this means that the electromagnetic field can be described by the classical equations of Maxwell. This can be stated that the probability amplitude for the absorption or emission of a photon by an atom is equal to the amplitude given by the absorption and emission of an electromagnetic wave described by the classical electrodynamics of Maxwell's equations. This statement must be accompanied by the stipulation that the classical wave is normalized. Then energy density of the wave is $\hbar\omega$ times the probability per unit volume for the occurrence of the photon, and the classical wave is broken into two complex components $e^{-i\omega t}$ and $e^{i\omega t}$ that represent the phase of an absorbed and emitted photon. These phases will, by the first stipulation, be multiplied by the appropriate probability amplitudes for absorption and emission. This sets us up for an examination of the semiclassical theory of radiation and its interaction with quantized atoms.

We have that the electromagnetic field is described within a box. This means that the number of states per unit volume is dependent upon the number of discrete modes per volume $|\mathbf{k}|^2(2\pi)^3\Delta|\mathbf{k}|$. This can easily be carried over to the continuous version if we let the wall of the cavity separate to arbitrary distances. The density of states is then $k^2(2\pi c)^3(dkd\Omega)/\hbar d\omega$. This describes the density of states that are available for an atom to interact with. We then have that if we have an atom in the state ψ_i that it may then enter into the state ψ_f, with respective energies E_i and E_f. The probability per increment in time is proportional to the transition probability for this event times the density of states. Assume that the time over which this transition occurs is far larger than the periodicity of the electromagnetic field. The transition probability is then proportional to the modulus square of the vector potential when averaged over many periods of the field. This then gives the Fermi Golden Rule[4]:

$$\frac{\Delta}{\Delta t}Prob_{\psi_i\rightarrow\psi_f} = \frac{2\pi}{\hbar^2}|A|^2(\omega^2/(2\pi c)^3)d\Omega. \tag{6.1.11}$$

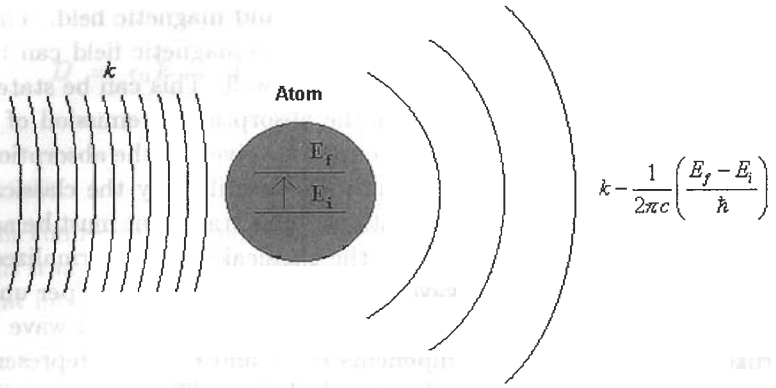


Figure 6.1

All we need to do is to estimate the average of the potential. To do this the form of the electric and magnetic fields are used in the normalized energy density of the electromagnetic field

$$E = \hbar\omega = \frac{1}{8\pi}(|\mathbf{E}|^2 + |\mathbf{B}|^2) = \frac{1}{8\pi}\frac{A^2\omega^2}{c^2}, \tag{6.1.12}$$

which gives the averaged potential as $A = \sqrt{8\pi\hbar c^2/\omega}$ this gives us the transition probability per unit time

$$\frac{\Delta}{\Delta t} Prob_{\psi_i \to \psi_f} = \frac{2}{\pi}\frac{e^2\omega}{\hbar c}. \tag{6.1.13}$$

This gives us an order of magnitude estimate for this transition. It assumes that the potential is absorbed or emitted with no regard to its components $e^{i\omega t}$ and $e^{-i\omega t}$. As such this can only be regarded as a rather crude estimate. However, we are beginning to make progress in our understanding of how electromagnetic fields interact quantum mechanically with atoms.

Returning to equation 2, we express this according to the matrix element U_{ij} that will be determined explicitly,

$$\frac{\Delta}{\Delta t} Prob_{\psi_i \to \psi_f} = \frac{2\pi}{\hbar^2}|U_{if}|^2\omega^2/(2\pi c)^3 d\Omega. \tag{6.1.14}$$

This matrix element is the expectation of a time dependent perturbative or interaction Hamiltonian, $V = e^{i\omega t}U(\mathbf{r})$,

$$U_{fi} = \int d^3r \psi_f^* V \psi_i. \tag{6.1.15}$$

Since the Pauli-Schrödinger equation is of the form $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ we may write the wave functions as $\psi_{if} = e^{iE_{if}t/\hbar}\psi(0)_{if}$. We then have the transition matrix element written as,

$$U_{fi} = \int d^3r \psi(0)_f^* U(\mathbf{r})\psi(0)_i exp\Big(i\frac{(E_f - E_i)t}{\hbar} + i\omega t\Big). \tag{6.1.16}$$

The initial and final states of the system are $E_i - \hbar\omega$ and E_f. We expect that the interaction occurs where $E_i - \hbar\omega = E_f$. This means that we may set the phase equal to zero and interactions that are slightly off resonant are ignored, and

$$U_{fi} = \int d^3r \psi(0)_f^* U(\mathbf{r})\psi(0)_i. \tag{6.1.17}$$

The interaction Hamiltonian can be extracted from the Pauli Hamiltonian plus a dipole interaction Hamiltonian

$$H = \frac{1}{2m}\Big(\mathbf{p} - \frac{e}{c}\mathbf{A}\Big)^2 - \frac{e\hbar}{2mc}\sigma\cdot\nabla\mathbf{A}$$

$$= \frac{1}{2m}p^2 - \frac{e}{2mc}(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}) - \frac{e\hbar}{2mc}\sigma\cdot\nabla\mathbf{A} + \frac{e^2}{2mc^2}\mathbf{A}\cdot\mathbf{A}$$
$$+ \frac{e\hbar}{2mc}\sigma\cdot\nabla\mathbf{A}. \tag{6.1.18}$$

The second and third terms are the interaction terms that couple the atom, here modeled as a two-state system with Pauli matrices, to the electromagnetic field. We consider the momentum to be the operator $\mathbf{p} = \frac{\hbar}{i}\nabla$ and consider this operator as not only operating on the vector potential but on the wave function. Hence we find that

$$\nabla\times\mathbf{A} = i\mathbf{k}\times\mathbf{e}Ae^{i\mathbf{k}\cdot\mathbf{r} - i\omega t},$$

$$\mathbf{p}\cdot\mathbf{A} = \mathbf{A}\cdot(\mathbf{p}+\hbar\mathbf{k}). \tag{6.1.19}$$

This leads to a more complete form of the interaction Hamiltonian

$$U_{fi} = -\frac{e}{2mc}A\int d^3r\phi_f^*(0)\big(\mathbf{p}\cdot\mathbf{e}+\mathbf{e}\cdot\mathbf{p}-i\hbar\sigma\cdot(\mathbf{k}\times\mathbf{e})e^{i\mathbf{k}\cdot\mathbf{r}}\big)\psi_i(0). \tag{6.1.20}$$

This result is an exact expression for the transition matrix element. Physically we have a dipole interaction with the vector potential and a dipole interaction with the magnetic field modulated by a phase factor. The problem is that this integral is difficult to compute. An approximation can be invoked. The wave vector has a magnitude equal to $1/\lambda$. The position $\mathbf{r}$ is set to the position of an atom and is on the order of the radius of that atom. Thus $\mathbf{K}\cdot\mathbf{r} \simeq a/\lambda$. So if the wavelength of the radiation is much larger than the radius of the atom, which is the case with optical radiation, we may then invoke the approximation $e^{i\mathbf{k}\cdot\mathbf{r}} \simeq 1+i\mathbf{k}\cdot\mathbf{r}$. This is commonly known as the Born approximation. This first order term under this approximation is also seen to vanish in the first two terms as it multiplies the term $\mathbf{p}\cdot\mathbf{e}$. A further simplification occurs, since the term $\sigma\cdot(\mathbf{k}\times\mathbf{e})$ has only diagonal entries, and our transition matrix evaluates these over orthogonal states. Hence, the last term vanishes. We are then left with the simplified variant of the transition matrix,

$$U_{fi} = -\frac{e}{mc}A\int d^3r\phi_f^*(0)\mathbf{p}\cdot\mathbf{e}\phi_i(0)$$

$$U_{fi} = -\frac{e}{mc}A\langle f|\mathbf{p}|i\rangle\cdot\mathbf{e}. \tag{6.1.21}$$

The element $\mathbf{p}_{fi}\cdot\mathbf{e} = |\mathbf{p}_{fi}|cos(\theta)$, where θ is the angle between these two vectors. However this angle is $\pi/2$ different from the coordinate angle evaluated in $d^3r = r^2drdsin(\theta)d\phi$, so we set $\theta \rightarrow \theta+\pi/2$ This means that the transition probability per unit time assumes the form

$$U_{if} = -\frac{1}{\hbar}\Big(\frac{e}{mc}\Big)^2 A\int_0^{\pi}\int_0^{2\pi}|\mathbf{p}_{fi}\cdot\mathbf{e}|^2 sin^3(\theta)r^2d\theta d\phi. \tag{6.1.22}$$

Recognizing that $\mathbf{p}_{fi} = im\omega\mathbf{r}_{fi}$ and performing the integration we find that

$$\frac{\Delta}{\Delta t}P_{i\rightarrow j} = \frac{4}{3}\frac{e^2}{\hbar c}\frac{\omega^3}{c^2}|r_{fi}|^2 \tag{6.1.23}$$

As a final side note, the term $\alpha = \frac{e^2}{\hbar c} \simeq 1/137$ is the fine structure constant for the electromagnetic interaction. This is a dimensionless quantity that gives the interaction strength between photons and charged particles.

So far we have the transition probability per unit time. What is measured is the transition probability over a given time as measured from a statistical ensemble of identical systems. A quantum operator, O_{op} evolves in time according to the Schrödinger equation

$$i\hbar\frac{\partial O_{op}}{\partial t} = [H,\ O_{op}]\ . \tag{6.1.24}$$

For the matrix U_{ij} defined at a time t, we have the solution to the Schrödinger equation with this initial condition

$$a_{ij}(t) = -\frac{i}{\hbar}\int_0^T e^{iE_jt/\hbar}U_{ij}(t)e^{-iE_it/\hbar}dt \tag{6.1.25}$$

The use of the symbol a_{ij} is to indicate that this represents the absorption of a photon by an atom. Further, the matrix $U_{ij}(t) = e^{-i\omega t}U_{ij}(0)$, and when placed substituted into equation (6.1.25), we arrive at an expression for $a_{ij}(t)$. Now when $a_{ij}(t)$ is multiplied by its complex conjugate we have:

$$|a_{ij}|^2 = \frac{4sin^2(\Delta T/2\hbar)}{\Delta^2}|U_{ij}(0)|^2,\ \Delta = E_i - E_j - \hbar\omega. \tag{6.1.26}$$

This gives the probability for the absorption of a photon with a frequency ω traveling along a particular angle pair in spherical coordinates. This must then be integrated over by the solid angle $d\Omega$ and evaluated.

So far considerable progress has been made. We have a fairly reasonable understanding of how the electromagnetic field interacts with an atom, and have in hand an expression that gives the transition probability for the absorption and emission of a photon by an atom. This expression has been demonstrated to be remarkably accurate in its description of the interaction of light with atomic structure. Additional features may be included to account for the permutation symmetry of various photons that interact with an atom. Explicit consideration may also be given for the probability that the atoms may also emit a photon once in the excited state. These considerations can be found in many texts on quantum electrodynamics.

What has been presented here is a semiclassical theory of $U(1)$ quantum electrodynamics. Here the electromagnetic field is treated in a purely classical manner, but where the electromagnetic potential has been normalized to include one photon per some unit volume. Here the absorption and emission of a photon is treated in a purely perturbative manner. Further, the field normalization is done so that each unit volume contains the equivalent of n photons and that the energy is computed accordingly. However, this is not a complete theory, for it is known that the transition probability is proportional to n+1. So the semiclassical theory is only appropriate when the number of photons is comparatively large.

6.2 A BRIEF INTRODUCTION TO DIFFERENTIAL FORMS

Differential forms are a particularly convenient method for examining physics problems. A p-form, where p is less than the dimension of the space considered, is an object that that is formed from the antisymmetric product of basis elements that are 1-forms

$$\omega = \omega_{ij\ldots k} dx^i \wedge dx^j \wedge \ldots \wedge dx^k, \tag{6.2.27}$$

where there a p differential 1-forms dx^i. A basic property of 2-forms

composed of two 1-forms is

$$dx^i \wedge dx^j = -dx^j \wedge dx^i. \tag{6.2.28}$$

In general the wedge product between a p-form and a q-form is

$$\alpha \wedge \beta = (-1)^{p+q-1} \beta \wedge \alpha. \tag{6.2.29}$$

The operator $\boldsymbol{d}$ is called a boundary operator that acts on a p-form to give a $p+1$-form. Since $\boldsymbol{d}$ is essentially a 1-form we have, by the antisymmetry of wedge products of 1-forms, $\boldsymbol{d}^2 = 0$, which is often interpreted as stating that the "boundary of a boundary vanishes."

The simplest example is the action of $\boldsymbol{d}$ on a 0-form or scalar,

$$\boldsymbol{d}\phi = \frac{\partial \phi}{\partial x^i} \boldsymbol{d}^i x = \partial_i \phi \boldsymbol{d}^i x, \tag{6.2.30}$$

where the components associated with the 1-form define the standard gradient. The next example is the action of the differential form on a 1-form

$$\boldsymbol{d}\omega = \partial_i \omega_j \boldsymbol{d}x^i \wedge \boldsymbol{d}x^j. \tag{6.2.31}$$

The antisymmetry between exchange of 1-forms means that the interchange between the indices i and j is antisymmetric so the component of this 2-form is $\frac{1}{2}(\partial_i \omega_j - \partial_j \omega_i)$ which defines the curl operation in a space of three dimensions. This procedure can go further.

A further property of forms is that of duality. Given an n-dimensional space with a p-form, for $p < n$, it is easy to show that there exists an $(n-p)$-form that is dual to this p-form. To demonstrate this we write this p-form according to equation 8. Now the basis of this p-form $dx^i \wedge dx^j \wedge \ldots \wedge dx^k$ defines a p-dimensional cube in the n dimensional space that is dual to an $(n-p)$-dimensional cube since

$$*dx^i \wedge dx^j \wedge \ldots \wedge dx^k = \epsilon^{ij\ldots k}{}_{mn\ldots r} dx^m \wedge dx^n \wedge \ldots \wedge dx^r, \tag{6.2.32}$$

where their are $n-p$ differential 1-forms that comprise the right hand side. The symbol * is called the Hodge star duality operator. The term $\epsilon^{ij\ldots k}{}_{mn\ldots r}$ acts on the component of this p-form to give a component that has $(n-p)$ indices.

A moments reflection reveals why the standard vector operation work. In the case of $\nabla \times \mathbf{A} = \mathbf{B}$ we have an operation that inputs a vector and gives back a vector. In reality we have $\boldsymbol{d}\mathbf{A} = \mathbf{F}$, that is a two form. However, this differential 2-form is dual to a 1-form whose components, in three dimensional space, defines the magnetic field as a vector.
A final discussion is on the generalized Stokes law. It is presented here without proof as

$$\int_{\partial S} \omega = \int_S \boldsymbol{d}\omega, \tag{6.2.33}$$

where S is a subspace with boundary ∂S in the space considered. In the case that S is a two dimensional disk in 3 dimensional space we recover the standard Stokes law, and in the case it is a 3 dimensional ball in 3-space we recover the Gauss law. These are standard tools in electromagnetism.

6.3 THE PHYSICAL BASIS FOR NONABELIAN ELECTRODYNAMICS

The earliest study of electromagnetism was with optics. The earliest observations were the most elementary: an inverted image could be seen through a raindrop hanging from a leaf. This particular study was not academically studied extensively during the middle ages, but did become of interest to glass makers during the great age of cathedral building in Europe from the 11^{th} to the 14^{th} centuries. During this period glassmakers perfected the art of forming glass with the right staining that would display a brilliant light through cathedral windows. Sophistication grew to the point that windows along the naive of a cathedral would display various stations of the

cross at Lent before the feast of Easter with colors and imagery that reflected the crucifiction of Jesus Christ according to the Gospels. Along with this sophistication, came a growth in the knowledge of the refractive properties of glass. It was found that images could be found focused on a plate through a glass of an scene distant from the glass. This information was used to attempt to grind lenses that would correct the myopic difficulties that some people suffered from. Of course only those with the financial resources could afford to employ these services.

The subject reached a measure of academic importance with Willebrord Snellius (1591-1626). He spent years working on the principles of optics involved with the process of vision; apparently the need for corrective eye wear was a growing market, and somebody had to find a complete understanding of how optics could assist the physician. In his treatise *Di Optrice* he laid down the first law of optical refraction. He recognized that the angle of incidence, with respect to the normal of a material surface, that a light ray hit a medium was related to the angle at which that light ray went through the transparent medium. So the paths of light outside and outside the glass with respect to the normal were related to each other by a constant later called the index of refraction. This ushered in the law of sines. He further went on to derive equations for curved thin lenses, based on this principle that were able to determine the position at which an image would form. This is the elementary lens maker's formula learned in first year physics.

The theory of light reached its second step forward with Huygens, who demonstrated that light was a wave that obeyed various diffractive properties[5]. Of course there later came Faraday and then Maxwell who brought in the complete theory of classical electromagnetism. The wave aspect of light tended to eclipse the older geometric optical view of light intellectually. However, the art of

geometric optics grew into a very refined art. Before the advent of computers, it required dozens or hundreds of human "computers" to complete the calculations required to characterize a particular optical system of lenses. The issue of refractive optics appeared to be in a sort of state of completion and was a matter of "simple calculation" that could be done by a machine.

Reality is not so simple. Suppose that the index of refraction depends upon the intensity of the light, or in a modern setting the electromagnetic fields, that pass through it. Suddenly we are confronted with having to revise our notion of the index of refraction: it is not necessarily a constant. Snellius had to compute the paths of rays that passed through a thin lens by considering the geometry in the curvature of a lens. Today nonlinear optics is a study that has to consider the variable index of refraction that was dependent upon the field strengths of the optical radiation being transmitted. This has become an important issue in the modern world. Optical fibers that transmit information as pulses of light are developed to transmit shorter pulses so that the date transmission rate can be increased. An optical fiber with a constant index of refraction has serious limitations. The radiation transmitted will reflect off the sides of the fiber, but at various angles. There will then be a spread in the optical pulse as it travels down the optical fiber since various photons will be reflected at slightly different angles. However, an optical fiber that has an index of refraction that is dependent upon the field strength will tend to "bunch" these photons into a single stream and thus eliminate this unfortunate problem.

The laws of electromagnetism are based upon the theory of gauge fields. The electromagnetic vector potential defines components of a gauge connection 1-form. This gauge connection defines a field strength two-form

$$\boldsymbol{d}\boldsymbol{A} = \boldsymbol{F}. \tag{6.3.34}$$

In general this emerges because the differential operator $\boldsymbol{d}$ is gauge covariant when it acts on a section of the bundle, or physically when it acts on a wave function $\boldsymbol{d} \rightarrow \boldsymbol{d} + q\mathbf{A}$. The application of this covariant differential operator twice on a function gives,

$$\begin{aligned}(\boldsymbol{d} + q\mathbf{A}) \wedge (\boldsymbol{d} + q\mathbf{A})\psi &= \mathcal{D} \wedge \mathcal{D}\psi \\ &= q(\boldsymbol{d}\mathbf{A} + q\mathbf{A} \wedge \mathbf{A})\psi. \end{aligned} \tag{6.3.35}$$

If the gauge connection is abelian then the term $e\mathbf{A} \wedge \mathbf{A}$ vanishes by the antisymmetry of the wedge product. This means that $\mathcal{D}^2\psi = q\boldsymbol{d}\mathbf{A}\psi$. This is an example of an abelian gauge theory, defined according to that vanishing of commutators between gauge potentials.

In general gauge theories are such that there is more that one particular gauge potential or connection coefficient $\mathbf{A}^{\mathbf{a}}$, where a is an index that spans a Lie algebra, such as $SU(2)$ and $SU(3)$, so that $q\mathbf{A}^a \wedge \mathbf{A}^b$ is in general nonvanishing. The gauge theories for the weak and nuclear interactions are such nonabelian gauge theories. Physically the occurrence of these antisymmetric terms means that the gauge vector boson, the analog of the photon, carries a charge associated with the field sources. This causes the field lines, analogous to the electric and magnetic field lines, to attract each other. Thus the field lines between two particles, that are themselves sources of the field, tend to clump into a tube like structure. If the coupling constant, the term analogous to the electric charge, is very large this tube becomes a very tightly bound structure. In the case of quantum chromodynamics (QCD) there mesons consist of two quarks as sources of the field lines in such a flux tube of field lines, and baryons consist of three quarks that sit in a bubble or bag of such self confined field lines.

Such field theories are in general difficult to obtain real solutions from. These difficulties have two sources. The first is that in QCD

you have three quarks in the bubble, and such 3-body problems are not exactly solvable. This is further compounded by the fact that the virtual quanta are themselves carriers of the various charges and so one essentially has a many body problem as one computes higher order perturbative Feynman diagrams. The second is that if the coupling constant is strong then the perturbation terms in the expansion contribute equally to all orders. This means that in general one has to compute an infinite number of such perturbation terms to determine anything about the theory. Fortunately, in the case of QCD a process called quark antiscreening implies that at sufficiently high energies the quarks behave more freely as the coupling constant is renormalized to a smaller value and this problem is ameliorated. This does mean that nobody knows precisely how to compute the problem of a proton in free space with no interactions with other particles. Lattice gauge methods have been written as algorithms and run on computers and approximate answers have been garnered.

Electromagnetism is considered to be an abelian gauge theory. This is most often expressed according to Maxwell's equations. This theory is remarkably successful, but is called into question when one has nonlinear optical and electromagnetic systems. This occurs when electric permitivities are themselves a function of the electric field. So this term, most often treated as a constant, contributes some term that is a function of the electric field to some power greater than one. It is standard to consider these effects as phenomenology associated with atoms within the medium. However, one can view the occurrence of these atoms as effectively changing the electromagnetic vacuum, and so this physics is ultimately electromagnetic. These nonlinear terms then have the appearance as the magnitude of the elements of the 2-form $q\mathbf{A} \wedge \mathbf{A}$. This is suggestive that electromagnetism may in fact have a deeper nonabelian structure.

An illustration of this fact comes from the nonlinear Schrödinger

equation. This equation describes an electromagnetic wave in a nonlinear medium, where the dispersive effects of the wave in that medium are compensated for by a refocusing property of that nonlinear medium. The result is that this electromagnetic wave is a soliton. Suppose we have a Fabry-Perot cavity of infinite extend in the x direction that is pumped with a laser [6][7]. The modes allowed in that cavity can be expanded in a Fourier series as,

$$\mathbf{E}(x,y,z,t) \;=\; \sum_{m,n} \mathcal{E}(x,t)\phi_{mn}(y,z)e^{-i\omega_0 t} \;+\; h.c.. \qquad (6.3.36)$$

The fundamental wave equation to emerge from Maxwell's equation is

$$\Big(\partial_x^2 \;-\; \frac{1}{c^2}\partial_t^2\Big)\mathbf{E}(x,t) \;=\; \frac{1}{c^2}\partial_t \mathbf{P}(x,t). \qquad (6.3.37)$$

If we input the mode expansion into this wave equation we arrive at the wave equation

$$i\partial_t \mathcal{E} \;=\; -\frac{c}{2k_0}\partial_x^2 \mathcal{E} \;-\; \omega_0 \mathbf{P}(x,t)$$

$$+ \textit{ inhomogenous driving and dissipation terms.} \qquad (6.3.38)$$

We will ignore these inhomogenous terms. The polarization vector is going to have contributions from the linear electric susceptibility and the nonlinear electric susceptibility due to the nonlinear response of the atoms,

$$\mathbf{P} \;=\; \chi_l \mathcal{E}(x,t) \;+\; \chi_{nl}|\mathcal{E}|^2\mathcal{E}. \qquad (6.3.39)$$

With an appropriate redefinition of constants we arrive at the following wave equation for the propagation of field $\mathcal{E}$,

$$i\partial_t \mathcal{E} \;=\; -\frac{c}{2k_0}\partial_x^2 \mathcal{E} \;-\; \omega_0 n|\mathcal{E}|^2\mathcal{E}. \qquad (6.3.40)$$

The solution to this cubic Schrödinger equation is $\mathcal{E} \;=\; \mathcal{E}_l sech(\kappa x)e^{i\omega t}$, where $\kappa \;=\; k_0\sqrt{n|\mathcal{E}|^2}$ which is a soliton wave.

It is noted that the derivation of this equation involves the phenomenological concept of the nonlinear response of the atoms. This

equation is derived based upon the standard abelian theory of electromagnetism, which is linear, and where the nonlinearity obtains by imposing nonlinear material responses. The physical underpinnings of these nonlinearities are not completely described. This soliton wave corresponds to a diphoton, or photon bunches.

It is then advanced that electromagnetism is expanded into a theory with 3 vector potentials and the conjugate product that determines an additional magnetic field,

$$\mathbf{B}^{(3)} = \frac{ie}{\hbar}\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}. \tag{6.3.41}$$

$\mathbf{A}^{(1)}$ is the complex vector potential field and $\mathbf{A}^{(2)} = \mathbf{A}^{(1)*}$ of the electromagnetic field. This additional magnetic field $\boldsymbol{B}^{(3)}$ has been described through the physics of fermion resonance, and with empirical evidence for this magnetic field as given by the optical conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$[8]. This magnetic field may enter into Dirac's theory of the electron so that the interaction of a fermion with this field is

$$E_{int} = -\frac{e\hbar}{2m}\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{B}^{(3)}. \tag{6.3.42}$$

A complete derivation involves a complete expansion of the Pauli Hamiltonian and the recognition that for the two complexified vector potentials $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ that one has the term

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})^2 = \boldsymbol{A} \cdot \boldsymbol{A}^* + i\boldsymbol{\sigma} \cdot \boldsymbol{A} \times \boldsymbol{A}^*. \tag{6.3.43}$$

This ansatz tends to conform to various data, and as will be later pointed out gives predictions of various nonlinear optical effects as well as vortex effects and photon bunching.

This 3-magnetic field has some striking effects. It is easy to see that there are the complex valued electric fields

$\boldsymbol{E}^{(1.2)} = \frac{\partial \boldsymbol{A}^{(1.2)}}{\partial t} = \omega \boldsymbol{A}^{(1.2)}$ So we then see that the magnitude of the optical conjugate product is then I/ω^2 for $I = |\boldsymbol{A} \times \boldsymbol{A}^*|$ defined as the intensity of electromagnetic radiation or optical beam. An exact expression for this magnetic field is then seen to be

$$\boldsymbol{B}^{(3)} = \frac{e\mu_0 c}{\hbar}\frac{I}{\omega^2}\boldsymbol{e}^{(3)} = 5.723 \times 10^{17}\frac{I}{\omega^2}\boldsymbol{e}^{(3)}, \tag{6.3.44}$$

where the constants are evaluated with SI unites. This has some rather aggregious consequences. For visible light this effect is quite small. For a beam of $10 watts/cm^2$ at the visible wavelength $\lambda = 500nm$ the magnetic field is on the order of a nanotesla. However, for a 10 MHz radio frequency wave this magnetic field is 14.5 megatesla. This apparently is a way of generating rather large magnetic fields without the need of massive electromagnets.

The occurrence of the nonlinear Schrödinger equation is then a fairly generic result. For the $\mathbf{A}^{(1)}$ potential we have the magnetic field that is easily seen to be

$$\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)} + \frac{ie}{\hbar}(\boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)}) \times \boldsymbol{A}^{(1)}. \tag{6.3.45}$$

The last term vanishes since the $\mathbf{A}^{(3)}$ photon is found to be very massive in an examination of this approach to electromagnetism embedded in an extended standard model. These issues will be discussed later. This photon decays away and so the $\mathbf{A}^{(3)}$ potential is very short ranged $\simeq 10^{-17}cm$ and is of no consequence to quantum optics. Let $\nabla \times \mathbf{A}^{(1)} = \mathbf{B}_0^{(1)}$. Now compute Maxwell's equation, where $\mathcal{D} = \nabla + (ie/\hbar)(\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)})$ is a covariant form of ∇

$$\mathcal{D} \times \mathbf{B}^1 = \nabla \times \mathbf{B}_0^{(1)} + \frac{ie}{\hbar}(\boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)}) \times \boldsymbol{B}_0^{(1)}, \tag{6.3.46}$$

where $\nabla \times \mathbf{B}^{(3)} = 0$. Now compute $\mathcal{D} \times \mathcal{D} \times \boldsymbol{B}^{(1)}$ to find the covariant wave equation,

$$\mathcal{D} \times \mathcal{D} \times \boldsymbol{B}^{(1)} = \nabla^2 \boldsymbol{B}_0^{(1)} + \left(\frac{e}{\hbar}\right)^2 (|\boldsymbol{A}^{(1)}|^2 + |\boldsymbol{A}^{(2)}|^2)\boldsymbol{B}_0^{(1)}. \tag{6.3.47}$$

Now use $|\boldsymbol{A}^{(1,2)}| = (1/k)|\boldsymbol{B}^{(1,2)}|$ to find

$$\mathcal{D}\times\mathcal{D}\times\boldsymbol{B}^{(1)} = \nabla^2\boldsymbol{B}_0^{(1)} + 2\Big(\frac{e}{k\hbar}\Big)^2|\boldsymbol{B}_0^{(1)}|^2\boldsymbol{B}_0^{(1)}. \qquad (6.3.48)$$

Now $\mathcal{D}\times\mathcal{D}\times\boldsymbol{B}^{(1)} = (1/c^2)\mathcal{D}^2\boldsymbol{E}_0^{(1)}/\partial t^2$ which means that we arrive at the nonlinear equation

$$\nabla^2\boldsymbol{B}^{(1)} + 2\Big(\frac{e}{k\hbar}\Big)^2|\boldsymbol{B}^{(1)}|^2\boldsymbol{B}^{(1)} = \epsilon\mu\frac{\mathcal{D}^2\boldsymbol{E}^{(1)}}{\partial t^2}. \qquad (6.3.49)$$

Now we write the same Fourier of expansion for the electric field and write everything according to the magnetic field intensity $\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$ and we find with the case that $(e/\hbar)A_0 \simeq \omega$ the amplitude fixed to the wavelength as is the case for some solitons, for Gaussian packets, we arrive at the same cubic Schrödinger equation:

$$\frac{c}{k}\partial_x^2\mathcal{H}^{(1)} + 2\Big(\frac{e\mu}{k\hbar}\Big)^2\omega|\mathcal{H}^{(1)}|^2\mathcal{H}^{(1)} = -\frac{i}{c}\frac{\partial\mathcal{H}^{(1)}}{\partial t}. \qquad (6.3.50)$$

The solution to this equation is $Asech(kx)e^{i\omega t}$ which is a soliton solution. In the case where we have nonlinear optics and the occurrence of the cyclic electromagnetic fields the Maxwell's equations for the propagation of an electromagnetic wave are covariant and then give rise to soliton wave equations.

The difference this derivation has in comparison to the previous derivation of the nonlinear Schrödinger equation is that the nonlinearity is more fundamentally due to the nonabelian wave function rather than from material coefficients. In effect these material coefficients and phenomenology behave as they do because the variable index of refraction is associated with nonabelian electrodynamics. Ultimately these two views will merge, for the mechanisms on how photons interact with atoms and molecules will give a more complete picture on how nonabelian electrodynamics participates in these processes. However, at this stage we can see that we obtain nonlinear terms from a nonabelian electrodynamics that is fundamentally nonlinear. This is in contrast to the phenomenological approach that

imposes these nonlinearities onto a fundamentally linear theory of electrodynamics.

6.4 THE QUANTIZED U(1) AND $O(3)_b$ ELECTROMAGNETIC FIELD

The electromagnetic field is quantized as a set of harmonic oscillators. Maxwell's equations, and the resulting wave equations, are described by partial differential equations that formally have an infinite number of degrees of freedom. Physically this means that the electromagnetic field is described by an infinite number of harmonic oscillators, where one sits at every point in space. The modes of the electromagnetic field are then completely described by this ensemble of harmonic oscillators.

The harmonic oscillator has a long history in physics. Galileo noticed, starting as a youth who watched a chandelier swing in the cathedral at Pisa, that a mass attached to a light weight string executed swings through a small angle with a period that was independent of the mass. This oscillation was completely understood with Newton's laws by Robert Hooke. The Hamiltonian for this one dimensional system is

$$H = \frac{1}{2}(p^2 + \omega^2q^2), \qquad (6.4.51)$$

where p and q are the momentum and position variables of the system. Quantum mechanically these variables are replaced by quantum operators $p \to \hat{p}$ and $q \to \hat{q}$. These variables are combined to form ladder operators known as the lowering and raising operators, more often called absorption or annihilation and emission or creation operators,

$$a = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}),\ a^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}). \qquad (6.4.52)$$

These operators allow for the description of the quantum harmonic oscillator that is very parsemonous. The quantum harmonic oscilla-

tor has evenly spaced eigenstates and the state of the system may be changed according to

$$a|n\rangle = \sqrt{n}|n-1\rangle,\ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{6.4.53}$$

It is easy to see that the number operator is written as $N = a^\dagger a$ that are diagonal with respect to the eigenvalues $N|n\rangle = n|n\rangle$ and also define the energy levels for the system since the Hamiltonian is

$$H = \hbar\omega\Big(N + \frac{1}{2}\Big) = \hbar\omega\Big(a^\dagger a + \frac{1}{2}\Big) = \hbar\omega\Big(n + \frac{1}{2}\Big). \tag{6.4.54}$$

A curious aspect to this is that the $n = 0$ state is one that has a nonzero energy $\frac{1}{2}\hbar\omega$.

Now consider an ensemble of harmonic oscillators in three dimensions. Each of these harmonic oscillators has a different frequency $\omega = |\boldsymbol{k}|c$, their own hamiltonian and raising and lowering operators

$$a_{\boldsymbol{k}} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q}_{\boldsymbol{k}} + i\hat{p}_{\boldsymbol{k}}),\ a^\dagger_{\boldsymbol{k}} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q}_{\boldsymbol{k}} - i\hat{p}_{\boldsymbol{k}}). \tag{6.4.55}$$

We then have a description of an infinite number of harmonic oscillators with every possible mode at every point in space. The electromagnetic field is quantized in a cavity with a volume V by defining annihilation and creation operators by redefining these raising and lower operators as

$$a_{\boldsymbol{k}} \to \sqrt{\frac{\hbar}{4\epsilon_0 V}}a_{\boldsymbol{k}},\ a^\dagger_{\boldsymbol{k}} \to \sqrt{\frac{\hbar}{4\epsilon_0 V}}a^\dagger_{\boldsymbol{k}}. \tag{6.4.56}$$

This allows for the expansion of the vector potential into spacial eigenmodes

$$\mathbf{A} = i\sum_{\boldsymbol{k}}\sqrt{\frac{\hbar}{2\omega\epsilon V}}\mathbf{e}(a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} - a_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}). \tag{6.4.57}$$

Here ϵ is the electric permativity and ω is the frequency of the eigenmodes. The abelian magnetic field is then defined by

$$\mathbf{B} = \sum_{\boldsymbol{k}}\sqrt{\frac{\hbar}{2\omega\epsilon V}}\boldsymbol{k}\times\mathbf{e}(a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} + a_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}). \tag{6.4.58}$$

and the electric field is defined by

$$\mathbf{E} = \sum_{\boldsymbol{k}}\sqrt{\frac{\hbar}{2\omega\epsilon V}}\boldsymbol{e}\omega(a^\dagger_{\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} + a_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}). \tag{6.4.59}$$

This is the abelian theory of quantum electrodynamics as a free field uncoupled to charged particles and fermions.

Since there is a nonabelian nature to this theory we return to the nonrelatistic equation that describes the interaction of a fermion with the electromagnetic field. The Pauli Hamiltonian is modified with the addition of a $\boldsymbol{B}^{(3)}$ interaction term [9]

$$H_{B^{(3)}} = H + \frac{e^2}{2m}(\sigma\cdot\boldsymbol{A})(\sigma\cdot\boldsymbol{A}^*), \tag{6.4.60}$$

which may be rewritten according to the algebra of Pauli matrics

$$H_{B^{(3)}} = \frac{e^2}{2m}(\boldsymbol{A}\cdot\boldsymbol{A} + i\sigma\cdot\boldsymbol{A}\times\boldsymbol{A}^*). \tag{6.4.61}$$

If we write this interaction Hamiltonian according to creation and annihilation operators we find that this term can be written as

$$H_{B^3_{int}} = \frac{4\pi e^2\hbar c^2}{m}\sum_{\boldsymbol{k}}\Big(\omega_{\boldsymbol{k}}^{-1}\boldsymbol{I}a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + \sum_{q}(\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}-q})^{-1/2}\boldsymbol{\sigma}^{(3)}\cdot\boldsymbol{n}(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}-q} + a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}-q})\Big). \tag{6.4.62}$$

This interaction Hamiltonian describes the exchange of a photon that results in the change of the spin of the fermion. This process is equivalent to the absorption of a photon in the atomic state transition

$i \rightarrow j$ and the absorption of a photon in the atomic transition $j \rightarrow i$.

Normally one does not worry about the free Hamiltonian term $\frac{1}{2}B^2$, but in the case of the $\boldsymbol{B}^{(3)}$ field this luxury is not afforded to us. This term is written according to the field operators as,

$$H_{B^{(3)}} = \frac{e}{2\omega_q \epsilon_0 V} \sum_{k,k',q} (a^\dagger_{k+q} a_k a^\dagger_{k'-q} a_{k'}). \tag{6.4.63}$$

This term is crucial to the concept of nonabelian electrodynamics. Essentially it describes the interaction between four photons. It describes the absorption of photons with the modes $k+q$ and $k'-q$ and the emission of photons with the modes k and k'. Physically this is a process where two photons mutually interact and exchange momenta. A classical analog of this process is to think of two photons as possessing $\boldsymbol{B}^{(3)}$ fields that are mutually coupled. This is one aspect of nonabelian electrodynamics that is different than standard electrodynamics. An analogous situation occurs with gluons in quantum chromodynamics. Here gluballs can exist which are self-bound states of gluons that are mutually interacting. The nonabelian electrodynamic effect is far simpler since there is no issue of confinement, but the situation is one where photons can interact. This effect is what is a part of the $|\mathcal{H}|^2\mathcal{H}$ term that counters the dispersive effects of an electromagnetic wave as governed by the nonlinear or cubic Schrödinger equation. This is a form of self-focusing or photon bunching that results from this form of mutual interaction between photons.

6.4.1 NUMERICAL RESULTS

This Hamiltonian was numerically examined by the authors. The problem is of considerable difficulty to be solved analytically, and so numerical methods were employed to understand the evolution of a system that obeys this Pauli Hamiltonian with the $\boldsymbol{B}^3$ magnetic

field. To perform the numerics it is necessary to consider only a finite number of possible photon modes. The minimal number to consider is two possible modes where $k_2 = 2k_1$ and where the can be twice as many photons in the k_1 mode as in the k_2 mode. Further, the difference in energy between these modes is set equal to the energy difference between the two spin states of the fermion. The next more complex situation to consider is with the modes $k_3 = 3k_1$ and $k_2 = 2k_1$. Here the k_1 mode can have up to 6 photons, the k_2 mode can have up to 3 photons, and the k_3 mode can have 2 photons. Again the energy difference between these modes is set equal to the energy required to flip the spin state of the fermion. It was this second choice that was numerically examined. To set up this algorithm one start with the wave function $\psi(r,t) = \langle \mathbf{r}|\psi\rangle$ for the state vector

$$|\psi(t)\rangle = \sum_{n_1,n_2,n_3} (C(n_1,n_2,n_3,+)e^{-i(3n_1+2n_2+n_3+\frac{1}{2}\omega)t}|n_1,n_2,n_3,+\rangle$$
$$+ C(n_1,n_2,n_3,-)e^{-i(3n_1+2n_2+n_3-\frac{1}{2}\omega)t}|n_1,n_2,n_3,-\rangle. \tag{6.4.1.64}$$

These amplitudes have photon indices that have a maximum $C(7,4,3,\pm)$, which includes the vacuum state for each mode, and where these indices give the vacuum at $n_{1,2,3} = 1$. This state vector enters into the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \tag{6.4.1.65}$$

This must be computed for each of the finite number of states considered. This results in a set of differential equations for each of the amplitudes $C(n_1,\ n_2,\ n_3,\ \pm)$ that are found by equating terms according to equal state vectors. This set of coupled differential equations is not written explicitly here, but they can be found by the reader without intellectual difficulty, but with some measure of algebraic work.

To start this study it is best to start with the simplest situation with only one n_1 photon that interacts with the fermion. In this problem

there will be no interaction that can couple this photon with the n_2 and n_3 photons by conservation of energy. On physical grounds we expect that this process will be a modified Rabi flopping[10][11]. A numerical examination of the probabilities for the spin up and spin down states yields the result in figure 6.2. The fast Fourier transform (FFT) of the spin state is illustrated in figure 6.3. This roughly has the $1/\omega$ type of spectrum that is typical of the RFR effect. The quantum oscillation of the k_1 photon state is similarly the expected result illustrated in figure 6.4.

An examination of the system that initially starts with a k_2 photon in a cavity with the fermion in the spin up state illustrates that the spin executes more of a nutating oscillation. The FFT of the spin state has the typical $1/\omega$ spectrum, on a gross scale, but with a stair step structure that is found to recur in many of these numerical exercizes. This reflects the fact that the amplitudes of these quantum nutations are do not repeat with each oscillation. An examination of the k_1 photon oscillation reveals a pattern that is regular in periodicity, but irregular in amplitude. This oscillation involves amplitudes for the generation of 1 and 2 photons with this mode. These results are illustrated in figures 6.5-8

The quantum oscillation of the $|2\rangle_1$ state is also regular in periodicity, with the occurrence of "half beats" and "quarter beats" in the oscillation, but irregular in amplitude. Since the photon is coupled to the fermion it is apparent that this beating phenomenon is the result of the stairstep structure seen in the Fourier transform. This randomness also appears for all times in the dynamical evolution. It is apparent that the decay of the fermion state and the $|1\rangle_2$ state generates the state $|2\rangle_1$. Similarly a system that obtains for a system that starts out with 2 photons in the k_1 mode gives generically similar results, illustrated in figures 6.9-12. There does not exist any generation of k_3 photons.

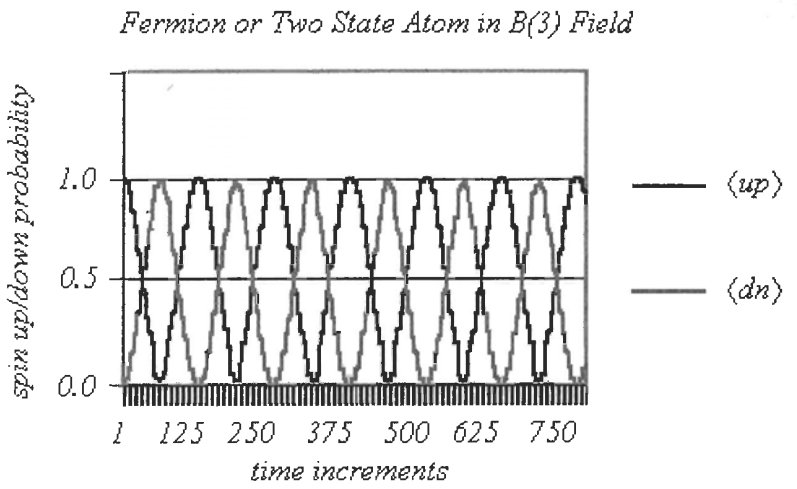


Figure 6.2

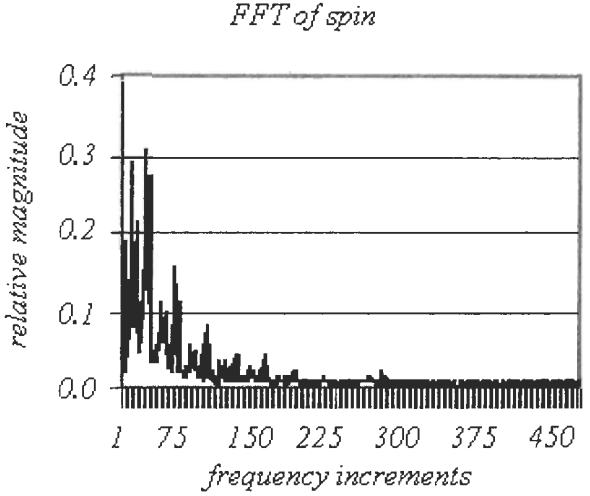


Figure 6.3

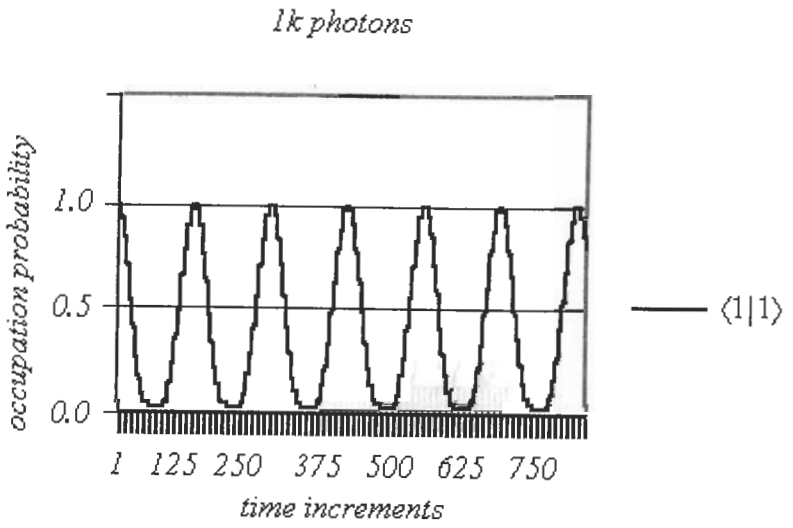


Figure 6.4

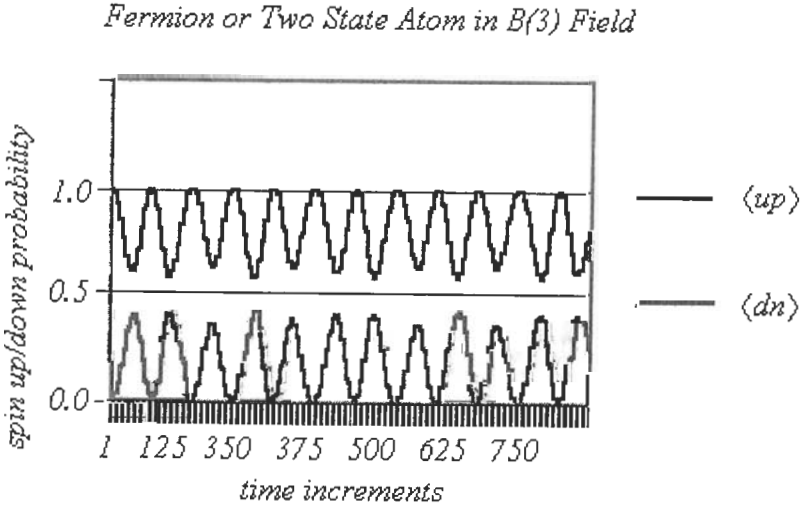


Figure 6.5

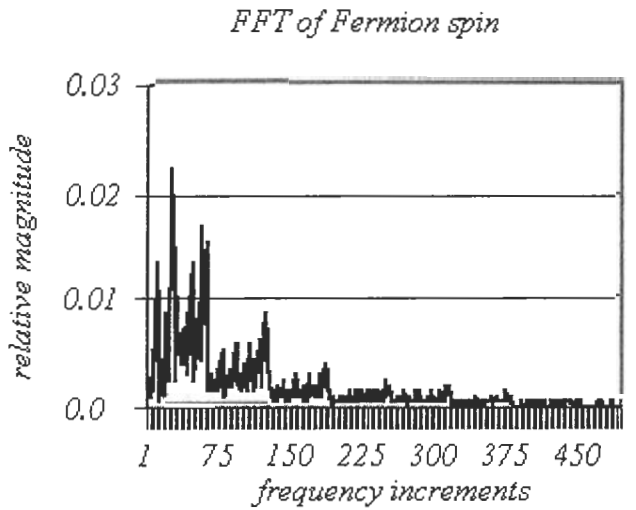


Figure 6.6

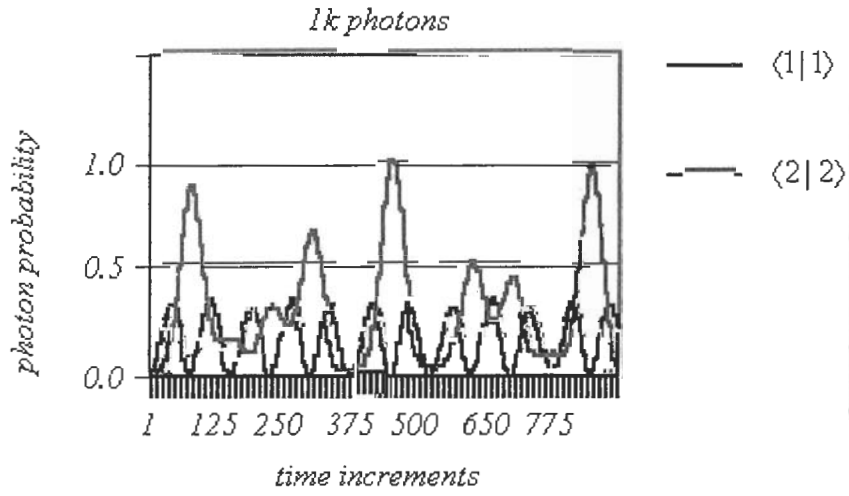


Figure 6.7

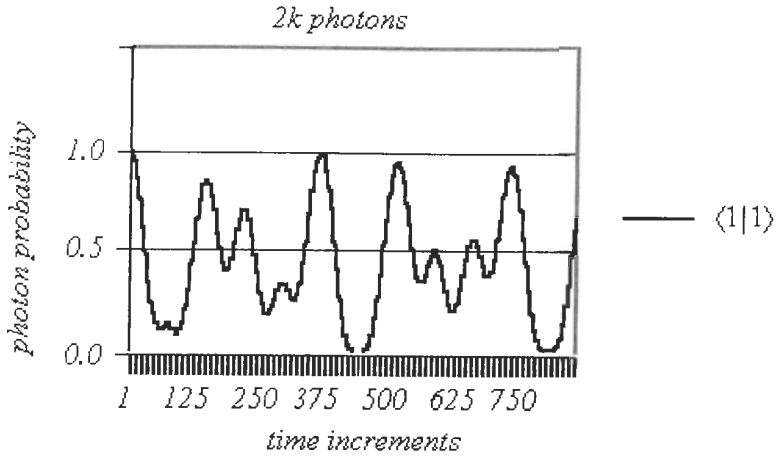


Figure 6.8

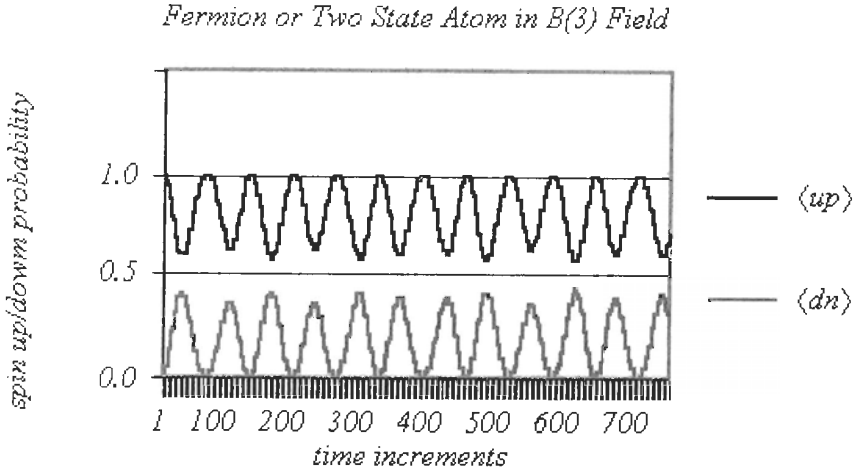


Figure 6.9

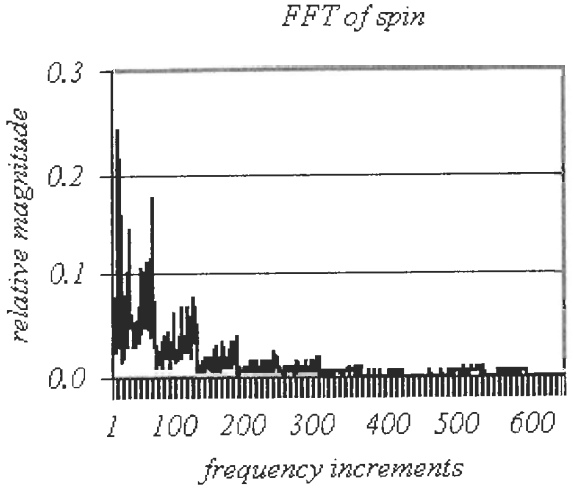


Figure 6.10

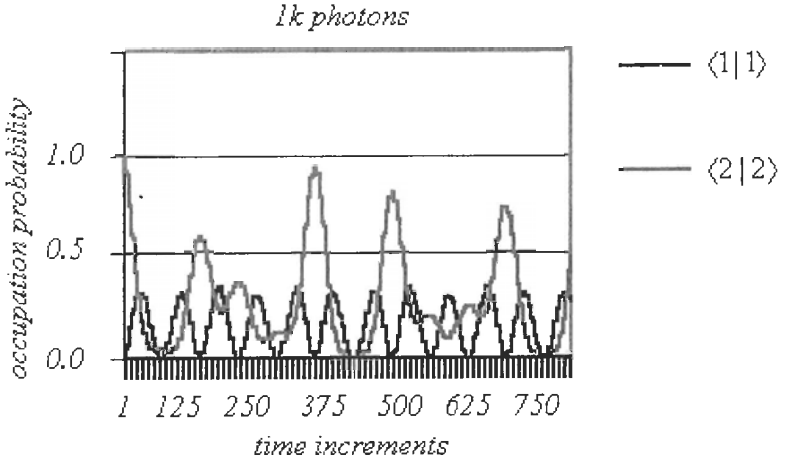


Figure 6.11

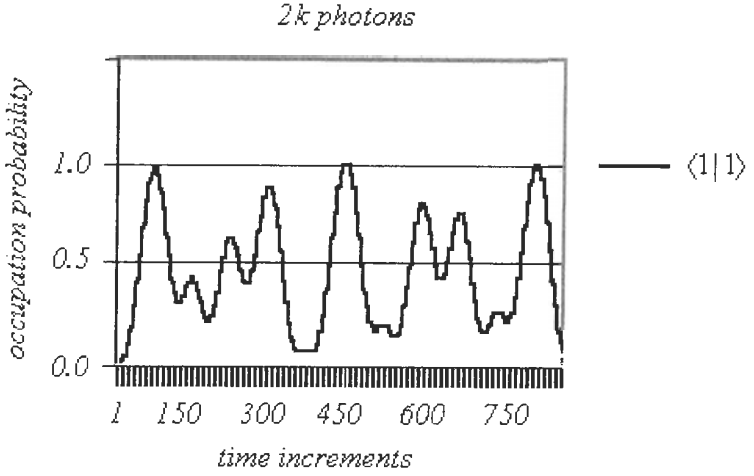


Figure 6.12

This state of affairs continues as we increase the number of initial photons. An example is the case where we start out with 2 photons with the k_1 mode and 1 photon in the k_2 mode. The probability for the spin up and the spin down states is illustrated in figure 6.13. The frequency of mode transition has increased significantly. Further the randomness in amplitude values has markedly increased. The occurrence of maxima is still periodic, but the large maxima occur randomly with these periods. It is apparent that there is an emerging stochasticity in the occurrence of these oscillations. The Fourier transform of the spin states, figure 6.14, reveals the same stair step structure on top of the $1/\omega$ frequency trend. However, each of the steps is marked by the appearance for other peaks within these steps. This will be seen as an important component towards a phase transition in quantum oscillations. The probability for the occurrence of photons in the k_1 and $2k$ modes, in figures 6.15-16, similarly reveals an increasingly rococo structure in the time domain. Here the vacuum state probability is also included. What is interesting is that we begin to see the occurrence of photons with the k_3 modes in figure 6.17. This is even though we did not start out with these photon in the initial state. This process indicates that the Hamiltonian term in equation 6.4.63 couples the k_3 modes with the other modes and generates photons there. The probability of finding a k_3 photon is quite small, but finite.

We now consider the case where there are 2 k_1 photons and 2 k_2 photons in the cavity with the fermion. The spin of the fermion executes a set of oscillations, with some gross measure of recurrence, that appear comparatively random and then abruptly settle into a new set of oscillations that exhibit serious departures. The final state appears to have little resemblance to the quantum oscillations found just after the initial conditions, figure 6.18, and the self similarity of the Fourier transform, figure 6.19, appears to be breaking down. Similarly the probabilities for the occurrence of photons exhibits a similar dramatic change in its dynamics, as illustrated in figures 6.20-22.

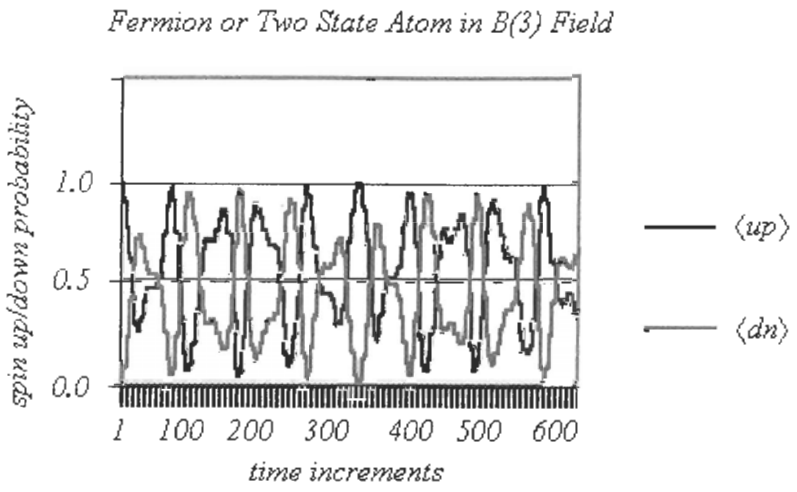


Figure 6.13

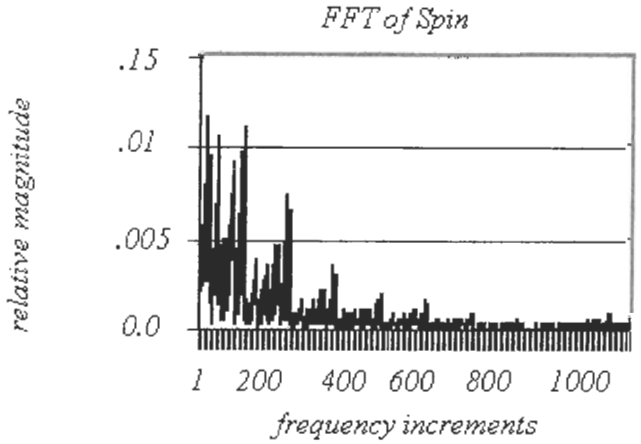


Figure 6.14

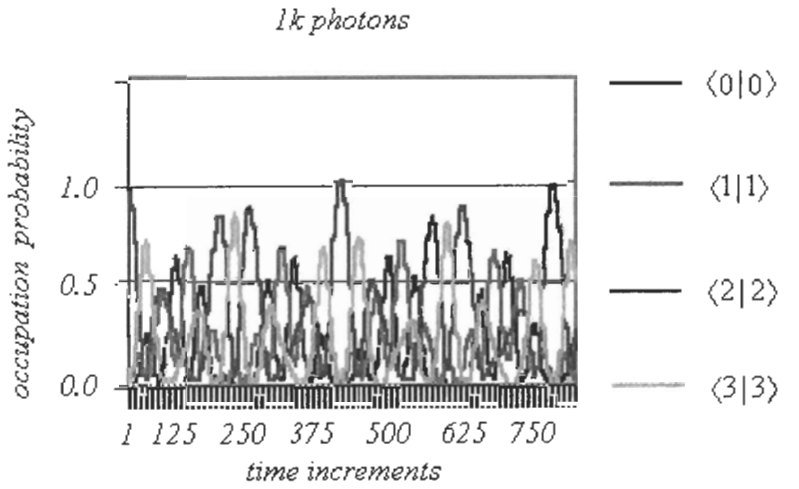


Figure 6.15

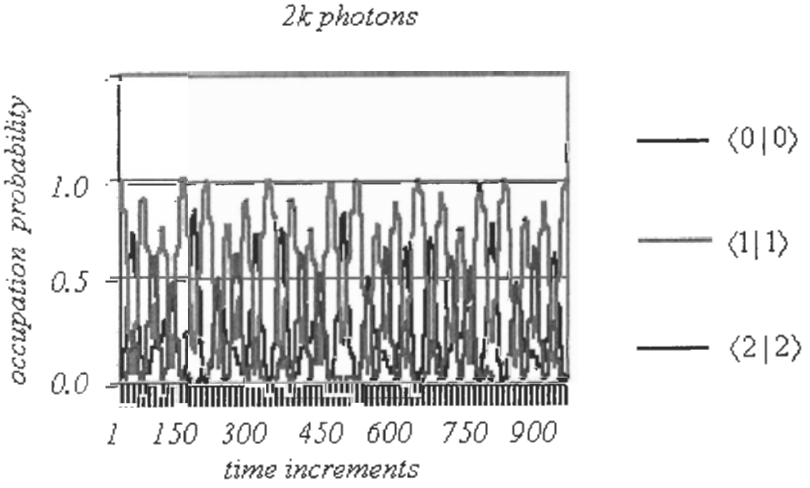


Figure 6.16

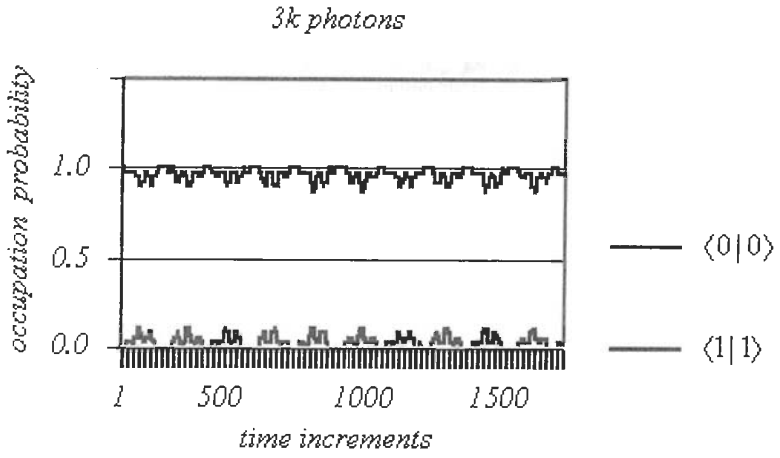


Figure 6.17

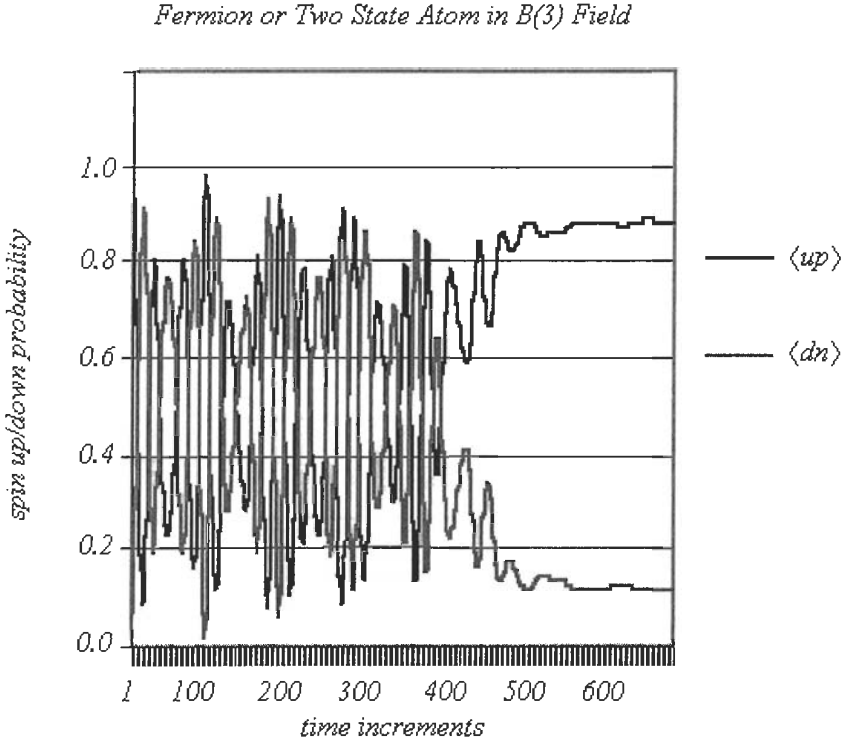


Figure 6.18

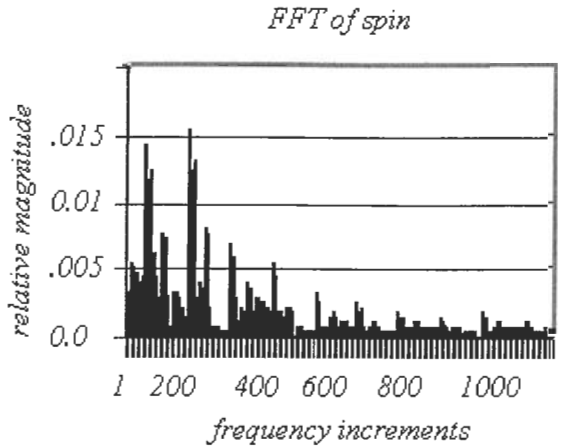


Figure 6.19

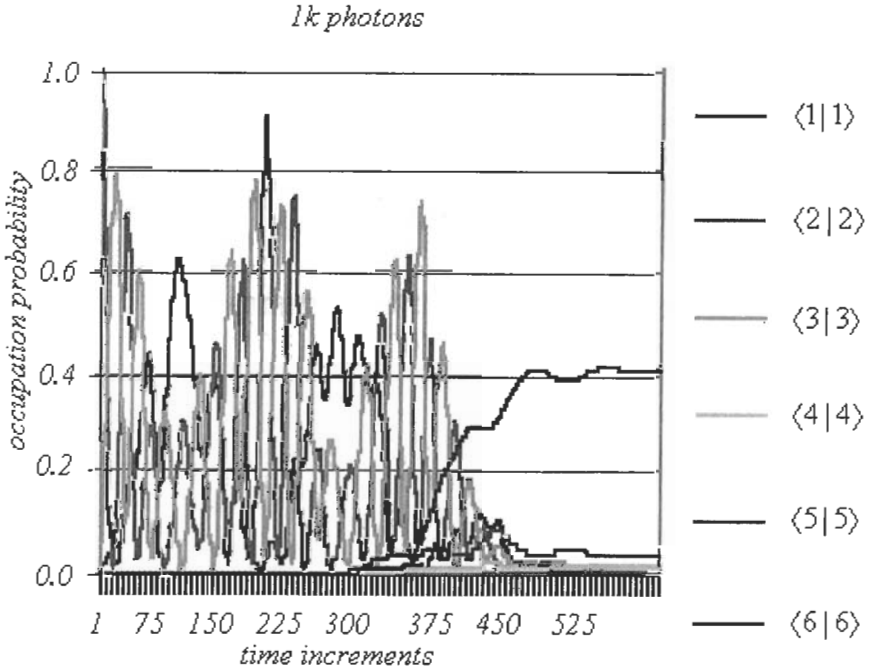


Figure 6.20

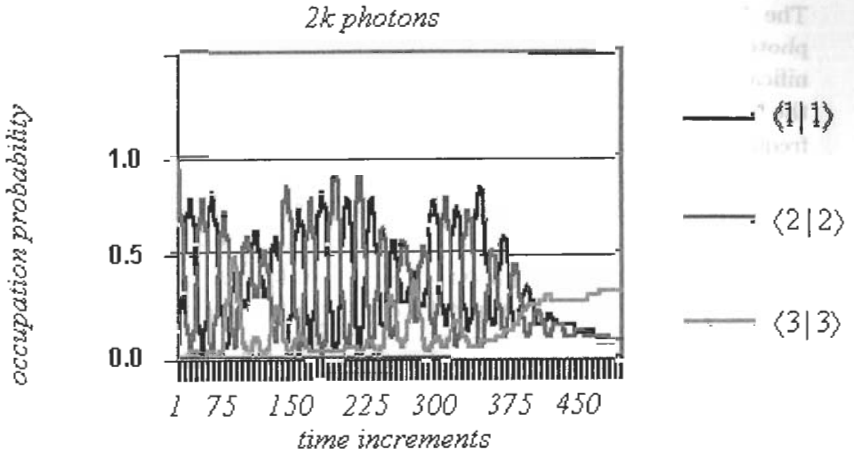


Figure 6.21

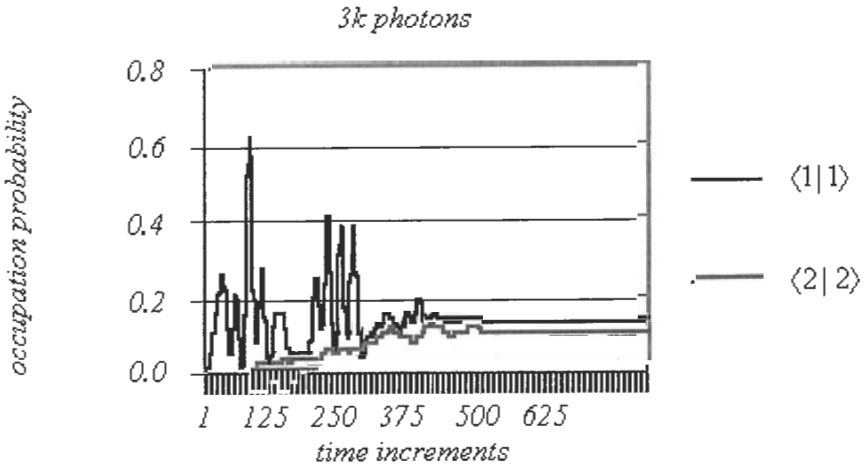


Figure 6.22

The k_1 photons end up primarily in the $n = 6$ state and the k_2 photons end up primarily in the $n = 3$ state. There is also a significant probability for the occurrence of a $n = 1$ photon state in the k_3 mode. In effect we have produced photons with a higher frequency from lower frequency photons. By considering the energies of these states and their probabilities there is an over all energy conservation. What is of primary interest is that the final quantum oscillations exhibit behavior that is utterly independent of the initial conditions. This has the appearance of a transition in behavior that is accompanied by the increased occupation of states with the k_3 modes. This process is governed by the Hamiltonian term in equation 6.4.63. This Hamiltonian term is quartic and it, along with the quadratic Hamiltonian (equation 6.4.62) act as a sort of Landau-Ginsburg potential, or a ϕ^4 type of field in the Higgs mechanism, that induces a transition in the behavior of the system. An examination of the Fourier transform of the spin oscillations, figure 6.18, also demonstrates that there is a near complete breakdown in the staircase structure to the spectrum. This staircase is been broken into a group of peaks representing certain oscillation frequencies that have a degree of self-similarity. However the staircase appears to be replaced this self-similar grouping of frequencies that correspond to a birfurcation in the frequency spectra of these quantum oscillations.

This trend continues as we increase the occupation numbers. In the case that initially we have 4 k_1 photons and 2 k_2 photons we have the same generic behavior, illustrated in figures 6.23-27. Further if the occupation numbers are still increased, in the case below $n_{k_1} = 5, n_{k_2} = 3$, and $n_{k_3} = 1$ it is found that this transition occurs more rapidly. At this point the staircase in the frequency spectrum for the quantum oscillations has turned into a set of recurrent peaks. These final results are contained in figures 6.28-31.

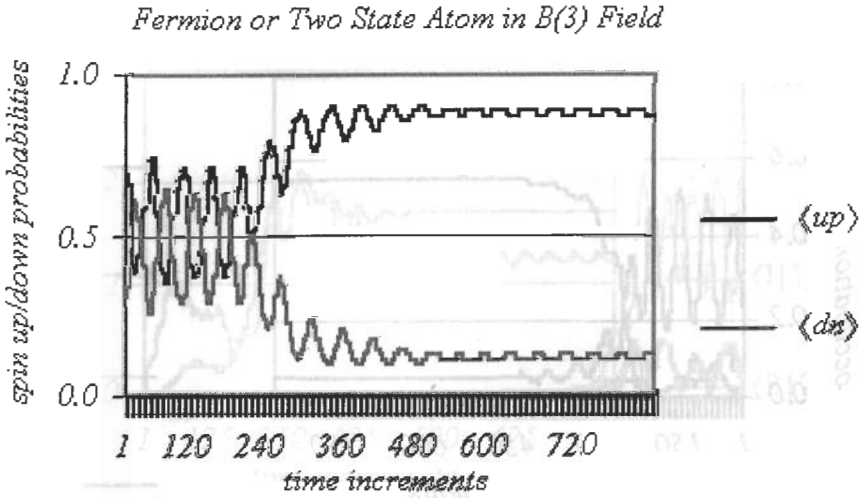


Figure 6.23

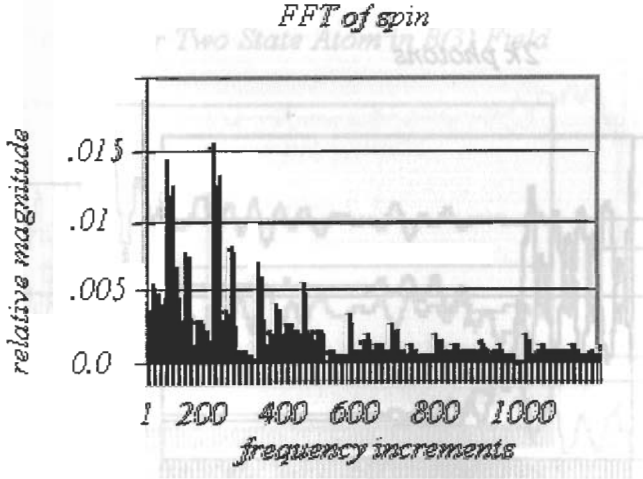


Figure 6.24

1k photons

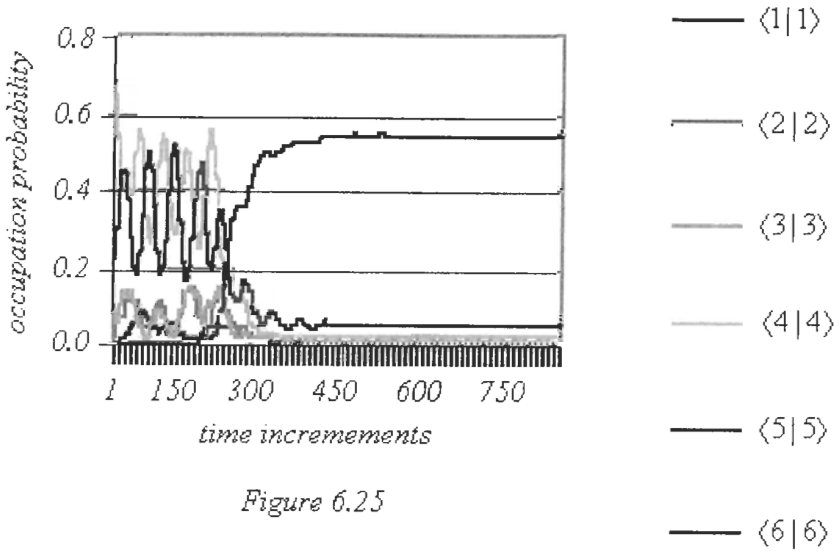


Figure 6.25

2k photons

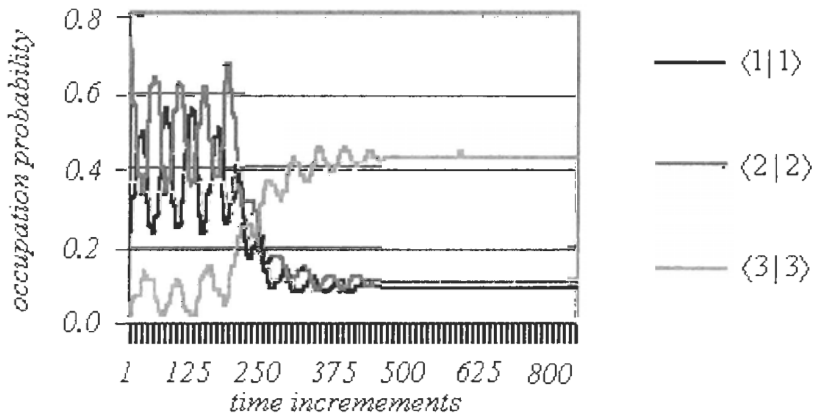


Figure 6.26

3k photons

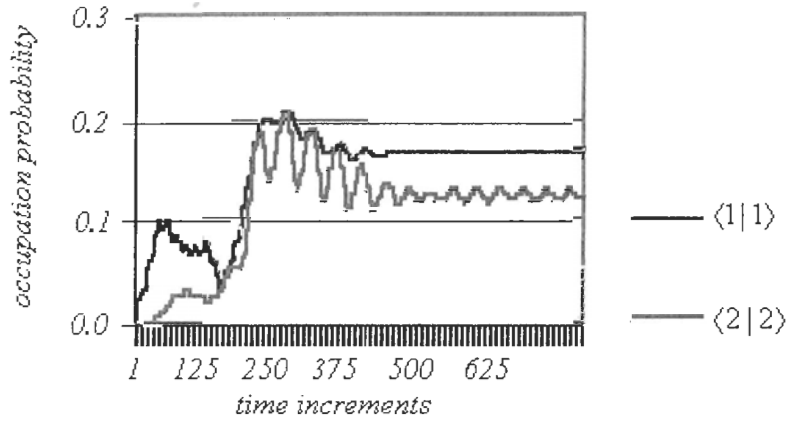


Figure 6.27

Fermion or Two State Atom in B(3) Field

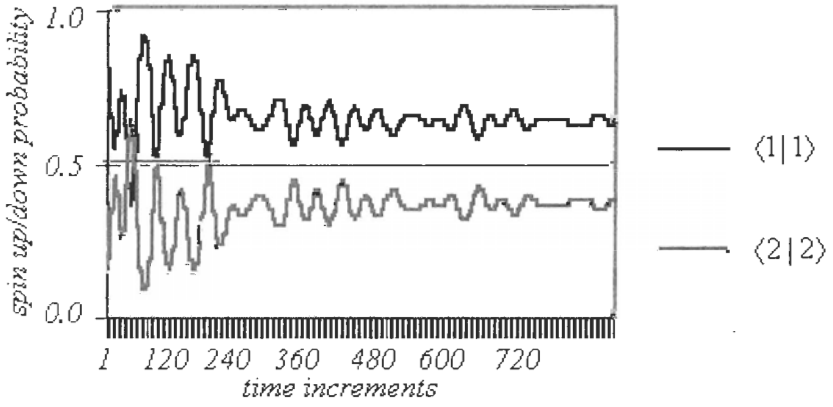


Figure 6.28

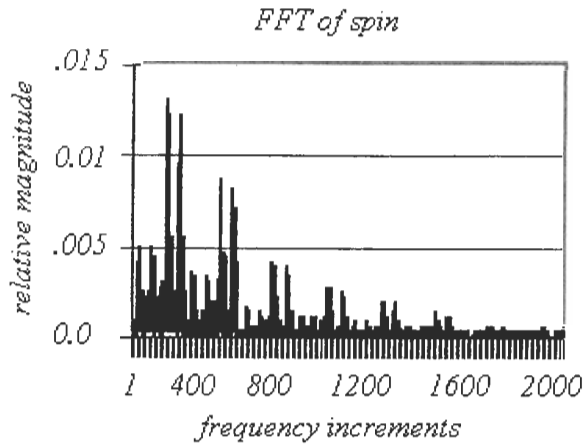


Figure 6.29

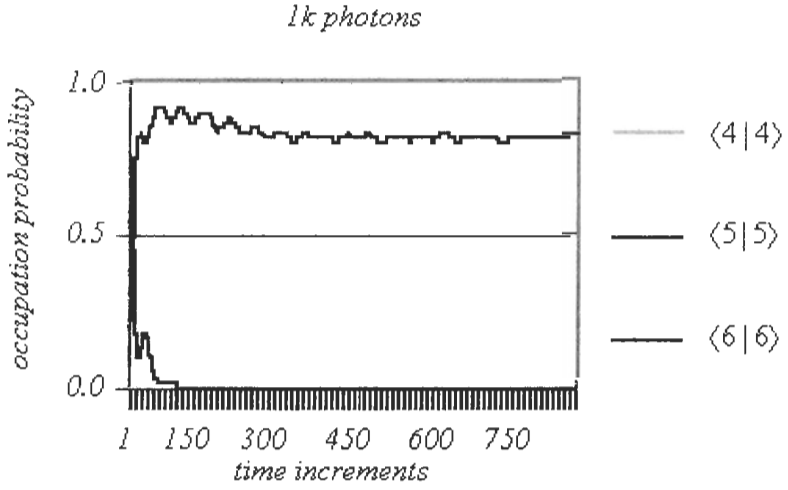


Figure 6.30

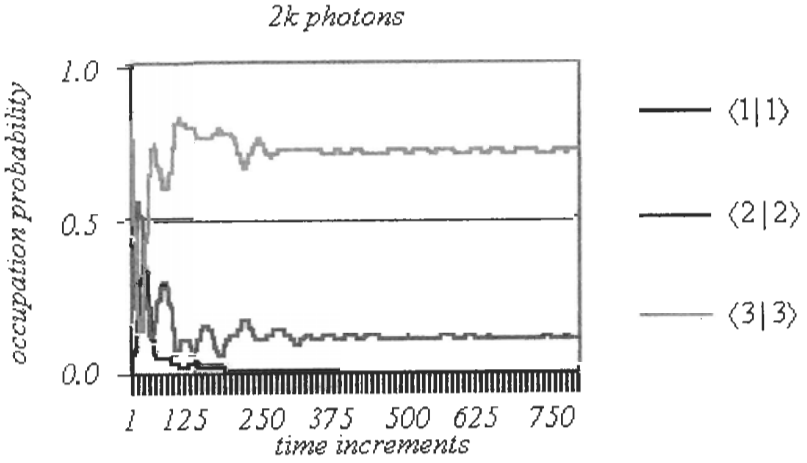


Figure 6.31

6.4.2 DISCUSSION OF THESE RESULTS

A physical description of this process is probably best given by an analogy with the onset of turbulence as given by Landau and Hopf[12][13]. A Hopf bifurcation involves the introduction of new fundamental frequencies in a system. In a classical mechanics setting a cycle, with a certain frequency, as represented by a circle is "blown up" into a torus with a new frequency determined by a cycle that corresponds to the circle that is topologically distinct from the original circle. A succession of such blow up produces a set of frequencies $\omega_1, \omega_2, ..., \omega_n, ...$ In the limit that there are an infinite number of such new frequencies, the classical system transitions into chaotic behavior. It appears as if this quantum system is exhibiting such a set of frequency bifurcations. This can be seen with the comparison of the FFT between the $n_1 = 0, n_2 = 1, n_3 = 0$, $n_1 = 2, n_2 = 2, n_3 = 0$ and $n_1 = 5, n_3 = 2, n_3 = 1$ cases. Here the stairstep spectra is replaced by a repeating set of frequency spikes. This appearance has the suggestion that chaos is involved with this phase transition.

The increased aperiodicity in quantum oscillations suggests that these oscillations correspond classically to windings on tori that approach a torus of irrational winding, similar to the continued fraction expansion advanced by Greene [14]. The abrupt change in behavior indicates that there is some form of breakdown in symmetries and a loss of quantum information. If one time reverses the computer program after this transition the system does not behave in a time reversed manner. The initial transient behavior is not recovered. This appears to be a difficulty for the Schrödinger equation is a time reversal invariant evolution equation. However, in the case of general relativity there are time reversal invariant equations that predict the existence of black holes that have thermodynamic interpretations. It appears that in the case of Schrödinger quantum mechanics there is also a loss of quantum information. The issue of the loss of quantum information is difficult to address, for it is usually associated

with a unit of action h. Yet in quantum chaos it is quite possible that if there are exponential divergences of quantum paths that very small quantum fluctuations can influence the future evolution so that quantum information is continually shuffled through this unit of action.

This preliminary study of nonabelian quantum electrodynamics in a cavity indicates that the occurrence of the $\boldsymbol{B}^3$ field means that the Hamiltonian $\frac{1}{2}(\boldsymbol{B}^3)^2$ contributes to the dynamics of probability amplitudes so as to induce transitional behavior. The Hamiltonian is similar to a quartic potential, where the value of the $\boldsymbol{B}^3$ field determines the value of the potential. Such a potential has two minima, one where $\boldsymbol{B}^3 = 0$ and the other for a finite value of the $\boldsymbol{B}^3$ field. This determines two domains of behavior for the system. The potential minima for a nonzero value of the $\boldsymbol{B}^3$ field corresponds to states that are invariants of the Lagrangian, but not of the vacuum state. It is at this point that the $O(3)_b$ symmetry of nonabelian electrodynamics is broken. An alternative viewpoint is that the energy available is not sufficient to induce quartic photon transitions and so the value of the $\boldsymbol{B}^3$ field is frozen at a value corresponding to the nonzero minimum of the potential. For higher energies, or more and larger occupation numbers corresponding to photon probabilities the symmetry of the $O(3)_b$ symmetry is recovered.

At this stage it should be noted that the Fourier transforms of this system do not indicate a divergence in the limit that $\omega \to 0$. This is in spite of the fact the Hamiltonian has a $1/\omega$ dependency. This infrared diveregence is removed by the statistics with photons. In these cavity examples there simply do not exist the photons with such low energy. If the cavity were thought to expand to allow for the modes with lower frequency then the same will occur. There will still exist a fundamental cut off in the infrared divergence. If we let the cavity become infinite in size then the infrared diveregence can be

removed by the Poisson statistics of photons. $O(3)_b$ electrodynamics is therefore not divergent.

6.5 QUANTUM ELECTRODYNAMICS OF ELEMENTARY SCATTERING

It is often retold about the genius of Einstein, Bohr, Heisenberg and others and their contribution to theoretical physics that lead to the modern physics of quantum theory. Sadly it is not as often remarked that experimentalists contributed their genius towards not only this effort, but also in the way in which nature was to be probed. The first contribution was made by Ernest Rutherford, who demonstrated that the atom had a nucleus by bombarding a gold foil with alpha particles. The scattering of alpha particles in the opposite direction clearly indicated that there existed a hard central nucleus to the atom. This was the first controlled scattering experiment. OE Lawrence devised the first small device that would accelerate charged particles for the purpose of measuring the scattering that resulted when this beam of particles interacted with a target. This philosophy continues in a wide range of experimental devices from the particle accelerators that collide beams in the TEV range of energy to the electronmicrograph where images of biomolecules are imaged from Fourier transforms of the scattered patterns of electrons. This has even lead to the recent development from IBM labs towards the imaging of atoms. Scattering is the grand new experimental paradigm of modern physics that essentially played little role in classical physics.

The study of the electromagnetic field has proceeded along the experimental paradigm of scattering once its wave mechanics were understood. Classical electromagnetism was initially based on the types of experiments done by Michael Faraday with currents that ran through wires arranged in coils. Faraday was able to demonstrate inductive effects by demonstrating the production of transient currents in a coil adjacent to a coil just as a current is turned on. These types of apparatii ultimately lead to the Marconi device that employed

Maxwell's equations to transmit and receive electromagnetic signals. The development of wireless and radio occurred at the time that the modern theoretical and experimental paradigm of physics began to germinate. The first case of wave scattering was Rayleigh scattering of photons with atoms, which gives the prediction of a blue sky. Here the wavelength of the photon is much larger than the atom. Further, wave scattering studied involved the scattering of electromagnetic radiation with aerosol particles by Gustav Mie and the study of atmospheric optics and the understanding of rainbows, glories and St. Petersburg displays. One of the earliest examples of scattering theory involving charged particles and the production of electromagnetic radiation was the recognition that accelerated charges emitted Bremsstrahlung (braking) radiation. This was primarily responsible for the emission of radiation due to the high energy collisions of charged ions. This discovery lead to the X-ray machine and its early impact on noninvasive medical diagnostics. This was one of the earliest treatments of how the scattering between two charged particles would lead to the emission of radiation.

In quantum electrodynamics the entities examined are charged particles, most often electrons, and photons. Either, one examines the scattering between two electrons, and the subsequent electromagnetic response plus the scattering angles of the two electrons are measured, or the scattering between a photon and an electron in the case of Compton radiation. With the advent of the Dirac theory of the electron we are confronted with the negative energy state of the positron, or antielectron. In a fully relativistic setting one must include scattering that emits or absorbs positrons.

We then have four possible cases for the scattering of the electron state. In all four cases we have that the initial electron state ψ_i is scattered into the final electron state ψ_f. However, these electron states can be either postive or negative, corresponding to the

electron and positron. Further, we interpret the propagation of an electron state of negative energy as the positron of positive energy propagating from the future to the past. So we have the four cases with $\psi_i \rightarrow \psi_f$:
ψ_i in the past scatters to ψ_f in the future for electron scattering,
ψ_i in the future scatters to ψ_f in the past for positron scattering,
ψ_i in the past scatters to ψ_f in the past for electron-positron pair annihilation,
ψ_i in the future scatters to ψ_f in the future for electron-positron pair creation.
The third of these processes was first predicted by Paul Dirac and subsequently observed by Anderson[16].

In describing these processes we need to have a way of describing the propagation of these fields through space and time. We have the wave equations for these fields and we know that the wave function is the solution to this wave equation. A propagator tells us how to promote these wave fields from one point in space and time to another. So this propagator must be the kernal of the wave operator. In the case of nonrelativistic quantum theory the wave equation is the Schrödinger equation. In the case that we introduce a perturbing potential $\mathcal{V}$ we have that the wave function satisfies

$$\Big(i\hbar\frac{\partial}{\partial t} - H_0 - \mathcal{V}\Big)\psi = 0. \tag{6.5.66}$$

The free propagator is defined for $\mathcal{V} = 0$ and is the kernal for the amplitude that evolves from $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$

$$\Big(i\hbar\frac{\partial}{\partial t} - H_0(\boldsymbol{x}_2)\Big)G(\boldsymbol{r}_1, \boldsymbol{r}_2) = i\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2). \tag{6.5.67}$$

The propagator for this unperturbed Schrödinger operator is

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = Nexp\Big(\frac{1}{2}im\frac{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2}{t_2 - t_1}\Big)\Theta(t_2 - t_1), \tag{6.5.68}$$

where $\Theta(t_2 - t_1)$ is the Heaviside function that vanishes for $t_2 > t_1$ and N is the normalization factor $N = \sqrt{m/2\pi i(t_2 - t_1)}$. This

propagator is used to find a wave function $\psi(\boldsymbol{r}_2, t_2)$ given a wave function $\psi(\boldsymbol{r}_1, t_1)$ at an earlier time $t_2 > t_1$

$$\psi(\boldsymbol{r}_2, t_2) = \int G(\boldsymbol{r}_1, \boldsymbol{r}_2, t_2, t_1)\psi(\boldsymbol{r}_1, t_1)d^3\boldsymbol{r}_1. \tag{6.5.69}$$

This propagator is the trivial case, but can be used to estimate propagators in the case that $\mathcal{V} \neq 0$. In general we have a propagator that satisfies the equation

$$\Big(i\hbar\frac{\partial}{\partial t} - H_0 - \mathcal{V}\Big)G_{\mathcal{V}}(\boldsymbol{r}_1, \boldsymbol{r}_2) = i\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2). \tag{6.5.70}$$

In general this kernal can not be exactly computed in an analytical manner. In practice one must then find a series approximation for this kernal.

The complete kernal can be represented by the eigenstate solutions to the wave functions as

$$G_{\mathcal{V}}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \sum_n e^{-iE(t_2 - t_1)}\psi_n(\boldsymbol{r}_2))\psi_n^*(\boldsymbol{r}_1). \tag{6.5.71}$$

We feed the kernal for the unperturbed Schrödinger operator into the integral

$$G_{\mathcal{V}}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = G_0(\boldsymbol{r}_1 - \boldsymbol{r}_2) + i\int G_{\mathcal{V}}(\boldsymbol{r}_1 - \boldsymbol{r}_3)\mathcal{V}G_0(\boldsymbol{r}_2 - \boldsymbol{r}_3)d\boldsymbol{r}_3^3dt_3, \tag{6.5.72}$$

and proceed with these interated substutions to arrive at the series

$$G_{\mathcal{V}}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = G_0(\boldsymbol{r}_1 - \boldsymbol{r}_2) + i\int G_0(\boldsymbol{r}_2 - \boldsymbol{r}_3)\mathcal{V}G_0(\boldsymbol{r}_1 - \boldsymbol{r}_3)d\boldsymbol{r}_3^3dt_3 -$$

$$\int\int G_0(\boldsymbol{r}_1 - \boldsymbol{r}_3)\mathcal{V}(\boldsymbol{r}_3)G_0(\boldsymbol{r}_3 - \boldsymbol{r}_4)\mathcal{V}(\boldsymbol{r}_4)G_0(\boldsymbol{r}_2 - \boldsymbol{r}_4)d\boldsymbol{r}_4^3d\boldsymbol{r}_3^3dt_4dt_3. \tag{6.5.73}$$

Since the propagator allows us to find the quantum state of a system at a latter time this can further be used to find the probability

transition. The amplitude for state transitions as the wave function is propagated from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ is then

$$A_{1,2} = \int\int \psi(\boldsymbol{r}_2)G(\boldsymbol{r}_2,\ \boldsymbol{r}_1,\ t_2,\ t_1)\psi(\boldsymbol{r}_1). \tag{6.5.74}$$

The propagator may be expanded in a series.

We now make the leap to relativistic quantum mechanics. For a fermion the basic wave equation is the Dirac equation

$$i\gamma^\mu(\partial_\mu + qA_\mu)\psi + m\psi = 0 \tag{6.5.75}$$

Using the fact that $\gamma^0\gamma_i = \alpha_i$ we can write this wave equation as

$$i\frac{\partial}{\partial t}\psi = \Big(\boldsymbol{\alpha}\cdot(\boldsymbol{p} - q\boldsymbol{A}) + \mathcal{V} + m\beta\Big)\psi. \tag{6.5.76}$$

In the same manner a kernal exists that is the solution to the equation

$$\Big(i\frac{\partial}{\partial t} - \boldsymbol{\alpha}\cdot(\boldsymbol{p} - q\boldsymbol{A}) - \mathcal{V} + m\beta\Big)G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \delta(\boldsymbol{r}_2 - \boldsymbol{r}_1). \tag{6.5.77}$$

The free particle kernal is obtained in the case that the electromagnetic vector potential and the scalar potentials are constant and can be set to zero. And so we define the propagator for the free particle as the kernal that satisfies the equation,

$$\Big(i\frac{\partial}{\partial t} - \boldsymbol{\alpha}\cdot\boldsymbol{p} - m\beta\Big)G_0(\boldsymbol{r}_1, \boldsymbol{r}_2) = \delta(\boldsymbol{r}_2 - \boldsymbol{r}_1). \tag{6.5.78}$$

The kernal for this operator is defined by the spinor solutions to the Dirac equation

$$G_0(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \sum_n e^{-iE(t_2 - t_1)}\big(\psi^0_n(\boldsymbol{r}_2)\psi^{0*}_n(\boldsymbol{r}_1)|_+ + \psi^0_n(\boldsymbol{r}_2))\psi^{0*}_n(\boldsymbol{r}_1)|_-\big), \tag{6.5.79}$$

which involves a sum over positive energy and negative energy solutions. The perturbed propagator is then

$$G(\boldsymbol{r}_1 - \boldsymbol{r}_2) = G_0(\boldsymbol{r}_1 - \boldsymbol{r}_2) + i\int G_0(\boldsymbol{r}_2 - \boldsymbol{r}_3)\boldsymbol{\alpha}\cdot\boldsymbol{A}G_0(\boldsymbol{r}_1 - \boldsymbol{r}_3)d\boldsymbol{r}_3^3dt_3$$

$$- \int\int G_0(\boldsymbol{r}_1 - \boldsymbol{r}_3)\boldsymbol{\alpha}\cdot\boldsymbol{A}(\boldsymbol{r}_3)G_0(\boldsymbol{r}_3 - \boldsymbol{r}_4)\boldsymbol{\alpha}\cdot\boldsymbol{A}(\boldsymbol{r}_4)G_0(\boldsymbol{r}_2 - \boldsymbol{r}_4)d\boldsymbol{r}_4^3d\boldsymbol{r}_3^3dt_4 \tag{6.5.80}$$

This propagator describes the evolution of a wave function under the influence of the potentials $\boldsymbol{\alpha}\cdot\boldsymbol{A}$ interpreted as the amplitude per volume per second. The Dirac equation is a 4×4 matrix that is iteratively built up from the exterior products of the spinor that compose the wave function for the free fermion.

The simplest example is the case of coulomb scattering. In this case we have the coulomb potential $A_t = \Phi = e/r$ and the wave functions are $\psi_{1,2} = u_{1,2}exp(\boldsymbol{p}_{1,2}\cdot\boldsymbol{r})$. The unperturbed propagator is then

$$G(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \sum_n e^{-iE(t_2 - t_1)}\big(\psi_n(\boldsymbol{r}_2)\psi 0^*_n(\boldsymbol{r}_1)|_+ + \psi_n(\boldsymbol{r}_2))\psi^*_n(\boldsymbol{r}_1)|_-\big). \tag{6.5.81}$$

This must now be expanded. The first order amplitude for the transition from the i^{th} state to the j^{th} state is calculated to be

$$a_{ij} = -i(u_2^\dagger\gamma_0u_1)\int e^{-i\boldsymbol{p}_2\cdot\boldsymbol{r}}\frac{e^2}{r}e^{i\boldsymbol{p}_1\cdot\boldsymbol{r}}\int_0^t dtdt$$

$$= -i(u_2\gamma_0u_1)\Phi(|\boldsymbol{p}_2 - \boldsymbol{p}_1|)\Big(\frac{e^{i(E_2 - E_1)t'}}{i(E_2 - E_1)}\Big). \tag{6.5.82}$$

We have two integrals here. The integral over the momenta is a Fourier transform of the Coulomb potential $\Phi(|\boldsymbol{p}_2 - \boldsymbol{p}_1|) = 4\pi e^2/|\boldsymbol{p}_2 - \boldsymbol{p}_1|^2$, and the second integral gives the same result found in the Fermi golden rule. The transition probability for this process is then given by the same density of states argument that $prob(i \to j)/sec \simeq |a_{ij}|^2(density\ of\ states)$. This leads to the cross section for the scattering process

$$\sigma = \frac{4e^2}{(|\boldsymbol{p}_2 - \boldsymbol{p}_1|)^4}d\Omega|u_2\gamma_0u_1|^2. \tag{6.5.83}$$

The calculation of the term $u_2\gamma_0u_1$ comes from the relativistic Dirac equation. This is found to be $|u_2\gamma_0u_1|^2 = |(E+m)^2 + p_2p_1|^2/(E+m)^2$.

The difference between p_1 and p_2 is the phase given by the angle at which the two plane waves scatter off from each other. So for $p_2 = p_1 e^{-i\theta}$. This gives the scattering cross section

$$\sigma = \frac{4e^2E^2}{(|p_2 - p_1|)^4}(1 - v^2 sin^2(\theta/2))d\Omega. \tag{6.5.84}$$

This is the result for Compton scattering.

This is an illustration of how photons and electrons scatter according to the $U(1)$ theory of electromagnetic theory. This result just presented is a fairly basic result involving the tree level scattering of electrons and photons. In chapter 8 this subject will be more completely developed for $O(3)_b$ electrodynamics.

6.6 PHYSICS OF QUANTUM ELECTRODYNAMICS OF ELECTRONS AND PHOTONS WITH THE B(3) FIELD

The electromagnetic field in full generality involves the coupling of charged fermions with the photon field. This is the case with an electron in a region free of photons, as well as for a photon in a region free of fermions. An electromagnetic wave is essentially a disturbance in the electric field that propagates away from the source, and in so doing is associated with a disturbance in the magnetic field. So the wave is composed of an oscillating electric and magnetic fields that are perpendicular to each other, and where both are perpendicular to the direction the wave is traveling. The photon is basically the same thing, but where these fields are quantized.

Now on a somewhat deeper level these quantized electric and magnetic fields exist in the vacuum that is composed of virtual photons. These virtual photons are the result of Heisenberg uncertainty fluctuations in the electric and magnetic fields. These fluctuations form the first order terms that describe the vacuum. The second order

terms involve fluctuations with electrons and positrons (I am ignoring virtual pairs of other more massive particles). Now these virtual pairs are randomly oriented in the vacuum. However, if there is an electric field present these virtual pairs will preferentially occur so that the virtual charge separation is aligned, or polarized, with this electric field. So a photon, with its oscillating electric field, will be associated with these virtual pairs of electrons and positrons that are polarized with the photon electric field. In the more formal language of QED this is represented with Feynman diagrams.

In this picture it must also be pointed out that the magnetic field is oriented perpendicular to the plane inscribed by a completely polarized electron-positron pair. The virtual electron-positron accompanies a virtual electric and magnetic field. The charges of the virtual pair will separate under the influence of the photon electric field, and the magnetic field lines of the virtual electron-positron pair will preferentially align with the magnetic field of the photon. Let us assume that we have the vector potentials

$$\boldsymbol{A}^{(1)} = \frac{1}{\sqrt{2}}A^{(0)}(\boldsymbol{i} + i\boldsymbol{j})e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} = \boldsymbol{A}^{(2)\dagger}. \tag{6.6.85}$$

We then have that the magnetic field is $\boldsymbol{B}^{(1,2)} = i\boldsymbol{k} \times \boldsymbol{A}^{(1,2)}$ and that the electric field is $\boldsymbol{E}^{(1,2)} = i\omega\boldsymbol{A}^{(1,2)}$. Now we consider the vacuum fluctuations that surround these fields. Here we are saying that quantum theory is the action of the vacuum on particles and fields. This means that we have the fields $\boldsymbol{B}^{(1,2)} + \delta\boldsymbol{B}^{(1,2)}$ and $\boldsymbol{B}^{(1,2)} + \delta\boldsymbol{B}^{(1,2)}$, where the variational piece added on are the quantum fluctuations. Now we have in spacetime the differential form $\boldsymbol{F} = d\boldsymbol{A}$. This is written in spacetime as

$$\boldsymbol{F} = F_{\mu\nu}d\boldsymbol{x}^\mu \wedge d\boldsymbol{x}^\nu. \tag{6.6.86}$$

The Yang-Mills functional is defined by the integration of the wedge product $\boldsymbol{F} \wedge *\boldsymbol{F}$, where * is the Hodge dual star operator

$$8\pi^2 k = \int_{(\mathcal{M},g)} F_{\mu\nu}F_{\alpha\beta}d\boldsymbol{x}^\mu \wedge d\boldsymbol{x}^\nu \wedge d\boldsymbol{x}^\alpha \wedge d\boldsymbol{x}^\beta, \tag{6.6.87}$$

where k is the instanton number. Now the electric and magnetic fields on the spacial manifold of three dimensions are $E_i = \epsilon_{0ji}F^{0j}$ and $B_i = \epsilon_{kji}F^{kj}$. With the application of the antisymmetric symbols we find that the Yang-Mills functional is

$$16\pi^2 k = \int \Big([E_i,\ B_j] + [\delta E_i,\ \delta B_j] \Big) d^4x. \tag{6.6.88}$$

This leads to the equal time commutator

$$[\delta E_i^a(r,\ t),\ \delta B_j^b(r',\ t)] = \hbar\delta_{ij}\delta^{ab}\delta(r - r')\delta(t - t'), \tag{6.6.89}$$

where we have included the $O(3)_b$ indices. We see that quantum mechanically the electric and magnetic fields are the conjugate variables, and that the uncertainty relationship is dictated by the fluctuations in these fields in the vacuum.

These field fluctuations in the vacuum will couple to the photon's electric and magnetic fields. In the case of the magnetic field we will have a fluctuation in the interaction energy that is,

$$\delta E = \int \delta(j \cdot A) = \int H \cdot \delta B d^3r. \tag{6.6.90}$$

This fluctuation in the magnetic field can be estimated from the flux quantization of the magnetic field $\mathcal{F} = 2\pi\hbar/e$ that is obtained from SQUID and Josephson junction measurements. It is this term that is responsible for the Lamb shift in the energy level of a hydrogen atom. We may then say that the magnetic field fluctuation is the magnetic flux quanta times the small area enclosed by the electron-positron pair. This small area is determined by the coordinate fluctuation of the electron and positron. This can be estimated by using the energy fluctuation as $\delta E = \delta mc^2$, the uncertainty relation between the energy and time $\delta E\delta t = \hbar$ and the uncertainty in the position as $\delta x = c\delta t$. We find that the magnetic field fluctuation is approximately 5.6×10^9 gauss over a range of around $10^{-13}cm$ that occurs for around 10^{-23}seconds. We see that this magnetic field fluctuation

is quite strong, but short ranged and short lived. This means that electromagnetic physics that is performed above the scale of $10^{-13}cm$ will be effected less and less by these fluctuations as the scale of the physics measured increases above this range. The energy at which one needs to examine quantum electrodynamics on this scale is above 1 Mev. It is interesting to note that this occurs at the classical radius of the electron. Figure 6.32 illustrates this polarization of virtual electron-positron pairs due to a magnetic field.

Polarization of electron-positron pairs in the vacuum

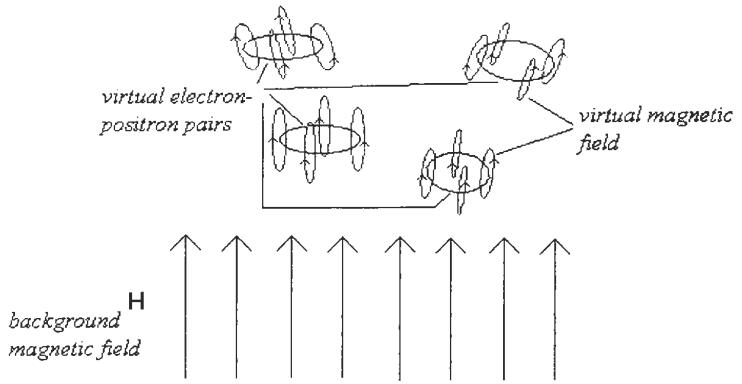


Figure 6.32

This can then be used to estimate the energy fluctuation in a photon associated with the mutual inductance between the photon magnetic field and the quantum fluctuation in the magnetic field. The magnetic field for a photon is given by the relationship $|B| = \sqrt{4\pi\hbar\mu_0\omega}$, which for a 500nm optical photon is around 3×10^{-9}gauss. This means that the virtual work done on this photon by the magnetic field fluctuation on the scale of $10^{-13}cm$, is then approximately 2.13×10^8 ergs. To estimate the effect of this perturbation on an atomic energy level requires a more complete calculation by computing the expectation of this over a long time wiht the atomic basis states. This is compared to the energy of the photon which is

6.3×10^{-18} ergs. On the scale of $10^{-13}cm$ the electric and magnetic field of the optical photon is literally lost in a chaotic sea of virtual field fluctuations. A photon has to have an energy on the order of 1Mev in order for it to have appreciable effects on this scale.

Since $O(3)_b$ electrodynamics predicts the existence of the $\boldsymbol{B}^{(3)}$ field this field must also have an effect on the stochastic motion of an electron on a fine scale. Again we have fluctuations of the $\boldsymbol{B}^{(3)}$ and the $\boldsymbol{E}^{(3)}$ fields,

$$[\delta E_i^{(3)}(\boldsymbol{r},\ t),\ \delta B_j^{(3)}(\boldsymbol{r}',\ t)] \ = \ \hbar\delta_{ij}\delta(\boldsymbol{r} \ - \ \boldsymbol{r}')\delta(t \ - \ t'). \quad (6.6.91)$$

So far there is no evidence pointing to the existence of a classical $\boldsymbol{E}^{(3)}$ field, but it should be expected that there will exist fluctuations of this field. We then expect that the fluctuation in the $\boldsymbol{B}^{(3)}$ field will be of the form

$$\delta\boldsymbol{B}^{(3)} \ = \ \frac{e}{\hbar}\Big(\delta\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \ + \ \boldsymbol{A}^{(1)} \times \delta\boldsymbol{A}^{(2)}\Big). \quad (6.6.92)$$

Now we use the fact that the magnetic vector potentials have the magnitude $|\boldsymbol{B}^{(1,2)}|/k$. So the magnitude of the $\boldsymbol{B}^{(3)}$ is then

$$|\delta\boldsymbol{B}^{(3)}| \ = \ \frac{2e}{\hbar k^2}\Big(|\delta\boldsymbol{B}||\boldsymbol{B}|\Big). \quad (6.6.93)$$

To obtain estimates we proceed with the same analysis as before. Again we consider the fluctuations in the $\boldsymbol{B}^{(3)}$. The fluctuation in the ordinary magnetic field in this expression is $\delta\boldsymbol{B} \ = \ (\pi/2)(\delta m)^2/(e\hbar)$, which again is around approximately 5.6×10^9 gauss. The magnetic field associated with the photon, without quantum fluctuations, is around 3×10^{-9}gauss, the $\boldsymbol{B}^{(3)}$ field fluctuation is then around .006 gauss. So the $\boldsymbol{B}^{(3)}$ field fluctuations that result from virtual electron-positron pairs is estimated to be ten orders of magnitude smaller than the standard magnetic field. This $\boldsymbol{B}^{(3)}$ fluctuations should then give measurable contributions to vacuum QED physics in the 10 Gev range of energy.

This means that the vacuum contribution to the virtual $\boldsymbol{B}^{(3)}$ field is likely to be a very small effect that will be difficult to measure. However, it will have a contribution to some measurable physics. The virtual energy of this fluctuation is estimated to be around 10^{-12}ergs. It is then estimated that the shift in the atomic level will be 10^{-6} that of the Lamb shift. This is a small effect to attempt to measure, but not an impossible one.

6.7 NONRELATIVISTIC ESTIMATE OF THE $B^{(3)}$ CONTRIBUTION TO THE LAMB SHIFT

The coupling of the radiation field to the an electron is given by the interaction Hamiltonian

$$H \ = \ \frac{e}{c}\int d^3r\boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}). \quad (6.7.94)$$

We used Ampere's law $\nabla\boldsymbol{H} \ = \ \boldsymbol{j}$ with the covariant definition of the curl operator $\nabla \ \rightarrow \ \mathcal{D}\times \ = \ \nabla\times \ + \ i(e/\hbar)\sum_i \boldsymbol{A}^{(i)}\times$. This means that since

$$\boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}) \ = \ \mathcal{D}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r})$$

$$= \ \boldsymbol{H}(\boldsymbol{r})\cdot\mathcal{D}\times\boldsymbol{A}(\boldsymbol{r}) \ + \ \mathcal{D}\cdot\boldsymbol{H}(\boldsymbol{r})\times\boldsymbol{A}(\boldsymbol{r}). \quad (6.7.95)$$

The last term is a boundary term that can be set to zero. We then find a $\boldsymbol{B}^{(3)}$ contribution to this term that is of the form

$$H \ = \ -i\frac{e^2}{\hbar c}\int d^3r\boldsymbol{H}\cdot\boldsymbol{A}^{(1)}\times\boldsymbol{A}^{(2)}. \quad (6.7.96)$$

This is the form of the interaction Hamiltonian that we will use to calculate the Lamb shift [15] contribution due to the $\boldsymbol{B}^{(3)}$ field. This interaction Hamiltonian will induce the spontaneous of a photon with wave number $\omega \ = \ ck$ and the transition of an atomic state $|n\rangle \ \rightarrow \ |n'\rangle$. We then examine the second order perturbation in the shift of energy

$$\Delta E_n \ = \ \sum_{n'}\sum_{k,\epsilon}\Big(\frac{|\langle n', k, \epsilon|H_{int}|n, 0\rangle|^2}{E_n \ - \ E_{n'} \ - \ ck}\Big), \quad (6.7.97)$$

where ϵ is the polarization state of the emitted photon. We will first consider the term $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, with $\boldsymbol{A} = A\epsilon$, in our computation. The matrix elements of the interaction hamiltonian is

$$\langle n', k, \epsilon | H_{int} | n, 0 \rangle = \frac{e^4}{\mu^2 c^2} A^3 \langle | \boldsymbol{p} \times \epsilon \cdot \epsilon \times \epsilon^* | \rangle. \tag{6.7.98}$$

If we let the sum over the photon numbers go to the continuum the energy shift is then computed to by

$$\Delta E_n = -\frac{e^4}{\mu^2 c^2} A^3 \int \frac{d^3k}{k^3} \sum_{n',\epsilon} \frac{|\langle n' | \boldsymbol{p} \cdot \epsilon^* | n, 0 \rangle|^2}{E_n - E_{n'} - ck}, \tag{6.7.99}$$

where the $1/(2\pi\hbar)^3$ is absorbed into $\boldsymbol{A}^{(3)}$. Now sum over the polarization states and put the integral in spherical coordinates, and we arrive at the integral in the form

$$\Delta E_n = -\frac{e^4}{\mu^2 c^2} A^3 \int_0^\infty \frac{dk}{k} \sum_{n'} \frac{|\langle n' | \boldsymbol{p} \cdot \epsilon^* | n \rangle|^2}{E_n - E_{n'} - ck}. \tag{6.7.100}$$

The integration of this leads to the result that

$$\Delta E_n = -\frac{e^4}{\mu^2 c^2} A^3 \sum_{n'} \frac{|\langle n' | \boldsymbol{p} \cdot \epsilon^* | n \rangle|^2}{E_n - E_{n'}} \lim_{k \to 0} ln\Big(1 + \frac{E'_n - E_n}{\hbar kc}\Big), \tag{6.7.101}$$

which is divergent. This situation appears to be terrible, for this means that we can not make any predictions, and that there is an infinitely downward shift in the energy of the electron. However, this integral is logarithmically divergent in the infrared spectrum. However, these divergences are easily dealt with. Essentially this can be seen to be fictional, for an electron can not be scattered without an electromagnetic field response. We recognize that the probability of emitting a photon is dependent upon the electron current as a function of the wave number. We then reassess the dipole approximation by writing the $|\langle n',\ k\epsilon | \boldsymbol{p} | n,\ 0 \rangle = |\langle n' | \boldsymbol{p} | n,\ 0 \rangle|^2 |j(k)|^2$, where $j(k)$ is the current for each wave number k divided by the total current. This current ratio reflect the percentage of photons that are emitted with a certain k. For a finite number of photons this will appear to

be Poisson distribution. It we assume that the numbers of possible photons, or the sample size of possible photons, is very large, but where the numbers of photons emitted in reality is far less, then we can assume that the current is $|j(k)| \simeq k$. We then arrive at the result that

$$\Delta E_n = -\frac{e^4}{\mu^2 c^2} A^3 \sum_{n'} \frac{|\langle n' | \boldsymbol{p} \cdot \epsilon^* | n \rangle|^2}{(E_n - E_{n'})} ln\Big(1 - \frac{\hbar kc}{E_n - E'_n}\Big)\Big|_0^\infty. \tag{6.7.102}$$

This integral is now ultraviolet divergent, but it is only logarithmically divergent.

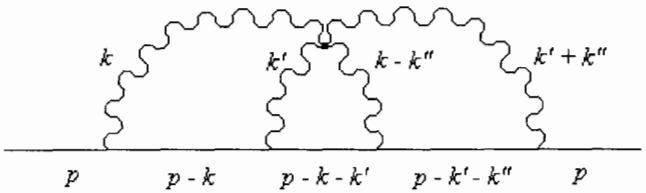
This ultraviolet divergence can be removed by countering it with a similar term that is similarly divergent. In the case of the free electron there is an infinite term

$$\Delta E_{e^-} = \frac{2e^2}{3\pi m^2 c^2} \sum_q \langle q | \boldsymbol{p} | p \rangle|^2 \int_0^\infty dk, \tag{6.7.103}$$

and this leads to the mass renormalization of the electron from the energy shift $E_{e^-} = E^0_{e^-} + \Delta E_{e^-}$

$$E_{e^-} = \frac{1}{p} \langle | \boldsymbol{p} | \rangle + \frac{2e^2}{3\pi m^2 c^2} \langle | \boldsymbol{p} | \rangle|^2 \int_0^\infty dk. \tag{6.7.104}$$

This process will cancel the divergence that occurs in the $U(1)$ theory of electromagnetism. The analogous process that needs to be cancelled in $O(3)_h$ theory involves the coupling of the electron with a nonlinear photon coupling



B(3) Lamb shift diagram. An electron is coupled to two mutually interacting photons.

Figure 6.33

which corresponds to the energy shift

$$\Delta E_{e-}^{B^3} = \sum_{k,n} |\langle n,\ k,\epsilon|p^2|A|^2 A|0,0\rangle|^2$$

$$= -\frac{8\pi}{3}\frac{\hbar^2 e^4}{m^2c^2\mu^4}A^3\sum_{n'}|\langle n'|p|n\rangle|^2\int_0^\infty dk. \qquad (6.7.105)$$

If we add this correction to the energy shift due to the $\boldsymbol{B}^{(3)}$ field we arrive at the integral:

$$\Delta E_{e-}^{B^3} = -\frac{8\pi}{3}\frac{\hbar^2 e^4}{m^2c^2\mu^4}\frac{1}{\hbar c}A^3\sum_{n'}|\langle n'|p|n\rangle|^2\int_0^\infty dk\Big(\frac{E_{n'} - E_n}{E_{n'} - E_n - \hbar kc}\Big). \qquad (6.7.106)$$

This integral is logarithmically divergent. This divergence is comparatively soft and can be removed by placing an effective cut off in the upper limit of the integration. However, this can be resolved by recognizing that the amplitudes are going to sharply drop off for processes that with frequencies $\hbar\omega > 2mc^2$, where m is the mass of the virtual electron and positron. This is the scale used in the quantum fluctuation estimates. This means that we can cut off the integral at this value and arrive at the final result that,

$$\Delta E_{e-}^{B^3} = -\frac{8\pi}{3}\frac{\hbar^2 e^4}{m^2c^2\mu^4}\alpha A^3\sum_{n'}|\langle n'|p|n\rangle|^2 ln\Big(\frac{2mc^2}{E_{n'} - E_n - \hbar kc}\Big). \qquad (6.7.107)$$

To finish the calculation we use $H|n\rangle = E_n|n\rangle$ and

$$\sum_{n'}|\langle n'|\boldsymbol{p}|n\rangle|^2(E_{n'} - E_n)^{-1} = \langle n'|\boldsymbol{p}(H_0 - E_n)^{-1}\cdot\boldsymbol{p}|n\rangle. \qquad (6.7.108)$$

The momentum operator acts on $(H_0 - E_n)^{-1}$ as

$$\boldsymbol{p}(H_0 - E_n)^{-1} = -(H_0 - E_n)^{-2}\boldsymbol{p}H_0, \qquad (6.7.109)$$

and so the action of the two momentum operators on the free Hamiltonian is

$$\boldsymbol{p}(H_0 - E_n)^{-1}\cdot\boldsymbol{p} = [\boldsymbol{p}\cdot[H_0,\ \boldsymbol{p}]]. \qquad (6.7.110)$$

The part that contributes to this commutator is the Coulomb potential between the proton and the electron in the hydrogen atom. The commutator with the free Hamiltonian becomes $(\hbar^2e^2/2)\nabla^2(1/r)$, which gives a delta function that is evaluated in the matrix element when written out by completeness as an integral over space,

$$\frac{e^2\hbar^2}{2}\langle n|\nabla^2(1/r)|n\rangle = \frac{e^2\hbar^2}{2}\int d^3r\psi^*(\boldsymbol{r})2\pi\delta(\boldsymbol{r})\psi(\boldsymbol{r}). \qquad (6.7.111)$$

For an atom in the s-state we have $|\psi|^2 = 1/\pi(na_0)^3$, where n is the principle atomic number and a_0 is the Bohr radius. This means that the final answer is

$$\frac{1}{3\pi^2}\Big(\frac{e^2}{a_0}\Big)\alpha^5 ln\Big(\frac{2mc^2}{E_m - E_n}\Big). \qquad (6.7.112)$$

This effect is going to be 5.33×10^{-5} that of the standard Lamb shift. This is compared with the quantum fluctuation estimate where this type of effect is to be a fifth of this. However, the two calculations are just outside an order of magnitude of each other. This discrepancy is discussed later in section 8.4, where various surprises are in store on the nature of nonabelian quantum electrodynamics. This result and our fluctuation estimates are an indication of a problem that lies in store. However, while this effect is small it should be detectable.

6.8 DERIVATION OF 1/f SPECTRUM FROM NONABELIAN ELECTRODYNAMICS

The 1/f spectrum occurs primarily in quantum electronic systems. It is a curious spectrum that has a reciprocal dependence with frequency. Further, the reciprocal nature of this spectrum means that there is a self-similarity to this spectrum under rescaling. This self similarity, or fractal like structure, to 1/f noise suggests that the underlying process is nonlinear[15]. Curiously, it has been discovered that 1/f noise appears in systems completely unrelated to quantum electronics[17]. The main approach to the problem of 1/f noise involves hopping transitions and the momenta of electrons that interact with lattice defects. The interaction between electrons and lattice defects can be modeled according to electron-phonon interactions. Lattice defects tend to induce "bunching," or an aggregation of lattice atoms[19], which causes phonons to aggregate. The physical statement advanced in this letter is that this is analogous to the bunching of photons in a nonlinear media. The relaxation rate of the lattice would then be analogous to the dispersion of photons. This process is then treated as analogous to nonlinear atomic responses to photons. Essentially the occurrence of lattice defects can be modeled in a simple manner by considering the phonon to be nonabelian. In this letter it is advanced here that these nonlinear phonons are basic to the physics, and that the electrons then interact with this nonlinear bath of phonons. The physical argument is that if electrodynamics is more fundamentally nonabelian, then optical phonons should naturally assume a similar nonabelian form. In a solid at thermal equilibrium the phonons exist in a thermal distribution and interact with electrons that exist in a distribution of states that reflect Fermi-Dirac statistics. The longitudinal magnetic field in nonabelian electrodynamics depends upon the frequency as $\simeq 1/\omega$ and this field is involved with the interaction between the nonlinear phonons, and with the interaction between phonons and electrons. This then results in a parsimonious description of the interaction between phonons and electrons that results in the 1/f spectrum of a current in condensed matter.

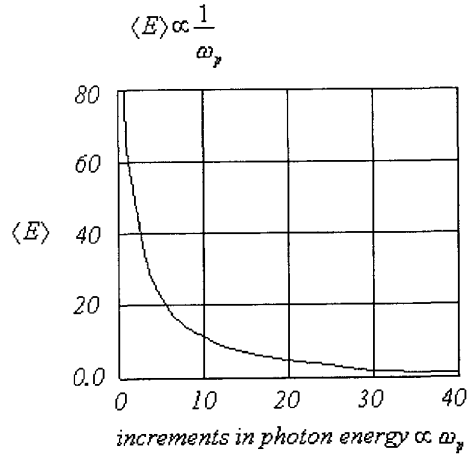


Figure 6.34

Given a nonabelian nature to this theory we return to the nonrelativistic equation that describes the interaction of a fermion of mass m with the electromagnetic field. The Pauli Hamiltonian is modified with the addition of a $\boldsymbol{B}^{(3)}$ interaction term[20][9]

$$H_{B^3_{int}} = \frac{e^2}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{A}^*), \tag{6.8.113}$$

which may be rewritten according to the algebra of Pauli matrics as

$$H_{B^3_{int}} = \frac{e^2}{2m}(\boldsymbol{A}\cdot\boldsymbol{A}^* + i\boldsymbol{\sigma}\cdot\boldsymbol{A}\times\boldsymbol{A}^*). \tag{6.8.114}$$

If we write this interaction Hamiltonian according to creation and annihilation operators we find that this term can be written as

$$H_{B^3_{int}} = \frac{e^2}{4m\hbar\epsilon_0 V}\sum_k\Big(\omega_k{}^{-1}\boldsymbol{I}a_k^\dagger a_k + \sum_q \omega_q{}^{-1}\boldsymbol{\sigma}^{(3)}(a_q^\dagger a_{k-q} + a_q a_{k-q}^\dagger)\Big). \tag{6.8.115}$$

Here $\boldsymbol{I}$ is the unit tensor, ω_p is the frequency of a mode with momentum p, $\boldsymbol{n}$ is a unit vector in the direction of the $\boldsymbol{B}^{(3)}$ field,and $a_p^\dagger$, a_p and the raising and lowering operators for the field expanded

in harmonic oscillator modes with momentum p. The bosonic field is now considered to be optical phonons that have the same form as photons in nonabelian electrodynamics. This interaction Hamiltonian then describes the exchange of a phonon that results in the change of the spin of the fermion. This process is analogous to the absorption of a phonon in the state p and the emission of a phonon in the state $p + k'$ and the electron state transition $k \rightarrow k + k'$. The process is represented in the following diagram:

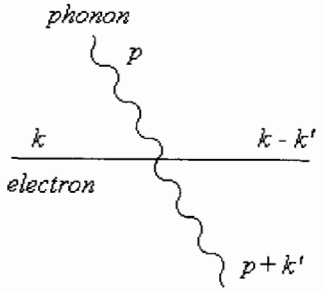


Figure 6.35

This problem is a modification of nonabelian electrodynamics where the interactions between photons and their interactions with charged fermions in $O(3)_b$ electrodynamics is identical in form to that of mutually interacting phonons that also interact with many electrons in a crystalline lattice. We assume that optical phonons, which involve charge separations in the lattice, are identical in form with the nonabelian electromagnetic field. Thus, it is assumed that the longitudinal $\boldsymbol{B}^{(3)}$ field is associated with phonons. The second adjustment that must be made is that $\boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{n}$ is replaced by the creation and annihilation operators of the fermion field. This is initially done with the expectation,

$$\langle \boldsymbol{\sigma}^{(3)} \rangle = \int \psi^*(\boldsymbol{r}) \boldsymbol{\sigma}^{(3)} \psi(\boldsymbol{r}), \tag{6.8.116}$$

with $\psi(\boldsymbol{r}) = \sum_k \psi(\boldsymbol{r})_k b_k$, and $\psi^\dagger(\boldsymbol{r}) = \sum_k \psi^*(\boldsymbol{r})_k b_k^\dagger$. The operators $b_k^\dagger$, b_k are the fermion raising and lowering operators that obey

Fermi-Dirac statistics with the anticommutator $\{b_k^\dagger, b_{k'}\} = \delta_{k,k'}$ The expection of $\boldsymbol{\sigma}^{(3)}$ is then

$$\langle \boldsymbol{\sigma}^{(3)} \rangle = \boldsymbol{\sigma}^{(3)} \sum_{k,k'} b_k^\dagger b_{k'} \int d^3 r \psi_k^*(\boldsymbol{r}) \psi_{k'}(\boldsymbol{r}). \tag{6.8.117}$$

The integral collapses to a delta function, and the result is diagonal. This means that we must generalize this expectation by considering $\frac{e^2}{2m} \langle \boldsymbol{\sigma}^{(3)} \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \rangle$. This is used to find the form of the Hamiltonian for the interaction of nonabelian phonons with electrons. Further, from equation 3 we have in the second term of the interaction Hamiltonian the frequency dependency $(\omega_k \omega_{k-q})^{-1/2}$. If we let the summation over $k \rightarrow k' + \frac{1}{2}q$, and restrict $\omega_q << \omega_k$ this factor is then approximated by $\omega_{k'}^{-1}$ and the summation may be replaced by the summation over k'. This assumption means that the electrons and phonons are weakly interacting, where the phonons transfer only a small amount of their momenta to the electrons. This leads to the interaction Hamiltonian for electrons and nonabelian phonons that acts on the Fock space of states

$$H_{B^3_{int}} = \mathcal{A} \sum_{k',k,p} \left({\omega_k}^{-1} b_{k'+q}^\dagger b_{k'} a_k^\dagger a_{k-q} + H.C. \right), \tag{6.8.118}$$

where $\mathcal{A}$ is an amplitude for the process.

We now examine this problem for the case that the momentum k is held fixed. This corresponds to a set of phonons that interact with the electrons by this interaction with one "input" set of phonons in one mode. We then compute the expectation

$$\langle n_{k'+q}, n_{k'}, N_k, N_{k-q} | H_{B^3_{int}} | n_{k'+q} - 1, n_{k'} + 1, N_k - 1, N_{k-q} + 1 \rangle =$$

$$\mathcal{A} \omega_k^{-1} \sqrt{n_{k'+q} n'_k N_k N_{k'-q}}, \tag{6.8.119}$$

where n_p are the number of fermions, or electrons, and N_p are the numbers of phonons. These formulae can be used to derive the Fermi-Dirac statistics of the electrons. If we let $k + q \rightarrow k$. We can

then have the numbers of excitations, n, n', with the modes k, k' respectively. We are then concerned with the transitions between these modes with the energy gap $\Delta E = E_k - E_{k'}$. If we assume that the transition rates between the two modes are in thermal equilibrium. This means that the Boltzmann factor f_1 for the probability for the transition $k \rightarrow k'$ is the inverse of the Boltzmann factor, f_2 for the transition $k' \rightarrow k$ and the rates of these transitions are the same. These rates are then

$$\mathcal{R}(k \rightarrow k') = n_k(1 \pm n_{k'})f_1 P_{k,k'}$$

$$\mathcal{R}(k' \rightarrow k) = n_{k'}(1 \pm n_k)f_2 P_{k',k}. \qquad (6.8.120)$$

This means that $f_1/f_2 = exp(\Delta E/kT)$. Since $P_{k,k'} = P_{k',k}$ the ratio of occupation numbers is

$$\frac{n_k(1 \pm n_{k'})}{n_{k'}(1 \pm n_k)} = exp(-\Delta E/kT). \qquad (6.8.121)$$

If we assume that the occupation number is completely dependent upon the energy of a mode ω_p we can perform the summation over the occupation numbers of fermions

$$n(E) = \frac{1}{exp(E_p/kT) + 1}, \qquad (6.8.122)$$

with $E \propto 1/\omega_p$ from the interaction Hamiltonian. Similarly, we have the expectation of the energy is given by the Fermi-Dirac statistical result

$$\langle E \rangle = \frac{E_p}{exp(E_p/kT) + 1} \propto \frac{1}{\omega_p(exp(const/\omega_p) + 1)}. \qquad (6.8.123)$$

This expectation for energy of the electrons then has the 1/f curve characteristic of noise encountered in quantum electronics. This

curve only deviates slightly from a strict 1/f curve. It then appears that nonabelian electrodynamics may be the foundation for the occurrence of 1/f noise in the quantum electronics of solids.

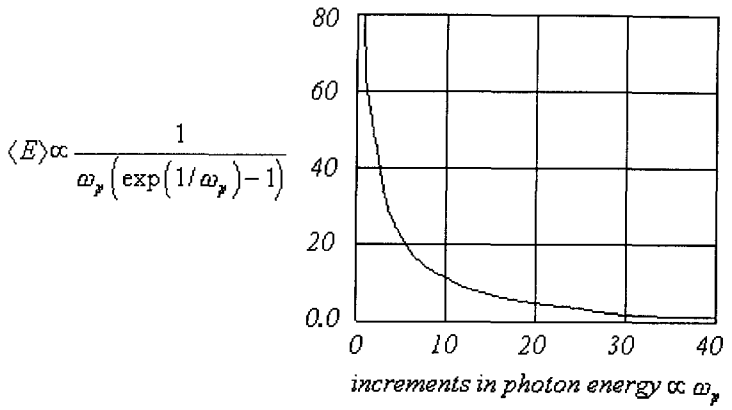


Figure 6.36

6.9 ANALOGY FROM CLASSICAL FIELDS TO NONABELIAN QUANTUM ELECTRODYNAMICS

It is well know that a material that is subjected to an electric field will exhibit charge polarization. The molecules that make up that material will be stretched slightly so that their positive and negative charges will separate. This added redistribution of charge contributes to the net field within this material. This will result in a redefined effective charge. By Gauss's law the effective charge at a radius away from a test charge inside a medium will be less than the test charge itself due to the opposite charges attracted to the test charge. So the effective charge will be $q_{eff} = q_{test}/\epsilon$. In effect as one penetrates closer to the inner charge the effective charge will increase.

Now the appearance of a test charge in the medium creates a "disruption" in the medium. In other words there must be a bubble in the medium around the charge. Deposited on the outer walls of the bubble, inside the medium, there will be a net distribution of

charge that equals the test charge within. This net charge will be maximally separated from each other as they exert an opposing force against one another. The result is that this electrostatic repulsion will tend to make the bubble expand. However, the net force of attraction between the molecules in the medium will prevent this and the bubble will assume a radius that is in energetic equilibrium with the medium.

We now treat this model as the being an electron in the vacuum. The bubble represents an energetic cut off in the physics, illustrated in figure 6.37. Just as we cut off the integral in the nonrelativistic computation of the energy shift due to the $B^{(3)}$ field, the energy involved with maintaining the "tension" on this bubble is precisely this cut off. The cut off in virtual frequencies due to the size of the bubble describe electron positron pairs coupled to virtual photons below a frequency cut off given by the scale of the bubble. Now consider the passage of a wave function for another electron interacting with this system. We treat this wave function as a distribution of charge in space that interacts with the bubble. As this charge distrubution will be repelled by the charge distrubution on the skin of the bubble, and will also repel the charges on the bubble. This process will lead to the overall constriction in the size of the bubble while the wave function passes through it. The shorter the wavelength of the incident electron and the denser is the repulsive charge density and the further the bubble will contract in size. Similarly, we shall see that the occurence of a photon will contribute to the total energy of this bubble and change the energy at which it reaches equilibrium. We have that the electric displacement everywhere equals the electric field inside the bubble

$$D_{in} = D_{out} = E_{in} = \frac{q}{R}, \tag{6.9.124}$$

and the electric field outside the bubble is

$$E_{out} = \frac{e}{\epsilon R}, \tag{6.9.125}$$

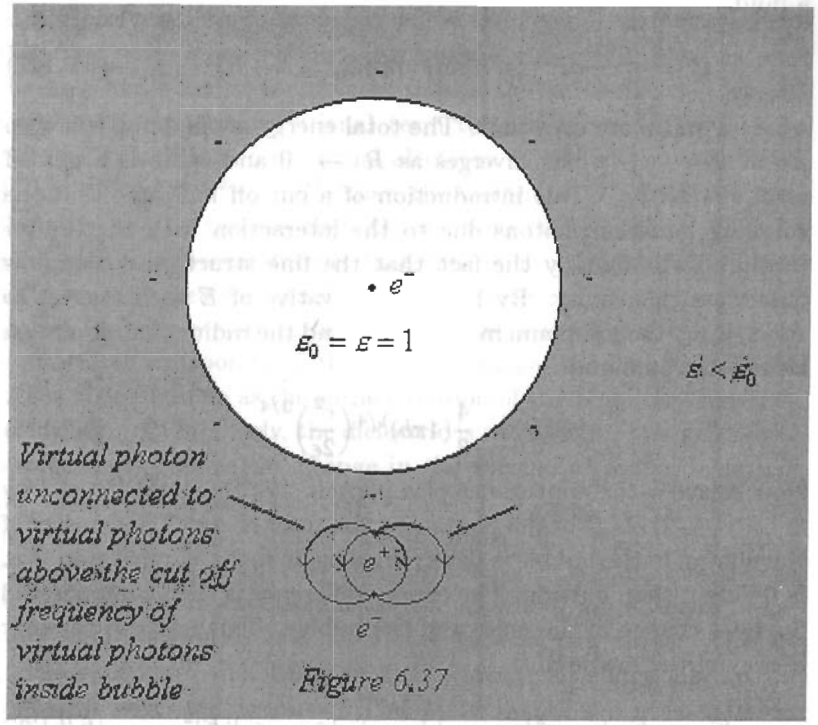


Figure 6.37

where R is the radius of the bubble. The electric polarization is defined by the change in the energy due to the medium $\mathcal{K}$ as the gradient $\boldsymbol{P} = (1/4\pi)(\boldsymbol{D} - \boldsymbol{E}) = \nabla\mathcal{K}$. This gives us the energy of the medium as

$$\mathcal{K} = \frac{1}{2}e^2\Big(\frac{1}{\epsilon} - 1\Big)\frac{1}{R}. \tag{6.9.126}$$

Similarly, the bubble has an energy term that is given by the volume of the bubble and its surface, a term analogous to surface tension in a fluid,

$$\mathcal{B} = \Big(\frac{4\pi}{3}aR^3 + 4\pi bR^2\Big)\Theta(R_{max} - R), \tag{6.9.127}$$

where a and b are constants. The total energy of the bubble is then $E = \mathcal{K} + \mathcal{B}$ which diverges as $R \to 0$ and exhibits a cut off as $R \to R_{max}$. This introduction of a cut off indicates that the coupling between photons due to the interaction with the bubble medium is limited by the fact that the fine structure constant is much less than unity. By find the derivative of E with respect to $R < R_{max}$ the minimum in the energy and the radius that minimum occurs can be found

$$E_0 = \frac{4}{3}(4\pi b)^{1/4}\Big(\frac{e^2}{2\epsilon}\Big)^{3/4}. \tag{6.9.128}$$

Now consider the appearance of a photon. We then add the energy density $(1/2)\boldsymbol{D}\cdot\boldsymbol{E}$ of the photon by the total energy. The energy of the photon in the bubble is determined by an index of refraction that is different than outside. The energy difference is then compensated for by a change in the energy of the bubble. This leads to the new energy of the medium

$$E' = E + \frac{1}{2}e^2\Big(\frac{1}{\epsilon} - 1\Big)\frac{1}{R} + (\epsilon - \epsilon_0)|\boldsymbol{E}|^2. \tag{6.9.129}$$

The occurrence of a photon then changes the over all energy of equilibrium of the bubble.

The addition of the photon energy and the renormalization of the bubble energy might not appear to be an argument for nonlinear

electrodynamics, but we can consider the situation where vacuum modes enter the this region with the bubble (figure 6.38). We then look at the bubble with a cut off in energy is perturbed by energies, real or virtual, exceeding this cutoff. We treat the vacuum as a set of modes that are randomly distributed in their phase

$$\boldsymbol{A}(\boldsymbol{r},t) = A\int d^3k\omega_k exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega_k t + \theta(k)). \tag{6.9.130}$$

If we have vacuum modes that enter the region near the charge that are not accounted for, due to the cut off in the scale of the hole and thus in the polarization of the vacuum, then we are forced to add in the additional term to the vacuum energy. Now if there is the creation of virtual electrons by these stochastic fields $2mc^2 = (1/2)(E^2 + B^2)$ then the electric field associated with this pair will be zero so that $\boldsymbol{D} = 0$ on the boundary. This is analogous to superconductors that can repel magnetic fields. Now an interacting electromagnetic wave will polarized these charges and change the energy within the bubble. If we assume that this bubble is elastic and responds to this energy change, then we change the radius of the bubble as the surface tension of the boundary responds to this change. Similarly, the dielectric is renormalized as well as the bubble energy, since the change in the volume or surface shape of the bubble will change the relative distances between the molecules near the surface. This means that the dielectric constant in a region just outside the bubble may then be changed. We think of this as being similar to a wave on the surface of a fluid. In effect the electron is being modeled as a membrane between two vacua. Without going deeply into the theory of fluid waves, it is commonly known that such waves are nonlinear. The interaction of this wave with the bubble then adjusts the energy of this bubble by inducing nonlinear oscillations. We then can model this shift by considering a dielectric of the form $\epsilon = \epsilon(|A|^2)$ to model the upward shift in the total energy. So we have the electric displacement vector defined as

$$\boldsymbol{D} = \epsilon(|\boldsymbol{A}|^2)\boldsymbol{E} = (1 + 4\pi\chi|\boldsymbol{A}|^2)\boldsymbol{E}. \tag{6.9.131}$$

To account for this change we write $\chi = \chi_0 + \beta|\boldsymbol{A}|^2$. This means that a photon that interacts with the bubble will obey a soliton wave

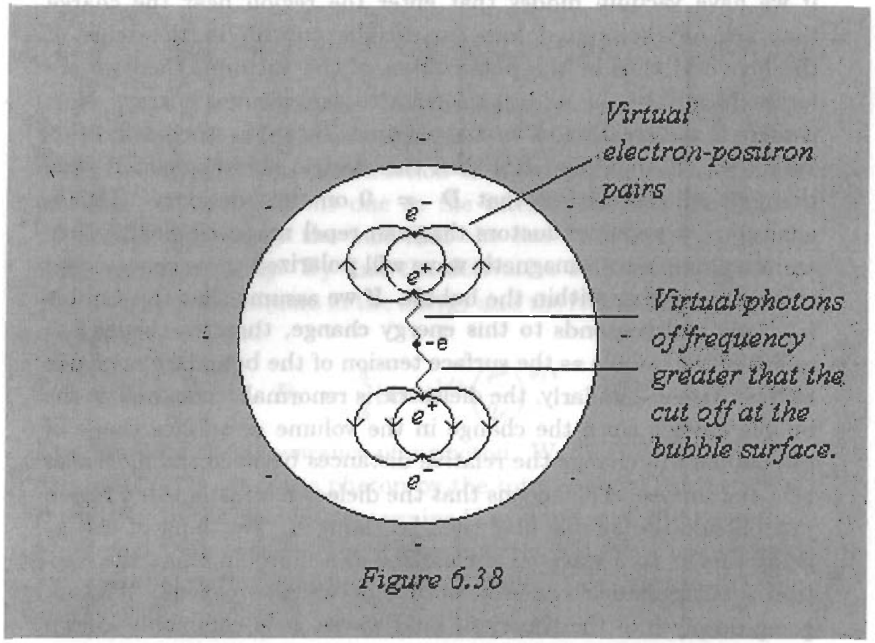


Figure 6.38

equation such as the cubic Schrödinger equation. The existence of scale lengths that one can cut off electrodynamics will itself lead to nonlinear field effects.

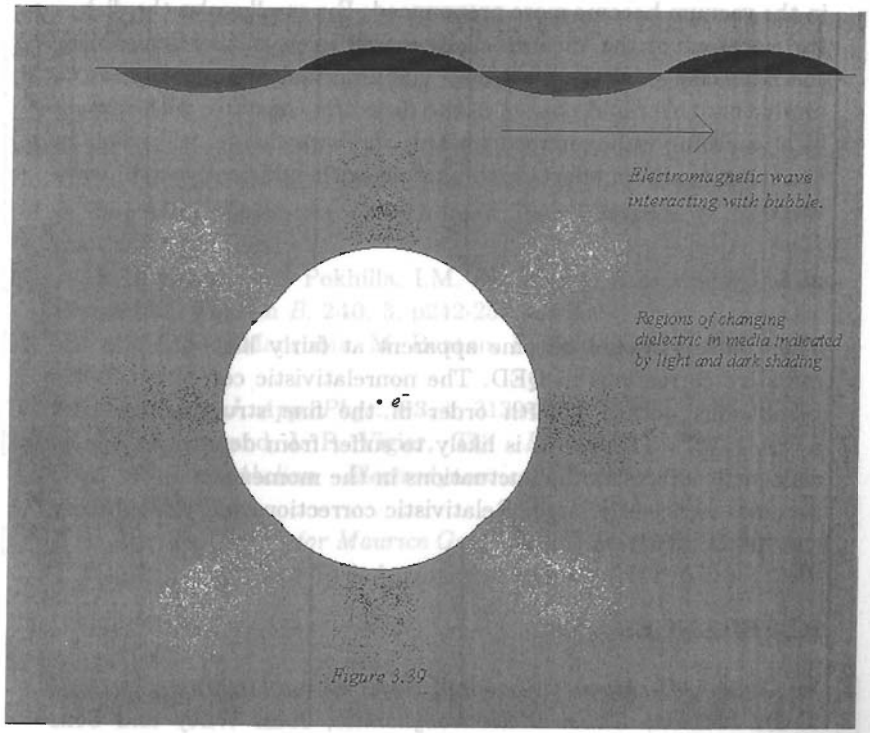


Figure 6.39

Based on this heuristic argument it would appear that electrodynamics allows for the existence of nonabelian fields. This argument

is similar to arguments for QCD confinement[21]. The one big difference is that the confinement exists only for virtual fields within the bubble. The electric field of the inner bare charge is longitudinal and escapes the bubble, unlike the situation in QCD where the color fields of the quarks are also confined by the bag. Further, it is reasonable to assume that for a large bubble, or equivalently a cut off in lower energies, that these nonlinear effects are comparatively small. For a small bubble, we then would likely see these nonlinear effects in the vacuum become more pronounced. For small scales the dielectric constant of the vacuum changes over even smaller scales near the bare charge of the electron. This informal argument tends to imply that this rapid change in the dielectric constant, with respect to a vanishing radius inward towards the bare charge, will result in nonlinear quantum effects with photons with sufficiently small wave length.

These effects should become apparent at fairly high order in the radiative corrections in QED. The nonrelativistic correction to the Lamb shift occurs at fifth order in the fine structure constant $\alpha \simeq 1/137$. This result is likely to suffer from departures due to relativistic effects as the fluctuations in the momentum $\delta p = \hbar/\delta x$ becomes sufficiently large. Relativistic corrections will undoubtedly change the situation.

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CHAPTER 7
QUANTUM CHAOS, TOPOLOGICAL INDICES, AND GAUGE THEORIES

7.1 INTRODUCTION

The numerical studies of $\boldsymbol{B}^{(3)}$ with a Fermion indicated that there was chaotic behavior. The interaction of the system according to the $\boldsymbol{B}^{(3)}$ Hamiltonian for larger photon occupation numbers appears to have induced highly random quantum oscillations. Here is a preliminary examination of topological index numbers and its possible application to the quantization of chaos within the Bohm's view of quantum mechanics, or the particle view of quantum mechanics, with a comparison with the Bohr interpretation of quantum mechanics. This view of quantum mechanics is used here in the context of quantum chaos, since classical chaos theories are formulated according to the separation of particles with nearby initial conditions. The Bohr and Bohm views of quantum theory are essentially dual pictures, and in this case the Bohm theory appears to be more applicable. These analyses are done without any specific reference to the $\boldsymbol{B}^{(3)}$ field until the end. Rather, this is done with consideration of conjugate variables and then later this is discussed in the light that $\boldsymbol{p} \rightarrow \boldsymbol{p} + \boldsymbol{A}$.

Let us write the Schrödinger equation as

$$i\frac{d\psi}{dt} = H\psi, \tag{7.1.1}$$

where H is the Hamiltonian. Then the differential of the wave func-

tion is

$$d\psi \;=\; -iH\psi dt. \tag{7.1.2}$$

Now the wave function lives in a Hilbert space, $\mathcal{H}$, and the projective Hilbert space is $\mathcal{PH}$. The elements in the projective Hilbert space are related to those in the Hilbert space by

$$\psi \;=\; e^{iy}\hat{\psi}, \tag{7.1.3}$$

then the above differential becomes

$$e^{iy}(d\hat{\psi}) \;+\; idye^{iy}\hat{\psi} \;=\; -iHe^{iy}\hat{\psi}dt. \tag{7.1.4}$$

Now multiply through by $\psi*$ and we are left with

$$\psi^* d\psi \;=\; idy \;-\; i(\psi^* H\psi)dt. \tag{7.1.5}$$

The first term on the right had side gives the geometric phase of M. Berry and the second is the dynamical phase [**1**]. Once integrated over a closed path the integral $\int_C dy$, gives the first Chern class of the manifold of dynamics [**2, 3**]

7.2 TOPOLOGICAL NUMBERS AND QUANTUM VORTICES

If one wants to look for topological numbers associated with the quantum force or the amplitude, the best approach to the problem is to use differential forms. This means that one should write for a wave function written as $\psi \;=\; \rho^{1/2}e^{iS}$ that,

$$\oint_C \psi^* d\psi \;=\; \oint_C \psi^*(\frac{1}{2}\rho^{-1}d\rho \;+\; idS)\psi, \tag{7.2.6}$$

where C is a closed loop the system traverses. Now Stokes' law gives the result

$$\oint_C \psi^* d\psi \;=\; \int_{\mathcal{A}} \psi^*(\frac{1}{2}\rho^{-1}d\rho \;-\; idS)\wedge(\frac{1}{2}\rho^{-1}d\rho \;+\; idS)\psi, \tag{7.2.7}$$

where C is a closed curve and $\mathcal{A}$ is any area bounded by this curve. The two-form may be expanded, and since $\rho \;=\; \rho(x,t)$, the two-form $d\rho\wedge d\rho$ is a product of gradients multiplied by the two-form $dx^j\wedge dx^i$. The total time derivative of the action is:

$$\frac{dS}{dt} \;=\; \frac{\partial S}{\partial t} \;+\; \frac{dx^i}{dt}\partial_i S \;+\; \frac{dp^i}{dt}\partial_{p^i} S. \tag{7.2.8}$$

The Hamilton-Jacobi equation $\frac{\partial S}{\partial t} \;=\; -H$ may be used to write the differential of the action as

$$dS \;=\; -\int_0^t dt' d\tilde{H}_{t'}, \tag{7.2.9}$$

and an expansion of the differential of the Hamiltonian in the basis of forms $dx^i\wedge dp^j$. Our loop integral is then

$$\begin{aligned}\int_C \psi^* d\psi \;&=\; (1/8)\int_{\mathcal{A}} \rho^{-2}[\partial_j\rho,\; \partial_i\rho]dx^j\wedge dx^i \\ &+ i\int_{\mathcal{A}} \psi^* \int_0^t dt'\{\tilde{H},\; \rho\}^{PB}_{ij} dx^j\wedge dp^i\psi \\ &+ (1/2)\int_{\mathcal{A}} \psi^* \int_0^t dt' \int_0^t dt''\{\tilde{H}(t'),\; \tilde{H}(t'')\}^{PB}_{ij} dx^j\wedge dp^i\psi.\end{aligned} \tag{7.2.10}$$

The second equation contains the commutator between the density and the classical Hamiltonian. The other two terms are the real valued portions, while the second is the imaginary portion. This equation is a differential-commutator equivalent forms of the usual decomposition of the Schrödinger equation. The expectation of this loop integral $\int_{\mathcal{R}^3} d^3r \int_C \psi^* d\psi$ is the Berry phase.

To those familiar with the particle view of quantum mechanics the real part to the Schrödinger equation is a modified Hamilton Jacobi equation and the imaginary part is a continuity equation for the flow of a "quantum hydrodynamic fluid," often refered to as the pilot wave. Physically this means that if $[\partial_j\rho,\; \partial_i\rho]$ is nonvanishing,

then the density has a rotational component to it. The quantum hydrodynamic fluid, or pilot wave, has a vortex. The value of this integral is a measure of the singularity or pole at the center of the whirlpool.

This is similar to Feynman's vortex theory of super fluids, Helium II [4]. This is a reasonable observation. The Feynman rotons are in effect quantum wave functions on a large scale. Another way of looking at this is according to interference experiments. Say we have a two slit experiment. Now the sum over experiments gets its statistics from the superposition of the two paths c_1 and c_2 that connect the source and a detected spot on the photoplate through the two slits. We can equally view this as a loop that $\psi = \rho^{1/2}e^{iS}$, $\hbar = 1$, sends a particle through. Now what does all of this mean? Let's go back to the Schrödinger equation

$$i\frac{d\psi}{dt} = H\psi, \tag{7.2.11}$$

and then look at $d\psi$

$$d\psi = -idtH\psi, \tag{7.2.12}$$

which tells us the change in ψ due to an increment in time. What our expression is telling us is that if we think of the particle as having traversed the loop between the source gun to the screen and back to the source gun that there is an increment in time, or δt between the two halves of the loop, weighted by the eigenspectrum. Another view is that this loop integral measures the interval of time as a path length for the parallel translate of $\rho^{-1/2}d\rho + idS$ around the loop. If the interval of time for $c_1 = c_2$ then the entire δt for traversal is zero and the value of our integral is zero, and this is destructive interference. Otherwise, where the integral is nonzero there is a nonzero measure for the δt associated with the particle traveling around the loop. This is a beginning to understand superposition according to the particle view of QM. The superposition principle can be looked at as a Gauss-Bonnet theorem for the δt a particle takes to traverse a loop.

Now back to the idea of superfluidity. This will be a heuristic introduction to what is to follow. Let the differential operator be

$$d \to d + iA. \tag{7.2.13}$$

Then our commutator of the density goes as

$$[\partial_j\rho,\ \partial_i\rho] \to [\partial_j\rho,\ \partial_i\rho] + \partial_jA_i - \partial_iA_j + [A_j,\ A_i], \tag{7.2.14}$$

where the additional part are the field components of a Yang-Mills gauge theory. If we are in a $U(1)$ domain then the last commutator vanishes and we have the magnetic field that satisfy Maxwell's equations. This is completely equivalent to a Berry phase that is induced by an electromagnetic field. The commutator describes the vortex motion of the quantum hydrodynamic fluid, or pilot wave, for the system. It has strong analogs to Feynman's rotons in liquid helium II [5].

7.3 DENSITY OPERATOR METHODS

Let $\hat{\rho} = |\psi\rangle\langle\psi|$ be the density operator that transforms according to the unitary group

$$\hat{\rho}(t) = e^{-i\sigma(t)/\hbar}\hat{\rho}(0)e^{i\sigma(t)/\hbar}. \tag{7.3.15}$$

The evolution equation for the density operator is then

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}(t),\ \frac{d\sigma}{dt}]. \tag{7.3.16}$$

The time derivative of the generator of the unitary transform is

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial t} + \frac{dx^i}{dt}\partial_i\sigma, \tag{7.3.17}$$

where for $\sigma = H_0t + \xi(x(t))$ with $H_0 = \frac{-\hbar^2}{2m}\nabla^2 + V(x)$ gives

$$\frac{d\hat{\rho}(t)}{dt} = \frac{i}{\hbar}[\hat{\rho}(t),\ H_0] + \frac{i}{\hbar}[\hat{\rho}(t),\ \frac{dx^i}{dt}\partial_i\xi]. \tag{7.3.18}$$

This may be expressed as the loop integration

$$\oint d\hat{\rho} \;=\; \frac{i}{\hbar}\int_0^T dt[\hat{\rho}(t),\; H_0] \;+\; \oint_{C=\partial\mathcal{A}}[\hat{\rho},\; d\xi]. \qquad (7.3.19)$$

By Stokes theorem the second loop integral is

$$\oint_{C=\partial\mathcal{A}}[\hat{\rho},\; d\xi] \;=\; \int_{\mathcal{A}}[d\hat{\rho},\; \wedge d\xi], \qquad (7.3.20)$$

and under the above transformation and a component by component breakdown of the commutator is

$$\oint_{C=\partial\mathcal{A}}[\hat{\rho},\; d\xi] \;=\; \int_{\mathcal{A}}\hat{\rho}d\xi\wedge d\xi \;+\; d\xi\wedge d\xi\hat{\rho}. \qquad (7.3.21)$$

Now let the time dependent generator of the time development operator according to a time dependent Hamiltonian term

$$\xi \;=\; \frac{1}{\hbar}\int_0^T dt\tilde{H}_t, \qquad (7.3.22)$$

so that

$$d\xi\wedge d\xi \;=\; \frac{1}{\hbar^2}\int_0^T dt\int_0^t dt' d\tilde{H}_t\wedge d\tilde{H}_{t'}. \qquad (7.3.23)$$

Now use the parameterization of this time dependent Hamiltonian according to the position and momentum $z \;=\; z(t),\; z \;=\; \{x,\; p\}$,

$$d\tilde{H}_t \;=\; \partial_i\tilde{H}_t dz^i. \qquad (7.3.24)$$

Then this two-form can be written according to the Poisson bracket

$$d\xi\wedge d\xi \;=\; \frac{1}{\hbar^2}\int_0^T dt\int_0^t dt'\{\tilde{H}_t,\; \tilde{H}_{t'}\}_{ij}^{PB}dz^i\wedge dz^j. \qquad (7.3.25)$$

Now the differential of the density matrix evaluated on a closed loop is

$$\oint_C d\hat{\rho} \;=\; \frac{i}{\hbar}\int_0^T dt[\hat{\rho}(t),\; H_0] \;+ \qquad (7.3.26)$$
$$+\;\int_{\mathcal{A}}\hat{\rho}(\frac{1}{\hbar^2}\int_0^T dt\int_0^t dt'\{\tilde{H}_t,\; \tilde{H}_{t'}\}_{ij}^{PB}dz^i\wedge dz^j)$$
$$+\;\int_{\mathcal{A}}(\frac{1}{\hbar^2}\int_0^T dt\int_0^t dt'\{\tilde{H}_t,\; \tilde{H}_{t'}\}_{ij}^{PB}dz^i\wedge dz^j)\hat{\rho}.$$

The intended us of this is to consider Hamiltonians of the form

$$\tilde{H}_t \;=\; \sum_i^\infty H_i e^{i(k_i x\;-\;\omega_i t)}, \qquad (7.3.27)$$

where ω_i are the angular frequencies. Then an examination of a punctured KAM surface and Cantori should give information of the extent to which the quantum hydrodynamic fluid diverges by the ensuing chaos or turbulence. As such the above time loops become unpredictable for the particle as it wildly dances in the Cantor dust of the shattered KAM surface.

At this point we must return to some analysis. Since the time dependent Hamiltonians evolve according to above unitary operator, $\tilde{H}_t \;=\; e^{-i\sigma(t)}\tilde{H}_0 e^{i\sigma(t)}$, and so the differential two-form is,

$$d\tilde{H}_t\wedge d\tilde{H}_{t'} \;=\; d\tilde{H}_t\wedge e^{-i\sigma(t')}d\tilde{H}_0 e^{i\sigma(t')}$$
$$\;=\; e^{-i\sigma(t')}d\tilde{H}_{t-t'}\wedge d\tilde{H}_0 e^{i\sigma(t')}. \qquad (7.3.28)$$

Now evaluate this two form under a trace, or sum over states

$$tr\int_0^T dt\int_0^t dt' d\tilde{H}_t\wedge d\tilde{H}_{t'} \;=\; tr\;\int_0^T dt\int_0^t dt' d\tilde{H}_{t-t'}\wedge d\tilde{H}_0. \qquad (7.3.29)$$

Now set $t - t' = \tau$ and this integral becomes

$$tr \int_0^T dt \int_0^t dt' d\tilde{H}_t \wedge d\tilde{H}_{t'} = tr \int_0^T dt' t' d\tilde{H}_\tau \wedge d\tilde{H}_0. \quad (7.3.30)$$

To conclude, this part our loop integral over the density matrix is now

$$tr \oint_C d\hat{\rho} = \frac{i}{\hbar} tr \int_0^T dt[\hat{\rho}(t), H_0] + \quad (7.3.31)$$

$$+ tr \int_{\mathcal{A}} \frac{1}{\hbar^2}(\hat{\rho} \int_0^T dt' t' d\tilde{H}_\tau \wedge d\tilde{H}_0 + d\tilde{H}_\tau \wedge d\tilde{H}_0 \hat{\rho}).$$

Since we are working with Hamiltonian systems the space phase volume of the system is conserved. The surface of this volume in phase space is determined by the conservation of energy, and defines the energy surface of the system,

$$\Omega = \int d^n z \Theta(H - E(z)), \quad (7.3.32)$$

where Θ is the Heavyside function. The energy defines the boundary of this volume in phase space,

$$\epsilon(z) = \nabla_z \Omega = \int d^n z \delta(H - E(z)). \quad (7.3.33)$$

The trace over any operator $\hat{O}$ is defined on the energy surface where expectation values exist,

$$tr\ (\rho \hat{O}) = \epsilon(z)^{-1} \int d^n z \hat{O} \delta(H - E(z)). \quad (7.3.34)$$

Now for $\hat{O} = d\tilde{H}_\tau \wedge d\tilde{H}_0$ we have

$$tr\ (\rho \hat{O} + \hat{O}\rho) = \epsilon(z)^{-1} \int d^{2n} z \{d\tilde{H}_\tau, \wedge d\tilde{H}_0\}_{PB} \delta(H - E(z)). \quad (7.3.35)$$

Now use the algebra of commutators to rewrite this as

$$\{d\tilde{H}_\tau, \wedge d\tilde{H}_0\}_{PB} \delta(H - E(z)) = \quad (7.3.36)$$

$$\{d\tilde{H}_\tau, \wedge d\tilde{H}_0 \delta(H - E(z))\}_{PB} - \{d\tilde{H}_\tau, \delta(H - E(z))\}_{PB} \wedge d\tilde{H}_0. \quad (7.3.37)$$

The first commutator is a surface term and thus vanishes. The second commutator is by the chain rule

$$\{d\tilde{H}_\tau, \delta(H - E(z))\}_{PB} \wedge d\tilde{H}_0 = \{d\tilde{H}_\tau, H\}_{PB} \wedge d\tilde{H}_0 \frac{\partial}{\partial z} \delta(H - E(z)). \quad (7.3.38)$$

Now the Poisson bracket here defines the time evolution of the Hamiltonian $d\tilde{H}_\tau$. Further, the derivative of the delta function when evaluated on a function just returns the derivative of that function at the stationary point. This then leads to for $\hat{O}$

$$\int_0^T dt'\ t'\ tr\ (\rho \hat{O} + \hat{O}\rho) = \int_0^T dt'\ t'\ \nabla_z (d\dot{\tilde{H}}_{t'} \wedge d\tilde{H}_0). \quad (7.3.39)$$

This is a fundamental equation, but some simplifications are required, since the appearance of t' is troubling. To eliminate this problem let us introduce the limit function $lim_{\epsilon \to 0} e^{-\epsilon t}$ into the integral. For $\epsilon \to 0$ it is a simple matter to show that

$$\int_0^\infty dt\ t\ \frac{df(t)}{dt} = -\int_0^\infty dt f(t), \quad (7.3.40)$$

and so our above result can be expressed in the more aesthetic form

$$\int_0^T dt'\ t'\ tr\ (\rho \hat{O} + \hat{O}\rho) = \epsilon(z)^{-1} \int d^{2n} z \int_0^T dt\ \nabla_z (d\tilde{H}_t \wedge d\tilde{H}_0). \quad (7.3.41)$$

Now if this is evaluated along the loop integration we have

$$\int_{\mathcal{A}} \int_0^T dt'\ t'\ tr\ (\rho \hat{O} + \hat{O}\rho) = \int_{\mathcal{A}} \epsilon(z)^{-1} \int d^{2n} z \int_0^T dt\ \nabla_z (d\tilde{H}_t \wedge d\tilde{H}_0). \quad (7.3.42)$$

7.4 HAMILTONIAN CHAOTIC SYSTEMS

Now let us consider chaotic systems. Let the time dependent Hamiltonian have the Fourier expansion

$$\tilde{H}_t = \sum_n H_n e^{-in\theta(t)}, \tag{7.4.43}$$

where we will work within the action angle variables J, θ so that

$$d\tilde{H}_t = \sum_n (dH_n - inH_n d\theta)e^{in\theta(t)}. \tag{7.4.44}$$

Now set $\theta(t) = \theta_0 + \omega(I)t$, with $d\theta/dt = \nabla_J H_0$. Then we have the two-form

$$d\tilde{H}_t \wedge d\tilde{H}_0 = i\sum_n ((n\cdot\nabla_J)H_0)H_n e^{-in\theta(t)} d\theta \wedge dJ. \tag{7.4.45}$$

The gradient along the action variables of this equation is then

$$\nabla_J d\tilde{H}_t \wedge d\tilde{H}_0 = i\sum_n ((n\cdot\nabla_J)H_0)\nabla_J H_n e^{-in\theta(t)} d\theta \wedge dJ. \tag{7.4.46}$$

Now we have $\nabla_J H_n = J_n$, and so this reduces to

$$\nabla_J d\tilde{H}_t \wedge d\tilde{H}_0 = i\sum_n (n\cdot J)J_n e^{-in\theta(t)} d\theta\wedge dJ = i\sum_n (J_n\cdot J)e^{-in\theta(t)} d\theta\wedge dJ, \tag{7.4.47}$$

where the last equality absorbs n into J_n. The term $J_n \cdot J$ is now a product between two different action variables. We could just as well write this as

$$J_n \cdot J = g_{ij} J_n^i J^j, \tag{7.4.48}$$

where g_{ij} is a metric for the energy surface.

$$\int_{\mathcal{A}} \hat{\rho} d\xi\wedge d\xi + d\xi\wedge d\xi\hat{\rho} = i\epsilon(z)^{-1}\int_{\mathcal{A}}\int d^{2n}z \int_0^T dt \sum_n g_{ij} J_n^i J^j, e^{-in\theta(t)} d\theta\wedge dJ, \tag{7.4.49}$$

is a curvature two-form evaluated on an enclosed area of the energy surface. Further, the curvature is proportional to the metric. This means that the space is an Einstein space, with symplectic coordinates.

Here is where chaos enters the picture. A dynamical system is chaotic if any two initial points arbitrarily close become widely separated within a finite period of time. Let the distance between any two points on the energy surface be given by a line element,

$$ds^2 = g_{ij} dx^i dx^j. \tag{7.4.50}$$

Then the separation of any two points is given by

$$\Delta x = \sqrt{g_{ij} dx^i dx^j} = \sqrt{g_{ij} v^i v^j}\, dt, \tag{7.4.51}$$

and the Liapunov exponent for a dynamical system is defined as

$$\lim_{t\to\infty}\lim_{\Delta x(t_0)\to 0} \frac{1}{t} ln \frac{\int_0^t \sqrt{g_{ij}v^i v^j}\, dt}{\int_0^{t_0} \sqrt{g_{ij}v^i v^j}\, dt}. \tag{7.4.52}$$

For a chaotic system this distance will separate at an exponential rate and the Liapunov exponent is nonzero. The tori of regular dynamics have elliptical curvatures, which means that any two intersecting geodesics are guaranteed to recross at some other point on the space. A manifold with a hyperbolic curvature will have divergent geodesics that will exponentially separate with time. The Gaussian curvature of a hyperbolic manifold is negative, and is a reasonable geometry to use for the study of chaotic systems. The break down of torii leads to regions of stochastic behavior. The positive Gaussian curvature of the torus is being made negative in regions of overlap across KAM surfaces of separation.

An easy example to demonstrate is with geodesics on the Poincare half plane. This two-dimensional space has the line element

$$ds^2 = \frac{1}{y^2} dx^2 + \frac{1}{y^2} dy^2. \tag{7.4.53}$$

The connection coefficients are,

$$\Gamma_{12}{}^{1} = \Gamma_{21}{}^{1} = \Gamma_{11}{}^{2} = \Gamma_{22}{}^{2} = -1/y. \tag{7.4.54}$$

The geodesic equations are

$$x_{tt} - \frac{2}{y}x_t y_t = 0,\ y_{tt} + \frac{1}{y}(x_t{}^2 + y_t{}^2). \tag{7.4.55}$$

The solutions to this equations are $x = \alpha + \beta tanh(t)$ and $x = \beta sech(t)$. The distance between two points y_1 and y_2 diverges as

$$\int_{y_1}^{y_2} = \int_{y_1}^{y_2} \frac{dy}{y} = ln(y_2/y_1) = \alpha \int_0^T dt. \tag{7.4.56}$$

These diverge as $y_1 \to 0$ and $y_2 \to \infty$, or equivalently as $T \to \infty$. Further, the Gaussian curvature is,

$$K = -\frac{1}{\sqrt{g}}\Big(\frac{\partial}{\partial x}\frac{\partial g_{11}/\partial x}{\sqrt{g}} + \frac{\partial}{\partial y}\frac{\partial g_{22}/\partial y}{\sqrt{g}}\Big) = -1. \tag{7.4.57}$$

This space is a reasonable toy that captures the hyperbolic structure of the geometry that occurs with the breakdown of a KAM surface.

7.5 QUANTUM GEOMETRY AND BOHM'S THEORY

Now let us examine this within the context of Bohm's particle plus pilot wave model of quantum mechanics. When the wave function is written in a polar form $\psi = exp(\beta/\hbar)$, $\beta = R + iS$ the Schrödinger equation splits into a real and imaginary parts[**6**],

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\{(\nabla S)^2 - (\nabla R)^2 - \hbar\nabla^2 R\} + V,$$

and

$$-\frac{\partial R}{\partial t} = \frac{1}{m}\nabla R\cdot\nabla S - \frac{\hbar}{2m}\nabla^2 S, \tag{7.5.58}$$

which are the quantum corrected Hamilton-Jacobi equation and continuity equation respectively. Now the particle's momentum is

$\boldsymbol{p} = \nabla S$, which defines the Lagrangian manifold of constant action in phase space. Now let the momentum operator $\hat{p} = \frac{\hbar}{i}\nabla$ act on the wave function $\psi = \rho^{1/2}e^{iS} = e^{R+iS}$:

$$\hat{p}\psi = (\nabla S - i\nabla R)\psi, \tag{7.5.59}$$

The expectation value of the momentum operator is

$$\langle\hat{p}\rangle = \int d^3 r\psi^* \frac{\hbar}{i}\nabla\psi = \int d^3 x\psi^*(\nabla S - i\nabla R)\psi. \tag{7.5.60}$$

Since $\nabla S = \boldsymbol{p}$ this is written in the more compact form

$$\langle\hat{\boldsymbol{p}}\rangle = \boldsymbol{p} - i\langle\nabla R\rangle. \tag{7.5.61}$$

This must mean that $i\langle\nabla R\rangle = 0$ for $\langle\hat{\boldsymbol{p}}\rangle = \boldsymbol{p}$. Now put $1 = tr\rho$ in the first term on right hand side of equation (7.6.61)

$$\langle\boldsymbol{p}\rangle = \langle tr\rho\boldsymbol{p}\rangle$$

$$= \langle\int d^3|\psi|^2\boldsymbol{p}\rangle = \langle\langle\hat{p}\rangle\rangle. \tag{7.5.62}$$

We now demand that $\langle\hat{\boldsymbol{p}}\rangle = \boldsymbol{p}$, so if one just peals one level of $\langle\ \rangle$ off we are left with

$$\hat{\boldsymbol{p}} = \langle\hat{\boldsymbol{p}}\rangle - i\nabla R. \tag{7.5.63}$$

The particle's momentum is the expected value plus a quantum correction to this momentum,

$$\delta\boldsymbol{p} = i\boldsymbol{\pi} = \frac{i\hbar}{2}\frac{\nabla\rho}{\rho} = -i\nabla R. \tag{7.5.64}$$

The particle's momentum is then the expectation $\boldsymbol{p} = \langle\hat{\boldsymbol{p}}\rangle$ plus the fluctuation,

$$\hat{\boldsymbol{p}} = \langle\hat{\boldsymbol{p}}\rangle + \delta\boldsymbol{p}. \tag{7.5.65}$$

It is easily see that $\langle\delta\boldsymbol{p}\rangle = 0$. Now define $\beta^{\pm} = \nabla(R \pm iS)$. Now the two equations become

$$\frac{\partial S}{\partial t} = \frac{1}{4m}\big(\beta_i^+\beta^{+i} + \beta_i^-\beta^{-i} - \frac{\hbar}{2}(\partial^i\beta_i^+ + \partial^i\beta_i^-)\big) + V = -H_s$$

and

$$\frac{\partial R}{\partial t} = \frac{i}{4m}\left(\beta_i^+\beta^{+i} - \beta_i^-\beta^{-i} - \hbar(\partial^i\beta_i^+ - \partial^i\beta_i^-)\right) = -H_r. \quad (7.5.66)$$

Bohm's quantum potential is then equal to the square of the quantum fluctuations plus a modified quantum potential $Q' = \frac{\hbar}{2}\nabla^2 R$

$$Q = (\delta p)^2 + Q'. \quad (7.5.67)$$

The equations of motion are found by taking the gradient of the modified Hamilton-Jacobi equation:

$$\partial_i\frac{\partial S}{\partial t} = \frac{\partial p_i}{\partial t} = \frac{1}{2m}\left(\beta_j^+\partial_i\beta^{+j} + \beta_j^-\partial_i\beta^{-j} - \frac{\hbar}{2}\partial_i(\partial^j\beta_j^+ + \partial^j\beta_j^-)\right) - \partial_i V \quad (7.5.68)$$

Now use the fact that

$$\beta^{j\pm}\partial_i\beta_j^\pm = \beta^{j\pm}\partial_j\beta_i^\pm + \beta^{j\pm}(\partial_i\beta_j^\pm - \partial_j\beta_i^\pm), \quad (7.5.69)$$

to find that the equation of motion is,

$$\frac{\partial p_i}{\partial t} = \frac{1}{2m}\left(\beta_j^+\partial^j\beta_i^+ + \beta_j^-\partial^j\beta_i^- + \beta^{j+}(\partial_i\beta_j^+ - \partial_j\beta_i^+) + \right.$$

$$\left.\beta^{j-}(\partial_i\beta_j^- - \partial_j\beta_i^-) + \frac{\hbar}{2}\partial_i(\partial^j\beta_j^+ + \partial^j\beta_j^-)\right) + \partial_i V. \quad (7.5.70)$$

Similarly the gradient on the continuity equation gives the fluctuation in the force as,

$$\frac{\partial \pi_i}{\partial t} = \frac{i}{2m}\left(\beta_j^+\partial^j\beta_i^+ - \beta_j^-\partial^j\beta_i^- + \beta^{j+}(\partial_i\beta_j^+ - \partial_j\beta_i^+) - \right.$$

$$\left.\beta^{j-}(\partial_i\beta_j^- - \partial_j\beta_i^-) + \frac{\hbar}{2}(\partial_i(\partial^j\beta_j^+ - \partial^j\beta_j^-))\right). \quad (7.5.71)$$

Now suppose that $\beta^\pm$ depends upon some internal set of coordinates that transform according to a Lie group $\mathcal{G}$, $\beta^\pm = \beta_a^\pm e^a$. Then the generalized velocities $\beta_i^\pm$ become

$$\beta_j^\pm = \partial_j\beta^\pm = (\partial_j\beta_a)e^a + g\beta_a^\pm A^a{}_{bj}e^b, \quad (7.5.72)$$

where $A^a{}_{bj}$ is a component of a gauge connection and g is a coupling constant for a Yang-Mills gauge field. It is then straight forward to calculate the action of commuting differential operators on $\beta^\pm$,

$$(\partial_i\partial_j - \partial_i\partial_j)\beta^\pm = g\beta_a(\partial_i A^a{}_{bj} - \partial_j A^a{}_{bi} + g[A^a{}_{cj}, A^c{}_{bi}])e^b = gF_{ij}. \quad (7.5.73)$$

These gauge field equations may be substituted into the equations of motion:

$$\frac{\partial p_i}{\partial t} = \frac{1}{2m}\left(\beta_j^+\partial^j\beta_i^+ + \beta_j^-\partial^j\beta_i^- - g(\beta^{j+}F_{ij} + \beta^{j-}F_{ij}^\dagger) + \right.$$

$$\left.\frac{\hbar}{2}(\partial_i(\partial^j\beta_j^+ + \partial^j\beta_j^-) + g(\partial^j F_{ij} + \partial^j F_{ij}^\dagger))\right) + \partial_i V. \quad (7.5.74)$$

and

$$\frac{\partial \pi_i}{\partial t} = \frac{i}{2m}\left(\beta_j^+\partial^j\beta_i^+ - \beta_j^-\partial^j\beta_i^- + \right. \quad (7.5.75)$$

$$\left.g(\beta^{j+}F_{ij} - \beta^{j-}F_{ij}^\dagger) + \frac{\hbar}{2}(\partial_i(\partial^j\beta_j^+ - \partial^j\beta_j^-) + g(\partial^j F_{ij} - \partial^j F_{ij}^\dagger))\right).$$

We may make these equations of motion more symmetrical by finding $\partial(p_i \pm i\pi)/\partial t$. Further, the internal symmetries are allowed to have time dependencies. Then we are left with the coupled set of differential equations,

$$-i\frac{\partial\beta_i^+}{\partial t} - g\frac{\partial A_i}{\partial t} = \quad (7.6.76)$$

$$\frac{1}{m}\left(\beta_j^+\partial^j\beta_i^+ - g\beta^{j-}F_{ij} + \frac{\hbar}{2}(\partial_i\partial^j\beta_j^+ + g\partial^j F_{ij})\right) + \partial_i V$$

$$-i\frac{\partial\beta_i^-}{\partial t} - g\frac{\partial A_i^\dagger}{\partial t} = \quad (7.5.77)$$

$$\frac{1}{m}\left(\beta_j^-\partial^j\beta_i^+ - g\beta^{j+}F_{ij} + \frac{\hbar}{2}(\partial_i\partial^j\beta_j^- + g\partial^j F_{ij}^\dagger)\right) + \partial_i V.$$

Physically the gauge field terms have several meanings. The terms $\beta^{j\pm}F_{ij}$ are Lorentz forces. The field strength terms F_{ij} are the magnetic field components of the gauge theory $\epsilon_{ijk}B^k = F_{ij}$ and the force is in a direction orthogonal to the generalized velocity $v_j + \delta v_j = \beta_j/m$. The second term $\partial^j F_{ij}$ are the currents, which are zero in a source free region. The terms $\partial A_i/\partial t$ are the electric field components.

At this point it should be pointed out that the gauge theory is seen to emerge from the quantum potential

$$Q = -\pi^i\pi_i - \frac{\hbar}{2m}\nabla_i\pi^i, \tag{7.5.78}$$

when $\beta^{\pm} = R \pm iS$ is dependent on internal degrees of freedom governed by a Lie algebra. The electric field components of the gauge theory are found in the potential in the Hamilton-Jacobi equation, $V = V_0 + \phi$, and $E_i = \partial_i\phi$. For the case where the gauge theory is the $U(1)$ electromagnetic field the abelian group structure is reflected in the vanishing of the commutator between the gauge connection.

Before we examine the geometry of a quantum system let us touch base with some basic notions of quantum mechanics. We have from the Bohm approach to quantum theory that the momentum of a particle is given by a classical part plus a fluctuation. By construction we have that $\langle\delta x\rangle = 0$ and

$$[\delta x,\ \delta p] = i\hbar. \tag{7.5.79}$$

The Heisenberg equations of motion assume the form

$$\langle\dot{x}\rangle + \delta\dot{x} = \frac{1}{m}(\langle p\rangle + \delta p)$$

$$\langle\dot{p}\rangle + \delta\dot{p} = -\nabla V(\langle x\rangle + \delta x). \tag{7.5.80}$$

The potential may be expanded in terms of $\langle\delta x\rangle$, and then the fluctuations obey the equations of motion

$$\delta\dot{x} = \frac{1}{m}\delta p$$

$$\delta\dot{p} = -\nabla^2 V(\langle x\rangle)\delta x - \frac{1}{2}\nabla^3 V(\delta x^2 - \langle x^2\rangle) + \ldots. \tag{7.5.81}$$

The Lyapunov exponent used to measure the sensitivity on initial conditions of a dynamical system. The exponential deviation between the positions of two particles in phase space generate this quantity,

$$L = lim_{t\to\infty}\frac{1}{t}ln\sqrt{\left(\frac{\delta x(t)}{\delta x(0)}\right)^2 + \left(\frac{\delta p(t)}{\delta p(0)}\right)^2}. \tag{7.5.82}$$

A nonzero value for Λ indicates that the trajectories for two different particles whose initial positions in phase space differ by a fluctuation $(\delta x(0),\ \delta p(0))$. Then if the fluctuation is propagated exponentially with time, then the system is chaotic. For classical systems this fluctuation represents a small deviation in the trajectories between two particles, or an uncertainty in the position of a single particle. This uncertainty exponentially propagates for a chaotic system. For a quantum particle this uncertainty is due to the stochastic fluctuations due to the quantum force on the particle.

The fluctuations are found from the equations of motion

$$\delta x(t) = -\frac{i}{m}\int_0^t dt'\pi(t') \tag{7.5.83}$$

$$\delta\dot{p}(t) = -i\dot{\pi}(t) = -\int dt(\nabla V(\langle x\rangle + \delta x) + \nabla V(\langle x\rangle))$$

$$= \frac{1}{2m}\int dt\big((\beta_j^+\partial^j\beta_i^+ - \beta_j^-\partial_j\beta_i^- + g(\beta^{j+}F_{ij} - \beta^{j-}F_{ij}^* +$$

$$\frac{\hbar}{2}(\partial_i(\partial^j\beta_j^+ - \partial^j\beta_j^-) + g(\partial^j F_{ij} - \partial^j F_{ij}^\dagger))) + \partial_i V.$$

The potential may be expanded in a Taylor series around $\langle x\rangle$ in powers of the fluctuation δx. Yet for a chaotic system the fluctuations grow exponentially and the Taylor series can not be satisfactorily truncated. As a result the second expression may be more useful, but difficult to apply.

Now let us examine the loop integral $\oint \psi^*\psi$ for $\psi = e^{\beta/\hbar}$,

$$\oint_{C=\mathcal{A}} \psi^*\psi = \frac{1}{\hbar}\int_{\mathcal{A}} \psi^* d\beta^* \wedge d\beta\psi. \tag{7.5.84}$$

The differential of the exponentiated term in the wave function may be written as

$$d\beta(t) = -\int_0^t dt'(dH_r(t') + i dH_s(t')). \tag{7.5.85}$$

With some analysis and the definition $H^\pm = H_r \pm iH_s$ the components of the two-form that projects through the area of integration is a Poisson bracket

$$d\beta^* \wedge d\beta = \int_0^t dt' \int_0^t dt''\{H^+(t'),\ H^-(t'')\}_{ij}^{PB} dx^i \wedge dp^j. \tag{7.5.86}$$

The Poisson bracket is evaluated with the Hamilton's equation for the position and momentum and their fluctuations,

$$\frac{\partial H^\pm}{\partial x^i} = -i\dot{\beta}_i^\pm,\ \frac{\partial H^\pm}{\partial p^i} = \frac{i}{m}\beta_i^\pm. \tag{7.5.87}$$

Use of these Hamilton's equations and the dynamical equations of motion lead to the two-form,

$$d\beta^- \wedge d\beta^+ = \frac{i}{m^2}\int_0^t dt' \int_0^{t''} (A_{ij} + B_{ij} + C_{ij} + h.c.) dx^i \wedge dp^j, \tag{7.5.88}$$

where

$$A_{ij} = \beta_i^-(t')\beta^{+k}(t'')\partial_k\beta_j^+(t'')$$
$$B_{ij} = g\beta_i^-(t')\beta^{+k}(t'')F_{jk}(t'')$$
$$C_{ij} = -\frac{\hbar}{2}\beta_j^+(t'')(\partial_k\partial_i\beta^{+k}(t') + g\partial^k F_{ik}(t')). \tag{7.5.89}$$

The two-form $d\beta^- \wedge d\beta^+$ is evaluated on a parallelogram on the energy surface of the system. This energy surface is the same as computed according to the Bohr interpretation, but where the "fuzziness" of quantum fluctuations $\pi\wedge\pi$ and $\pi\wedge p$ are explicitly included. The energy surface does not have the same sharpness given by a Heavyside function. The computation on the energy surface according to the Bohr approach is an expectation of the energy surface just computed according to the Bohm method. As such, the largest expected energy surface, or the energy surface of classical dynamics is one where $\langle\pi\rangle = 0$ has been taken. The quantum input from the expectation over the Bohm energy surface is due to terms of the form $\langle\pi_i\pi^j\rangle \neq 0$ that emerge from the quantum potential.

The same analysis on the time integration of the Poisson bracket $\{H^+(t'),\ H^-(t'')\}$ may be performed. Without need of weighty analysis the result shall just be stated:

$$d\beta^* \wedge d\beta = -\int_0^T dt\ \nabla_z(dH^+(t) \wedge d\tilde{H}^-(0)). \tag{7.5.90}$$

If we write these generalized Hamiltonians as $H^\pm = \sum_n H_n^\pm e^{\mp i\theta(t)}$ we arrive at a result that has the form

$$\nabla_J dH^+(t) \wedge dH^-(0) = i\sum_n (n\cdot J^-)J_n^+ e^{-in\theta(t)} d\theta \wedge dJ, \tag{7.5.91}$$

where $J^- = \nabla H^-(0)$ and $J_n^+ = \nabla H_n^+$. These action variables by construction contain quantum fluctuations, $J^\pm = J \pm i\delta J$. With this the two form is

$$d\beta^*\wedge d\beta = i\sum_n \{J\cdot J_n + i(J\cdot\delta J_n - \delta J\cdot J_n) + \delta J\cdot\delta J_n e^{-in\theta(t)}\} d\theta\wedge dJ, \tag{7.5.92}$$

where the terms linear in the fluctuations will have expectations that vanish.

For a chaotic system the Lyapunov function will involve the fluctuations. The fluctuation in the action variables will have their own contribution to the Liapunov exponent,

$$L = lim_{t\to\infty} lim_{\delta t_0 \to 0} \frac{1}{t} ln \frac{\int_0^t \sqrt{\sum_n \delta J \cdot \delta J_n}}{\int_0^{t_0} \sqrt{\sum_n \delta J \cdot \delta J_n}}, \tag{7.5.93}$$

which is nonzero for a chaotic system. In the chaotic case the influence of a fluctuation will proceed to grow with time. This is contrary to the usual notion of quantum fluctuations that are Markovian. Markovian fluctuations are such that the fluctuations in a system at one time are independent of the system configuration at any other time. Any transition between states, $|i\rangle \rightarrow |j\rangle$ through a fluctuation, such as tunneling, at a time t' occur through an interaction that is of the form $\langle i|V|j\rangle \simeq \omega_{ij}\delta(t - t')$. For a regular dynamical system the geometry of the energy surface is elliptic, and so any fluctuation the results in a deviation of the system will in a finite period of time intersect a path that would have resulted from another fluctuation. If the energy surface is hyperbolic then the effect of a fluctuation will be amplified according to $exp(L(t))$. This means that the dynamics of the particle, according to Bohm, will be highly dependent upon the initial configuration of the amplitude, or equivalently on the value of the quantum potential.

Since the quantum force can be written as

$$\nabla Q = \nabla V(\langle x\rangle + \delta x) - \nabla V(\langle x\rangle), \tag{7.5.94}$$

and that the wave packet oscillations are due to the behavior of the quantum potential we can write the wave function as

$$\psi(x) = F(S(x), x), \tag{7.5.95}$$

where F is a periodic function in time. Then given any observable $\hat{O}$ its expectation is

$$\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle = \sum_j \langle F_k|\hat{O}(x, j\nabla S - i\hbar\frac{\partial}{\partial x})|F_k\rangle + O(\hbar^2)$$

$$= \sum_j \int d^3x O(x, j\nabla S)|F_k|^2 - \tag{7.5.96}$$

$$i\hbar\sum_j \int d^3x \langle F_j(x)|\frac{\partial O}{\partial p}(x, j\nabla S)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial x}\frac{\partial O}{\partial p}(x, j\nabla S)|F_j\rangle + O(\hbar^2).$$

This construction means that the Lagrangian manifold for the classical system is associated with a foliation of Lagrangian submanifold

$$\mathcal{F}ol\psi = \cup_{j=-\infty}^{\infty}\{supF_j, p = j\nabla S\}. \tag{7.5.97}$$

This foliation of the Lagrangian surface is seen in the fuzziness of the energy surface due to the term $i(J\cdot\delta J_n - \delta J\cdot J_n)$ in the two form $\beta^*\wedge\beta$. For a chaotic system the value of these fluctuations grows exponentially. The observable role of this growth is unmeasurable as $\langle\delta p\rangle$, but is so as $\langle(\delta p)^2\rangle$.

7.6 DISCUSSION AND PROBLEMS

What we have is a demonstration of how fluctuations derived from Bohm's approach to quantum mechanics can be associated with a Lyapunov exponent for a Hamiltonian chaotic system. This fluctuation can involve both the momentum fluctuation of the particle and the associated gauge fields derived when the amplitude has a set of internal degrees of freedom. This promotion of a fluctuation in a quantum system goes against the often stated notion that quantum fluctuations are delta function correlated so that fluctuations at one time are independent of events at any other time. This approach to an understanding of quantum chaos advances the concept that such systems are nonMarkovian.

The gauge fields derived from the methods just presented are purely magnetic in nature. They could easily be magnetic fields, the magnetic analog in QCD, or $\boldsymbol{B}^{(3)} = -ie/\hbar \boldsymbol{A} \times \boldsymbol{A}^*$. For most practical issues this theory is designed for the electromagnetic interaction is likely to be used. This development indicates that since fluctuations may have their influence amplified with time that essentially the time evolution can not be time reversed to reproduce the initial conditions. In the numerical study of cavity QED with the $\boldsymbol{B}^{(3)}$ field we see that this results in a quartic potential that erases the initial conditions of the system. This nonMarkovian evolution of the effect of a fluctuation has the effect of erasing quantum information.

The apparent breakdown of the staircase of frequencies, or its bifurcation into new frequency spikes, may well be explained by the breakdown of a KAM surface. The method of Greene illustrates a continued fraction approach to the breakdown of KAM surfaces and their break up into Cantor set by self similar puncturing [7]. The above analysis indicates that there can be a form of quantum chaos that results in wild quantum oscillations. However, the orbit of a chaotic particle is still ultimately bounded, and if that particle evolution describes a path within a path integral, it is still correlated with other paths; in spite of any Lyapunov exponential divergence. As a result the quantum chaos will likely settle into some form of quantum oscillations that have commensurate frequencies. This would be established by constructive and destructive interferences between paths in the Feynman path integral. In effect this would lead to a sort of "order out of chaos," that is not seen in classical chaos. It appears that the numerical exercises illustrate quantum chaos as a sort of initial analytic transient that settles into ordered quantum oscillations.

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CHAPTER 8
FIELD THEORY OF $O(3)_b$ QED AND UNIFICATION WITH WEAK AND NUCLEAR INTERACTIONS

3.1 DISCUSSION

If electromagnetism is more fundamentally nonAbelian then this means that there must be changes in our fundamental views of quantum field theory. As indicated in chapter 6 there are deviations from Abelian quantum electrodynamics of optics and atomic interactions. These deviations indicate that there will exist measurable deviations if we admit that there are contributions to physics from $\langle \boldsymbol{A} \cdot \boldsymbol{A}^* \rangle$ on nonvacuum states. Even if this term contributes only to vacuum symmetries, it has been demonstrated that this provides a fundamental manner in which nonlinear effects in quantum optics are derived. In either case this has profound implications for our understanding of field theory and ultimately how electromagnetism is unified with the weak interactions and also at much higher energies into a Grand Unified Theory (GUT) with the nuclear interaction.

The first impact is with how we formulate relativistic QED. Feynman provided a regularization method for computing finite amplitudes in QED [1]. The method of renormalization allowed for the elimination of ultraviolet infinities. While some, notably Dyson, have been less than satisfied with this method [2], it stands as the acceptable method for computing in QED. Further, this method has been extended to other field theories. This method is the most acceptable for computing field theory at the rather modest energies we probe with when compared to the Planck energy of quantum gravity.

In our discussion in chapter 6 the role of field fluctuations was discussed. Further work was done to illustrate the role of a quantum fluctuation of the $B^{(3)}$ field. This is one major departure in the theory. Even though the value of this fluctuation was quite modest compared, 10^{-10}, to the fluctuation of the standard magnetic field. However, this will have subtle influences. As illustrated this results in a correction of 5×10^{-5} to the Lamb shift. This estimate is just outside the estimate based upon fluctuation estimates, but in section 6.4 this discrepancy is demonstrated to be due to our naive application and understanding of nonAbelian electrodynamics. Further, the $B^{(3)}$ field associated with the photon will interact with the magnetic fluctuations in the vacuum. This will have subtle effects and contribute the propagator function of the photon. Further, these influences will need to be considered in the renormalization of the theory.

8.2 BASICS OF RELATIVISTIC $O(3)_b$QED

Nonabelian electrodynamics has been presented in considerable detail in a nonrelativistic setting. However, all gauge fields exist in spacetime and thus exhibits Poincare transformation. In flat spacetime these transformations are global symmetries that act to transform the electric and magnetic components of a gauge field into each other. The same is the case for nonabelian electrodynamics. Further, the electromagnetic vector potential is written according to absorption and emission operators that act on element of a Fock space of states. It is then reasonable to require that the theory be treated in a manifestly Lorentz covariant manner.

The theory is defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^a{}_{\mu\nu}F^{a\mu\nu} \tag{8.2.1}$$

with the stress-energy tensor components defined according to the

gauge covariant derivative

$$F^a{}_{\mu\nu} = \partial_\nu A^a{}_\mu - \partial_\mu A^a{}_\nu + ig\epsilon^{abc}[A^b{}_\nu, A^c{}_\mu], \tag{8.2.2}$$

where the spacial components of the four-vector potential are Hermition $A^{a\dagger}{}_i = A^a{}_i$, $i \in \{1,2,3\}$, and the temporal parts are anti-Hermitian $A^{a\dagger}{}_0 = -A^a{}_0$. Here g is the coupling constant for the gauge theory. The upper latin index refers to the internal degrees associated with the gauge theory. The variational calculus with this Lagrangian density leads to the field equation

$$\partial_\mu F^{a\mu\nu} + ig\epsilon^{abc}A^b{}_\mu F^{c\mu\nu} = 0. \tag{8.2.3}$$

From the field stress tensor components we may write the electric and magnetic field components as

$$E^a{}_i = F^a{}_{i0} = -\dot{A}^a_i - \nabla_i A^a{}_0 + ig\epsilon^{abc}A^b{}_0 A^c{}_i \tag{8.2.4a}$$

$$\epsilon^k_{ij}B^a{}_k = \nabla_i A^a{}_j - \nabla_j A^a{}_i + ig\epsilon^{abc}A^b{}_i A^c{}_j. \tag{8.2.4b}$$

The components of the vector potential are then expanded in a Fourier series of modes with creation and annihilation operators that act on the Fock space of states. If this is done according to a box normalization, in a volume V, with periodic boundary conditions we have,

$$A^a{}_i(\boldsymbol{r},t) = \sum_k \frac{1}{\sqrt{2\omega V}}\left(e_i a^a(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + e_i a^{a\dagger}(k)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right). \tag{8.2.5}$$

Here we are only considering the transverse components of the vector potential. With these vector potential components written according to these operators the electric and magnetic fields within $O(3)_b$ electrodynamics are then,

$$E^a{}_i = \sum_k \frac{1}{\sqrt{2\omega V}}\left(\frac{|k|}{c}e_i a^a(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \frac{|k|}{c}e_i a^{a\dagger(k)}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

$$\epsilon^{k}_{ij}B^{a}{}_{k} = \sum_{k}\frac{1}{\sqrt{2\omega V}}\left(k_{[j}e_{i]}a^{a}(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + k_{[j}e_{i]}e_{i}a^{a\dagger}(k)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right) + \tag{8.2.6}$$
$$ige^{abc}\sum_{kk'}e_{[j}e_{i]}\left(a^{b}(k)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a^{b\dagger}(k)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)\left(a^{c}(k')e^{i\boldsymbol{k}'\cdot\boldsymbol{r}} + a^{c\dagger}(k)'e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}\right)$$

If is then apparent that the Hamiltonian for this nonAbelian field theory is going to contain quartic terms in addition to the quadratic terms seen in abelian field theory, such as $U(1)$ electromagnetism.

If we consider nonAbelian electromagnetism we have a situation where the vector potential component $A^{3}{}_{i}$ vanish and where $A(1)_{i} = A^{(2)*}{}_{i}$. The annulment of the components $A^{3}{}_{i}$ has been studied in the context of the unification of nonAbelian electromagnetism and weak interactions, where on the physical vacuum of the broken symmetry $SU(2)\times SU(2)$ the vector boson corresponding to $A^{3}{}_{i}$ is very massive and vanishes on low energy scales. This means that the 3-component of the magnetic field is then

$$\boldsymbol{B}^{3} = i\frac{e}{\hbar}\boldsymbol{A}^{1}\times\boldsymbol{A}^{2}. \tag{8.2.7}$$

It is apparent that for $A^{3}{}_{i} = 0$ that the electric field component does not contain a product of potential terms. In general the vanishing of this term occurs if there are no longitudinal electric field components. Within the framework of most quantum electrodynamic, or quantum optical, calculations this is often the case. The $\boldsymbol{B}^{3}$ field then is a Fourier sum over modes with operators $a^{\dagger}{}_{k-q}a_{q}$. The $\boldsymbol{B}^{3}$ field is then directed orthogonal to the plane defined by $\boldsymbol{A}^{1}$ and $\boldsymbol{A}^{2}$. The four dimensional dual to this term is defined on a timelike surface that has the interpretation, under dyad-vector duality in three dimension as, as an electric field or $\boldsymbol{E}^{3}$. The vanishing of the $\boldsymbol{E}^{3}$ can then be seen by the nonexistence of the raising and lowering operators a^{3}, $a^{3\dagger}$, where the $\boldsymbol{B}^{3}$ exists solely due to the occurrence of raising and lowering operators that $\boldsymbol{A}^{1}$ and $\boldsymbol{A}^{2}$ are expanded according to. This represents a breakdown of duality in four dimensions and the requirement that $\boldsymbol{B}$ be a longitudinal field.

This nonabelian gauge theory satisfies the usual transformation properties. If $\mathcal{M}$ is the base manifold in four dimensions then the gauge theory is determined by an internal set of symmetries described by a principal bundle. Let U_{α}, where $\alpha = 1,2,...,n$, be an atlas of charts on the $\mathcal{M}$. The transitions from one chart to another is given by $g_{\alpha\beta}: U_{\beta}\rightarrow U_{\alpha}$, where these determine the transition functions between sections on the principal bundle. The transform between one section to another is given by

$$s_{\alpha} = g_{\alpha\beta}s_{\beta} = e^{iX_{\alpha\beta}}s_{\beta}. \tag{8.2.8}$$

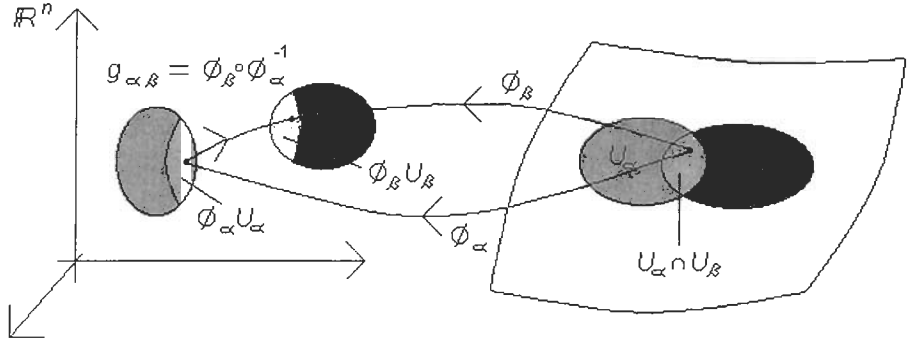

Figure 8.1

From this point we will suppress the chart indices to indicate sections and use the notation s, s' for the two charts with $gs = s'$. Now let the differential operator d act on s'

$$ds' = (gds + sdg). \tag{8.2.9}$$

Now define $g^{-1}dg$ as a connection coefficient A on the section s,

$$ds' = g(ds + ig^{-1}dg)s. \tag{8.2.10}$$

Now consider the action of g on $(d+A)s$ which equals $(d + A')s'$

$$(d + A')s' = g(d + A)s$$

$$= g(d + A)g^{-1}gs = (d + gAg^{-1} + gdg^{-1})s'. \quad (8.2.11a)$$

This is a fundamental definition for how a gauge connection transforms:

$$A' = gAg^{-1} + gdg^{-1}. \quad (8.2.11b)$$

Now we consider the group element g to be defined by algebraic generators so that $g = e^{iX}$. Further consider the transformation to be sufficiently small so that $e^{iX} \simeq 1 + iX$,

$$A's' = \big((1 + iX)A(1 - iX) - idX\big)s' = (A + i[X, A] - idX)s'. \quad (8.2.12)$$

If we are working with local gauge transformations where A is flat then we can work with the pure gauge term $(dg)d^{-1} = idX$ as the gauge connection.

Now to get the fields from this definition we have that the fields are defined to be under a gauge transformation

$$dA' = d(gAg^{-1} + (dg)g^{-1}). \quad (8.2.13)$$

From this we find that

$$dA' = g(dA + A \wedge A)g^{-1}. \quad (8.2.14)$$

which means that the fields transform homogenously under local gauge transformations. Just as the chart indices have been suppressed so have the indices for the internal symmetry space.

Now for nonAbelian electromagnetic field theory we have the 3-Lie index component of the field, and for the magnetic field $B^{(3)}$ it equals

$$\boldsymbol{B}^{[3]} = i\boldsymbol{A} \times \boldsymbol{A}^*, \quad (8.2.15)$$

where this is a component that emerges from the $A \wedge A$ term. We are working here with $\hbar = c = 1$. Then under local gauge transformations we will have that

$$\boldsymbol{B}^{(3)\prime} = ig(\boldsymbol{A} \times \boldsymbol{A}^*)g^{-1}, \quad (8.2.16)$$

where g is the group element for the$O(3)_b$theory. Then one can go on and write $g \simeq 1 + iX$ and find that

$$\boldsymbol{B}^{(3)\prime} = i(1+iX)(\boldsymbol{A}\times\boldsymbol{A}^*)(1-iX) = i\boldsymbol{A}\times\boldsymbol{A}^* - [X,[\boldsymbol{A},\boldsymbol{A}^*]] + O(X^2). \quad (8.2.17)$$

This can be written according to Lie derivative, and if X is a generator for a global gauge transformation then this double commutator vanishes. We are then left with

$$\boldsymbol{B}^{(3)\prime} = \boldsymbol{B}^{(3)} - iL_{\boldsymbol{X}}\boldsymbol{B}^{(3)}, \quad (8.2.18)$$

where the last term is the Lie derivative of $\boldsymbol{B}^3$ with respect to the variable X, here parameterized along a path.

In the case of quantum field theory the section determines the Hilbert space of states under a certain gauge. This choice of gauge then determines the unitary representation of the Hilbert space. We may then replace the section with the fermion field ψ, that acts on the Fock space of states. It is then apparent that a gauge transformation $A^a{}_\mu \to A^a{}_\mu + \delta A^a{}_\mu$ is associated with a unitary transform of the fermion field $\psi \to \psi + \delta\psi$. The unitary transformation of the fermion may be written according to $\psi' = U\psi$ where the unitary matrix is represented as the line integral along a path

$$U = \mathcal{T}e^{-ig\int^{\Gamma} A_\mu dx^\mu}, \quad (8.2.19)$$

where $\mathcal{T}$ is the time ordering operator that arranges fields in a product in a time ordered sequence. The application of the differential operator $\boldsymbol{d}$ on the unitary matrix gives,

$$\boldsymbol{d}U = -ig(A_\mu - A'_\mu)\boldsymbol{d}x^\mu U, \quad (8.2.20)$$

which leads to the result

$$\frac{i}{g}U^{\dagger}dU \; - \; U^{\dagger}(A_{\mu} \; - \; A'_{\mu})dx^{\mu}U \; = \; 0. \qquad (8.2.21)$$

This demonstrates the association between the unitary transformation of the Fermion field and the gauge theory.

More work is required to couple the gauge theory to the Fermion. We have the gauge field determined by its Lagrangian density, and the Fermion field determined by the Dirac Lagrangian density

$$\mathcal{L}_D \; = \; -\bar{\psi}(\gamma^{\mu}\partial_{\mu} \; + \; m)\psi. \qquad (8.2.22)$$

However, these two Lagrangian densities do not couple the two fields together. This requires that free field equation for the gauge field becomes,

$$\partial_{\mu}F^{a\mu\nu} \; + \; ig\epsilon^{abc}A^{b}{}_{\mu}F^{c\mu\nu} \; = \; j^{\nu}. \qquad (8.2.23)$$

Since this field equation is obtained by the Euler-Lagrange equation the inhomogenous term is the result of

$$j^{\nu} \; = \; \frac{\partial \mathcal{L}}{\partial A_{\nu}}, \qquad (8.2.24)$$

this implies the addition of an interaction Lagrangian density $\mathcal{L}_i \; = \; j^{\nu}A_{\nu}$. The current term is then determined by the Dirac field and is $j^{\nu} \; = \; \bar{\psi}\gamma^{\nu}\psi$. The subject of mass remormalization also requires that an additional interaction term be included $\bar{\psi}\gamma^{\nu}\psi\delta m$, where δm is the difference between the physical mass and the bare mass[3].

The total Lagrangian $\mathcal{L} \; = \; \mathcal{L}_G \; + \; \mathcal{L}_D \; + \; \mathcal{L}_i$ then involves the interaction between Fermions and the gauge field. The Dirac field will be generically considered to be the electron and the gauge theory

will be considered to be the nonAbelian electromagnetic field. The theory then describes the interaction between electrons and photons. A gauge theory involves the conveyance of momentum form one particle (electron) to another by the virtual creation and destruction of a vector boson (photon) that couples to the two electrons. The process can be diagrammatically represented as

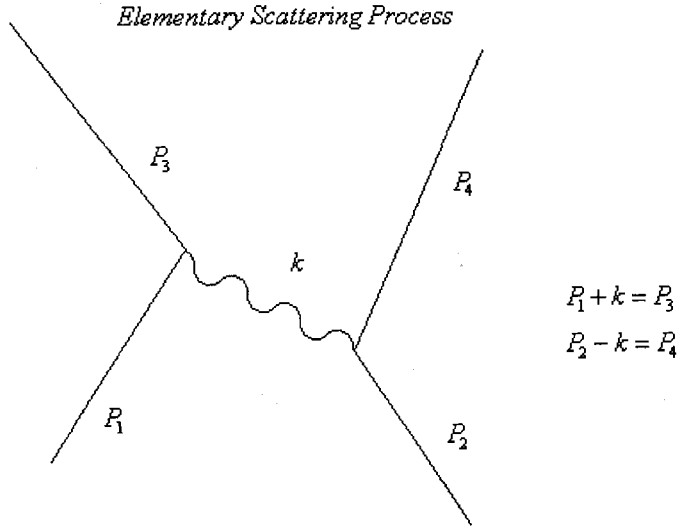

Figure 8.2

The process $p_1 \; + \; p_2 \; \rightarrow \; p_3 \; + \; p_4$ then involves the conservation of momentum, for there is no creation of any averaged momentum from the virtual quantum fluctuation. This process can be examined within the Coulomb gauge $\nabla \cdot \boldsymbol{A} \; = \; 0$. The field equation is then

$$\nabla \cdot \boldsymbol{E} \; = \; -\nabla^2 A_0 \; = \; ie\bar{\psi}\gamma^0\psi \; = \; e\rho, \qquad (8.2.25)$$

which has the solution

$$A_0(\boldsymbol{r},t) \; = \; e\int d^3r \frac{\rho(\boldsymbol{r}',t)}{4\pi|\boldsymbol{r} \; - \; \boldsymbol{r}'|}. \qquad (8.2.26)$$

The amplitude for this simple scattering process consists of the electromagnetic Hamiltonian and the interaction process. These two terms produce the amplitude

$$\frac{(-i)^2e^2}{2}\int d^4xd^4x'\mathcal{T}j_\mu A^\mu j_\nu A^\nu = \frac{(-i)^2e^2}{2}\int d^4xd^4x' j_\mu G^{\mu\nu}(x-x')j_\nu, \tag{8.2.27}$$

where $G^{\mu\nu}(x - x')$ is the propagator of the field that satisfies

$$\mathcal{T}A^\mu A^\nu G^{\mu\nu}(x - x') = \frac{-i\delta(\boldsymbol{r} - \boldsymbol{r}')}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{8.2.28}$$

For the purely transverse field the spacial components of the propagator are

$$G^{ij}(\boldsymbol{r} - \boldsymbol{r}') = \frac{-i}{4\pi^2}\int \frac{d^4k}{k^2 - i\epsilon}\left(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\right)e^{i\boldsymbol{k}(\boldsymbol{r} - \boldsymbol{r}')}, \tag{8.2.29}$$

where $k^2 = \boldsymbol{k}^2 - k_0^2$. This is seen to be the Fourier transform of the propagator in momentum space. The temporal components are then seen to be

$$\frac{1}{8\pi^3}\int \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\boldsymbol{k}^2} = \frac{1}{r}. \tag{8.2.30}$$

The amplitudes for the process are then evaluated on the initial and final states of the electrons. This then results in the matrix elements,

$$\frac{(-i)^2e^2}{2}\int d^4xd^4x'\langle p_2|j_\mu|p_1\rangle G^{\mu\nu}(x - x')\langle p_4|j_\nu|p_3\rangle \tag{8.2.31}$$

for the amplitudes. The amplitudes $\langle p_2|j_\mu|p_1\rangle$ and $\langle p_4|j_\nu|p_3\rangle$ are then represented as plane waves

$$\langle p_2|j_\mu|p_1\rangle = e^{i(p_1 - p_2)r}X_\mu, \tag{8.2.32a}$$

$$\langle p_4|j_\mu|p_3\rangle = e^{i(p_3 - p_4)r}Y_\mu, \tag{8.2.32b}$$

where X_μ and Y_μ are independent of the position coordinates. By momentum conservation we demand that $k_\mu = p_{1\mu} - p_{2\mu} = p_{4\mu} - p_{3\mu}$. The propagator acts on these matrix elements to give the amplitude,

$$\frac{(-i)^2e^2}{2}\int d^4xd^4x'\frac{-i}{k^2}X^\mu Y_\mu. \tag{8.2.33}$$

Finally, this expression can be evaluated for many possible gauges according to

$$\frac{(-i)^2e^2}{2}\int d^4xd^4x'\frac{-i}{k^2}(X_iY_i - \beta X_0Y_0)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{8.2.34}$$

where for $\beta = 0$ this is evaluated in the Feynman gauge, and for $\beta = -1$ this is evaluated in the Landau gauge.

This example, within $U(1)$ electrodynamics can then be seen in the light of nonabelian electrodynamics. This may simply be seen by the replacement $A_\mu \to t^aA^a{}_\mu$, where t^a is a structure constant that obeys $[t^a, t^b] = 2\epsilon^{abc}t^c$. Then the time ordered product may be written as

$$\mathcal{T}j_\mu A^\mu j_\nu A^\nu = \frac{1}{2}\mathcal{T}(\{t^a,\ t^b\} + [t^a,\ t^b])j_\mu A^{a\mu}j_\nu A^{b\nu}, \tag{8.2.35}$$

where the product of the structure constants is written according to its symmetric and antisymmetric parts. The physical requirement that $A^3{}_\mu = 0$ is then imposed. From this the symmetric part of the time ordered product yields the same result as found in the $U(1)$ case. The antisymmetric part is then easily seen to be

$$\mathcal{T}j_\mu A^\mu j_\nu A^\nu = \frac{1}{2}\mathcal{T}[t^a,\ t^b]j_\mu A^{a\mu}j_\nu A^{b\nu} = 0. \tag{8.2.36}$$

This means that on the tree level there are no contributions from the $\boldsymbol{B}^{(3)}$ field.

In order to compute an amplitude contribution from the $\boldsymbol{B}^3$ field a process that is second order must be considered. This involves a loop diagram of the form

Virtual Photon Loop Correlated with B^3 Field Fluctuation

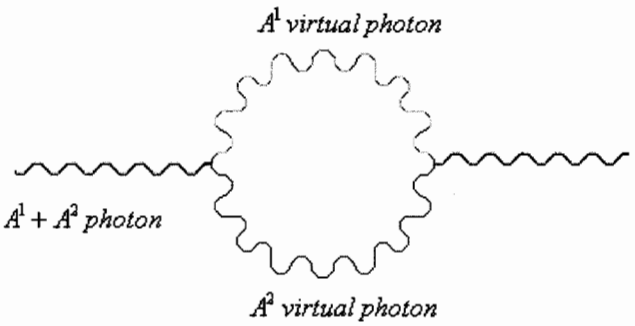

Figure 8.3

The propagator then assumes the form

$$G_{\mu\nu}(\boldsymbol{r} - \boldsymbol{r}') = \alpha^2 \mathcal{T} t^a t^b t^c t^d t^e t^f A^a{}_\mu(\boldsymbol{r}) \times$$

$$\int \frac{d^4k'}{(2\pi)^2} \left(A^{b\rho}(k') A^c{}_\sigma(k')\right) \int \frac{d^4k''}{(2\pi)^2} \left(A^{e\rho}(k'') A^{f\sigma}(k'')\right) A^f{}_\nu, \quad (8.2.37)$$

where the integrations exists since the vertices tied to the loop do not constrain momentum conservation. The four fields in the momentum integrals under the action of the antisymmetric portion of the structure constants contribute to,

$$\int \frac{d^4k'}{(2\pi)^2} B^3{}_\rho(k') \int \frac{d^4k'\prime}{(2\pi)^2} B^{3\rho}(k''). \quad (8.2.38)$$

This term is then a sum over all possible fluctuations of the $\boldsymbol{B}^{(3)}$ that couple to the virtual photon coupled to the electrons. This means that the propagator is of the form

$$G_{\mu\nu}(\boldsymbol{r} - \boldsymbol{r}') = \alpha^2 \mathcal{T} A^a{}_\mu(\boldsymbol{r}) \left(\int \frac{d^4k'}{(2\pi)^2} B^3{}_\rho(k') \int \frac{d^4k'\prime}{(2\pi)^2} B^{3\rho}(k'') \right) A^f{}_\nu. \quad (8.2.39)$$

This then contributes an amplitude,

$$\langle \ | \ \rangle_{B^3} = \frac{i}{k^2} X^\mu Y_\mu \int \frac{d^4k'}{(2\pi)^2} \int \frac{d^4k''}{(2\pi)^2} \Big(e^{i(\boldsymbol{k}' - \boldsymbol{k}'') \cdot \boldsymbol{r}} \times$$

$$\frac{1}{k^2} \Big(\boldsymbol{X} \cdot \boldsymbol{Y} - \frac{(\boldsymbol{k}' \cdot \boldsymbol{X})(\boldsymbol{k}'' \cdot \boldsymbol{Y})}{|\boldsymbol{k}|^2} \Big) - \frac{X_0 Y_0}{|\boldsymbol{k}|^2} \Big). \quad (8.2.40)$$

Here $|k|$ is the magnitude of the four vector, and $|\boldsymbol{k}|$ is the magnitude of the spacial part of the four vector k^μ. The integrals in this amplitude suffer from the usual ultraviolet divergence that can be removed through regularization techniques.

This is an introduction to the sort of process that may occur in $O(3)_b$ electrodynamics. In effect the $\boldsymbol{B}^3$ field produces quantum vortices that interact with electrons, as well as other charged particles, where these vortices are quantized states and exist as fluctuations in the QED vacuum. As mentioned earlier the dual of the $\boldsymbol{B}^{(3)}$ field does not exist as an electric field. These quantum fluctuations are easily seen to be associated with the $\boldsymbol{E}^{(1)}$ and $\boldsymbol{E}^{(2)}$ fields

$$\delta \boldsymbol{B}^3 = \frac{ie}{\hbar} (\delta \boldsymbol{A}^1 \times \boldsymbol{A}^2 + \boldsymbol{A}^1 \times \delta \boldsymbol{A}^2)$$

$$= \frac{ie}{\hbar} \frac{1}{\omega^2} (\delta \boldsymbol{E}^1 \times \boldsymbol{E}^2 + \boldsymbol{E}^1 \times \delta \boldsymbol{E}^2). \quad (8.2.41)$$

This indicates a number of things. The first is that the quantum fluctuations of the $\boldsymbol{B}^3$ field are accompanied by fluctuations in the standard electric field. Further, the ultraviolet divergence of the above integral is probably unimportant due to the $\frac{1}{\omega^2}$ relationship with the fluctuation. This tends to imply an infrared divergence; however the analysis with cavity QED indicates that the statistical occurrence of states is such that the divergence is damped. Infrared divergences are known to be of little trouble due to their statistical occurrence. This was essentially illustrated in section 6.7.

This approach to QED also suggests that methods for renormalization are applicable. If the quantum fluctuation of the electric field is associated in part with fluctuations in the $\boldsymbol{B}^{(3)}$ field, then the divergences that occur at the ultraviolet regime can possibly be absorbed into fluctuations with the $\boldsymbol{B}^{(3)}$ field. Physically these are damped out by the $\frac{1}{\omega^2}$ term in the Feynman path integral. This potentially leads to an additional physical understanding of the disappearance of divergences that occur at the high frequency domain of QED. This would be possible if the divergences in $U(1)$ electrodynamic processes, which exist as a subset of $O(3)_b$electrodynamic processes, can be absorbed into integrals that involve photon loop processes associated with fluctuations in the $\boldsymbol{B}^{(3)}$ field. These fluctuations appear to quench ultraviolet divergences by its $\frac{1}{\omega^2}$ behavior, and it may quench divergences for all processes if these divergences can be absorbed into $\boldsymbol{B}^{(3)}$ fluctuations.

At high energies it is reasonable to think that the electroweak theory is $SU(2) \times SU(2)$. The current $SU(2) \times U(1)$ theory is renormalizable since the vector boson propagator is "mixed" with the $U(1)$ field that is renormalizable. With an electroweak theory extended to include nonAbelain electrodynamics essentially the same will occur where the unphysical term $m\sigma^a A^{a\mu}\partial_\mu \zeta$, for ζ an unphysical field that oscillates around the Higgs minimum, is cancelled by the gauge fixing Lagrangian density,

$$\mathcal{L}_{gf} = \frac{1}{2\zeta}(T^a \partial_\mu A^{a\mu} + \xi m\zeta)^2. \tag{8.2.42}$$

Here for $\xi = 1$ we have the Feynman gauge, and $\xi = 0$ is the Landau gauge. This gauge fixing term will enter into the massive boson propagators for the $\boldsymbol{A}^{(3)}$ field. The propagator will be of the form

$$-\frac{i}{p^2 - m^2 + i\epsilon}\left(\delta_{\mu\nu} - \frac{\xi p_\mu p_\nu}{p^2}\right) \tag{8.2.43}$$

The existence of this propagator will be the largest addition to the physics of electroweak interactions when electromagnetism is non-

Abelian. Further discussion on the subject of $SU(2) \times SU(2)$ electroweak theory is given by the authors in [4]. Estimates on the mass of this boson are around four times the mass of the Z_0 boson and should be observable with the CERN Large Hadron Collider.

8.3 RENORMALIZATION OF $O(3)_b$ QED

Quantum electrodynamics involves the interaction of electrons, or other charged particles, and photons, where the interaction between two electrons involves the exchange of a virtual photon. Based upon equation (8.2.43) for a propagator that the interaction between electrons and photons means that the potential function may be written as [5]

$$A = \frac{\bar{v}_3 \gamma^\mu v_1 \bar{v}_4 \gamma_\mu v_2}{q^2}, \tag{8.3.44}$$

as illustrated in figure 8.1. Here v_i are the Dirac spinors for the electrons. This leads to the expectation that the potential in the Coulomb case is $\Phi \simeq e^2/r$. The issue of renormalization is apparent in that the potential and propagator is divergent in the limit that the distance between the electrons approaches zero. Further, as this distance decreases then by the Heisenberg uncertainty principle $\Delta p \Delta x \geq \hbar$ means that the momentum exchanged by the electrons due to fluctuations on that small scale becomes divergent. Carried further, this means that the vacuum is filled with virtual quanta that have enormously high momentum fluctuations. When these virtual quanta couple to systems they contribute divergences in the limit their wavelengths approach zero: the ultraviolet divergence.

There are two types of processes that are divergent due to this coupling to virtual quanta. There are the self-energy of the electron, vacuum polarizations, and vertex functions.

Self Energy of the Free Electron

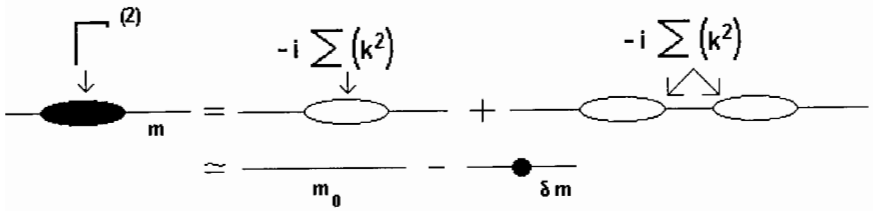

Figure 8.4

With $O(3)_b$ QED the major difference emerges from the effective photon bunching or interactions that can result in a photon loop, composed of an $A^{(1)}$ photon and an $A^{(2)}$ photon. This loop will be associated with a quanta of $B^{(3)}$ field with an intensity $e/\hbar$ mentioned in chapter 6. Equation (6.6.92) illustrates how this fluctuation in the $A^{(1)}$ and $A(2)$ potentials are associated with this magnetic fluctuation. The other renormalization techniques in $U(1)$ QED still apply, and are demonstrated below, and the renormalization of divergences associated with the $B^{(3)}$ magnetic fluctuation is also illustrated.

We will discuss at some length the interaction of a free electron with the vacuum, for this is similar to the renormalization problem presented by $O(3)_b$electrodynamics. An electron interacts with the vacuum according to the Dirac equation

$$\Big(\gamma^{\mu}(\partial_{\mu} - ieA_{\mu}) - m\Big)\psi = 0. \tag{8.3.45}$$

Even if there is no electromagnetic field present the vector potential exhibits fluctuations $A_{\mu} = \langle A_{\mu}\rangle + \delta A_{\mu}$, so that even if there is only the vacuum physics still involves this fluctuation. This is also seen in the zero point energy of the harmonic oscillator expansion of the fields. So an electron will interact with virtual photons. If we represent all of these interactions as a blob coupled to the path of an

electron, this blob may be expanded into a sum of diagrams where the electron interacts with photons. Each term is an order expansion and contributes a term on the order of $\alpha = e^2/\hbar c$. A single loop contributes the integral

$$\simeq \int_{-\infty}^{\infty} \frac{\gamma \cdot p - m}{p + k}, \tag{8.3.46}$$

which has an ultraviolet divergence as $k \to \infty$ for $\int_0^{\infty} dk\ k$ The standard approach amounts to imposing a cut off in the integral Λ so the integration is

$$\int_0^{\infty} dk\ k \to \int_0^{\Lambda} dk\ k, \tag{8.3.47}$$

so that for an electron of mass m this defines a mass counter term

$$\delta m = \frac{3\alpha m}{2\pi} log\Big(\frac{\Lambda}{m}\Big). \tag{8.3.48}$$

Then given the bare mass of the electron as m_0 we have the mass of the electron as $m = m_0 - \delta m$. By the Dirac equation this also contributes a counter term into the Lagrangian $\delta m\bar{\psi}\psi$

The counter term is computed by performing a perturbation expansion of the Green's function, or propagator for the free electron. The entire process is represented by $\Gamma^{(n)}$, which in general is determined by a time ordered product of fields,

$$\Gamma^{(n)}(p_1, p_2, \ldots, p_{n-1}) = \int \prod_i dx_i e^{ip_i x'} \langle 0|\mathcal{T}\phi_1\phi_2 \ldots \phi_{n-1}|0\rangle, \tag{8.3.49}$$

and generally describes processes of the type illustrated in figure 8.5

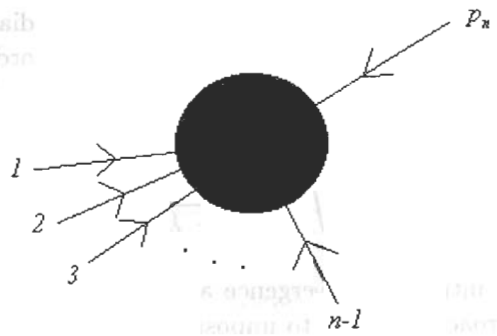

Figure 8.5

This process may also be computed from the path integral,

$$Z[J(x)] = \int \mathcal{D}\{\phi\} e^{iS + J\phi} \tag{8.3.50}$$

by the functional derivative the path integral according to the test source function $J(x)$

$$\Gamma^{(n)} = Z^{-1}[J(x)] \frac{\delta^n}{\delta J(x)^n} Z[J(x)]|_{J=0} \tag{8.3.51}$$

The propagator $\Gamma^{(2)}$ for the free electron is approximated by loops that are given by the function $-i\Sigma(k^2)$ connected by electron propagators of the form $i/(k^2 - m^2)$. So the propagator that computes the free electron with the mass counter term is then given by

$$\frac{i}{k^2 - m^2 - i\epsilon} = \frac{i}{k^2 - m^2} + \frac{i}{k^2 - m^2}(-i\Sigma(k^2))\frac{i}{k^2 - m^2}$$

$$+ \frac{i}{k^2 - m^2}(-i\Sigma(k^2))\frac{i}{k^2 - m^2}(-i\Sigma(k^2))\frac{i}{k^2 - m^2} + \ldots. \tag{8.3.52}$$

This is illustrated in figure 8.4. This series may then be written in a more compact form with

$$\Gamma^{(2)}(k^2) = \frac{i}{k^2 - m_0^2 - \Sigma(k^2)}, \tag{8.3.53}$$

where $\Sigma(k^2)$ is Taylor expanded around the mass m_0 with the result that

$$\Gamma^{(2)}(k^2) = \frac{i}{k^2 - m_0^2 - \delta m^2}. \tag{8.3.54}$$

This is a matter of replacing all of the correction terms with a finite number, in this case one, counter terms that may be evaluated

We now add a term to the harmonic oscillator Lagrangian of the form $\lambda\phi^4$, where ϕ is a field that represents the field coupled to the electron. To evaluate the amplitude we then have the integral of the form

$$I^2 = \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - \mu^2}. \tag{8.3.55}$$

To perform a dimensional regularization of this integral we replace the integral with

$$I^2(\omega) = \frac{\lambda}{2} \int \frac{d^{2\omega}k_e}{(2\pi)^4} \frac{1}{k_e^2 - \mu^2}, \tag{8.3.56}$$

where the dimension of the system has been replaced by 2ω. We then use

$$d^{2\omega}k_e = \frac{2\pi^\omega}{\Gamma(\omega)} k_e^{2\omega-1} dk_e, \tag{8.3.57}$$

and $\lambda = (M^2)^{2-\omega}\lambda_\omega$ to obtain the solution to the integral as

$$I^2(\omega) = -\frac{\lambda_\omega}{2}\left(\frac{\pi}{2\pi}\right)^\omega (M^2)^{2-\omega}\lambda_\omega (m^2)^{\omega-1} \frac{\Gamma(1-\omega)}{\Gamma(1)}. \tag{8.3.58}$$

Now the "trick" used is to identify $\omega = 2 - \epsilon$ to obtain,

$$I^2(\omega) = \frac{\lambda m^2}{32\pi^2}\left(1 - \gamma - log\left(\frac{m^2}{4\pi M^2}\right)\right), \tag{8.3.59}$$

from which the mass counter term is defined. Here γ is the Euler-Mascheroni constant $\gamma = .5772\ldots$

A similar divergent process exists with $O(3)_b$ electrodynamics with fluctuations associated with the $\boldsymbol{B}^{(3)}$ field. It is associated with computing the propagator for a photon loop, as illustrated in figure 8.3. The integral involved is of the form

$$I^4 = l^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i\epsilon)((k+q)^2 + i\epsilon)}. \tag{8.3.60}$$

By letting $4 \rightarrow 2\omega$ we arrive at an integral of the form

$$I^4 = \lambda^2 \int \frac{d^{2\omega}k}{(2\pi)^4} \frac{1}{(k^2 + i\epsilon)((k+q)^2 + i\epsilon)}, \tag{8.3.61}$$

then if the self interaction term is written as

$$\lambda = 4 - 2\omega = (Q^2)^{2-\omega}\lambda_\omega, \tag{8.3.62}$$

and the integration measure is redefined as

$$d^{2\omega}k = k^{2\omega-1}dk_\omega, \tag{8.3.63}$$

then

$$I^4 = l_\omega (Q^2)^{2-\omega} \times \int \frac{dk_\omega}{(2\pi)^4} \frac{k^{2\omega-1}}{(k^2 + i\epsilon)((k+q)^2 + i\epsilon)}. \tag{8.3.64}$$

This integral may then be evaluated as:

$$I^4 = const * q^{\omega-2} \frac{\Gamma(\omega)\Gamma(2-\omega)}{\Gamma(2)}, \tag{8.3.65}$$

where *const* is the constant

$$\kappa = \frac{l_\omega (Q^2)^{2-\omega}}{(2\pi)^{2\omega}}. \tag{8.3.65a}$$

So in a more compact this leads to the result

$$I^4 = \frac{\lambda_\omega}{(2\pi)^{2\omega}} \left(\frac{Q}{q}\right)^{4-\omega} \times \frac{\Gamma(\omega)\Gamma(2-\omega)}{\Gamma(2)}, \tag{8.3.66}$$

or

$$I^4 = \frac{\lambda_\omega}{(2\pi)^{2\omega}} \left(\frac{Q}{q}\right)^{4-\omega} \times \left(\frac{1}{3+\omega} + 1 - \gamma\right)\left(\frac{1}{1-\omega} + 1 - \gamma\right). \tag{8.3.67}$$

Then we have that since $\omega = 2 - \epsilon$, the trick of dimensional regularization then we see that

$$\Gamma(2 - \omega) = \Gamma(\epsilon)$$

$$\Gamma(\omega) = \Gamma(2 - \epsilon). \tag{8.3.68}$$

Now this integral can be put in the form

$$I^4 = \frac{1}{5}\frac{q}{32\pi}^4 \times \left(1 - 6\left((\gamma + 4log(\frac{q}{Q})\right)\right), \tag{8.3.69}$$

which is a finite quantity. Here γ is the Euler-Mascheroni constant.

This calculation demonstrates that the loop fluctuation of a photons, which correlated to a virtual quanta $e/\hbar$ of $\boldsymbol{B}^{(3)}$ field, can be calculated to be finite with out divergence. So the virtual fluctuation of a $\boldsymbol{B}^{(3)}$ field does not lead to an ultraviolet divergence, and thus $O(3)_b$ QED is renormalizable by dimensional regularization.

The issues of vacuum polarization and vertices may be computed in the same manner as seen with $U(1)$ electrodynamics. Effectively $O(3)_b$quantum electrodynamics appears to be, based on these initial regularization exercises, to be free of intractable ultraviolet divergences. The calculation of the Lamb shift also indicates that $O(3)_b$QED is also free of such divergences in the infrared region. This is a good sign that the theory at least is not frought with computational intractabilities that cast sever doubts on a theoretical level.

8.4 $B^{(3)}$ FIELD AS A VACUUM SYMMETRY

It is standard to adopt the rule in $U(1)$ electrodynamics that with the Hamiltonian of the form

$$H = \frac{1}{2m}|p - eA|^2, \tag{8.4.70}$$

that the gauge potential acts only once. The quadratic term is basically eliminated and ignored. This rule is essentially what $O(3)_b$ electrodynamics challenges. We then consider the role of $\boldsymbol{A}\cdot\boldsymbol{A}^* = \boldsymbol{A}^{(1)}\cdot\boldsymbol{A}^{(2)}$ within this Hamiltonian. This Hamiltonian leads to the evolution operator $U = e^{-iHt}$, that has the form

$$U = e^{-iH_0t}e^{\boldsymbol{A}^{(1)}\cdot\boldsymbol{A}^{(2)}}, \tag{8.4.71}$$

where H_0 is the Hamiltonian without the term quadratic in the potentials. The vector potentials may then be written as

$$\boldsymbol{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(e_x + ie_y)(a_k e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t} - a_k^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{r} + i\omega t}), \tag{8.4.72}$$

so that the modulus square of this operator is

$$\boldsymbol{A}^{(1)}.\boldsymbol{A}^{(2)} = {A^{(0)}}^2(a^\dagger a + \frac{1}{2} - \frac{1}{2}(a^{\dagger 2}e^{-2i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + a^2 e^{2i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)})). \tag{8.4.73}$$

This result is very interesting for the first two terms on the right hand side are just the standard harmonic oscillator Hamiltonian for the electromagnetic field H_{em}, and the latter are terms easily seen to be incommensurate with that Hamiltonian under commutation. As a result the evolution operator is then

$$U = e^{-i(H_0 + H_{em})t}e^{(za^{\dagger 2} + z^*a^2)}, \tag{8.4.74}$$

for $z = te^{-2i(\boldsymbol{kr} - \omega)t}$ The operator $S(z) = e^{(za^{\dagger 2} + z^*a^2)}$ is a squeezed state operator, which involves symmetries that lie outside those defined strictly by the Hamiltonian.

There are some reasons for supposing that the $\boldsymbol{B}^{(3)}$ may correspond to such symmetries. In the next two sections a presentation is given on how nonAbelian electrodynamics is unified with the weak interactions. Below it will be concluded that there is a duality between the $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ fields. From this it can be easily seen that the Lagrangian for these two fields vanishes. This is evidently a curious situation where there should exist a field, but where it has no Lagrangian. This would imply that there is no dynamics associated with this field. The argument is made that the existence of a massive $A^{(3)}$ field breaks this duality. This is then invoked to justify that $\boldsymbol{E}^{(3)} = 0$. However, this then creates a further difficulty. Electric and magnetic fields trasnform by the Lorentz group as $E_z{}' = \gamma(E_z - \beta B_y)$ and $B_x{}' = \gamma(B_x - \beta E_y)$. This leads to the unsettling prospect that if $\boldsymbol{B}^{(2)} > 0$ and $\boldsymbol{E}^{(2)} = 0$ that there is then a breaking of the Lorentz symmetry to spacetime. This means that unless there is an associated 3-electric field that we may have to conclude that $\langle\boldsymbol{B}^{(3)2}\rangle = \langle\boldsymbol{E}^{(3)2}\rangle = 0$. This then gives weight to the prospect that nonAbelian electrodynamics corresponds to non-Lagrangian symmetries or operators in electrodynamics.

It is still possible to have the Sagnac effect. If we consider the counter-rotating portion of the nonabelian contribution to,

$$\theta = i\oint \mathcal{D}_\mu dx^\mu = i\int\int [\mathcal{D}_\mu, \mathcal{D}_\nu]d\sigma^{\mu\nu}, \tag{8.4.75}$$

we have that

$$\theta = i\int (a^{\dagger 2}e^{-2i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} + a^2 e^{2i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)})J^3. \tag{8.4.76}$$

An evaluation of the integral and using the fact that $J^3 \propto \Omega$, the rotation of the platform it is then apparent that

$$\Delta\theta = \langle lnS(z)\rangle \tag{8.4.77}$$

where $z \simeq e^{4\omega\Omega\mathcal{A}/c^2}$, where the doubling of the frequency argument occurs from the existence of two path A and C. This apparently gives the Sagnac effect according to the squeezing of light.

Would this mean that most everything presented in this monograph is wrong? No, but it does mean that the classical results are purely pedagogical tools. The quantum results may still hold. For instance, with the cavity QED work the term $H_{B^3{}_{int}}$, equation (6.4.62) corresponds to the absorption of one photon and the emission of another as the atom changes its internal state. Similarly, the term $H_{B^{(3)}}$, equation (6.4.63) corresponds to the absorption of two k, k' mode photons and the emission of $k + q$ and $k' - q$ by the atom during a quantum fluctuation between its two atomic states.

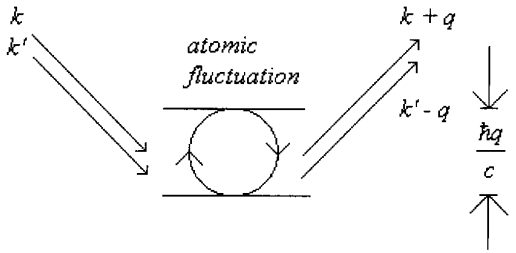

Figure 8.6

This indicates that the quantum mechanical aspects of this theory is valid, and the Hamiltonian, $H_{B^{(3)}}$, then involves quantum fluctuations in the atomic states. It must also be noticed that this interaction Hamiltonian is only real since it involves the introduction of a quantum system. In the absence of this atom we would no longer obtain this photon-photon coupling. In the case of photon loops this process must then be considered as attached to a fermion line, where the fermion has a fluctuation in its momentum to give rise to this photon graph. Similarly the Lamb shift is changed by a factor of $\alpha = 1/137$, as the current coupling in equation (6.7.94) involves a field fluctuation. This means that the Lamb shift due to nonAbelian electrodynamics is 3.89×10^{-7}. This is curiously at the lower end of our estimate based on fluctuation estimates and the result in section 6.6.

This means that states are not completely described as eigenstates of the Hamiltonian. While they posses kinematical properties of states,

but the squeezing of these states are not determined by diagonal operators and are thus not Hamiltonian or Lagrangian symmetries. This is an interesting result, for this implies that squeezed states in QED are connected to an underlying nonAbelian symmetry. This would continue to be the case even if there is no classical $B^{(3)}$ field, or if $H_{B^{(3)}} = 1/2|B^{(3)}|^2 = 0$. Under this condition the nonabelian symmetry of QED would be manisfested as vacuum squeezed states. In this case electrodynamics is then a $U(1)$ gauge theory, as described by a Lagrangian, plus additional nonLagrangian symmetries.

8.5.1 $SU(2) \times SU(2)$ ELECTROWEAK THEORY WITH ONE HIGGS FIELD

The first attempts at extending $SU(2) \times U(1)$ electroweak theory to incorporate the fundamental B^3 field of $SU(2)$ electrodynamics were made in [6] in which an $SU(2) \times SU(2)$ electroweak theory was suggested but not developed. In this paper a more rigorous, but incomplete, $SU(2) \times SU(2)$ electroweak theory is developed to find several results which are missing from the original electroweak theory, because the latter did not incorporate B^3 and restricted consideration to a $U(1)$ symmetry electromagnetic sector. The extension is achieved in such as way as to maintain agreement with experimental data on weakly interacting vector boson and their masses, while still assuming for the sake of argument, a massless photon. The $SU(2) \times SU(2)$ electroweak theory developed in this Letter reproduces the fundamental relation between B^3 and the conjugate product of nonlinear optics, the relation responsible for the potentially very useful technique of radiation induced fermion resonance [7] and suggests the existence of a longitudinal E^3 field dual to B^3.

Nonlinear optics causes physics to consider electrodynamics to be an $SU(2)$ field theory with a magnetic field defined by the conjugate

product[6],

$$\mathbf{B}^3 \;=\; -i\frac{e}{\hbar}\mathbf{A}^1 \times \mathbf{A}^2, \qquad (8.5.1.78)$$

where $\mathbf{A}^1$ and $\mathbf{A}^2$ are conjugates or duals of each other. This magnetic field will couple to a Fermi $\frac{1}{2}$ spin field according to the Hamiltonian[6],

$$H \;=\; -\frac{e\hbar}{2m}\boldsymbol{\sigma}^3 \cdot \mathbf{B}^3. \qquad (8.5.1.79a)$$

The occurrence of the B^3 field, in addition to the observation of nonlinear effects due to the conjugate product of connection coefficients, should be apparent through the spin resonance effect with a Fermi field.

We will examine the $SU(2) \times SU(2)$ electroweak model first with an analysis that is similar to that used with the $SU(2) \times U(1)$ electroweak model. Here we will have one Higgs field for both parts of the twisted bundle. We will then look at the consequences of that and determine what is wrong. From there the requirements to fix the theory are discussed. These corrections to the theory are then to be presented in the second paper.

8.5.2 THE SU(2) × SU(2) EXTENDED STANDARD MODEL

Consider an extended standard model to determine what form the electromagnetic and weak interactions assume on the physical vacuum defined by the Higgs mechanism. Such a theory would then be $SU(2) \times SU(2)$. The covariant derivative will then be

$$\mathcal{D}_\mu \;=\; \partial_\mu \;+\; ig'\boldsymbol{\sigma}\cdot\mathbf{A}_\mu \;+\; ig\boldsymbol{\tau}\cdot\mathbf{b}_\mu, \qquad (8.5.2.79b)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are the generators for the two $SU(2)$ gauge fields represented as Pauli matrices and $\mathbf{A}$, $\mathbf{b}$ are the gauge connections defined on the two SU(2) principal bundles. There is an additional

Lagrangian for the ϕ^4 scalar field[8],

$$\mathcal{L}_\phi \;=\; \frac{1}{2}|\mathcal{D}_\mu(\phi)|^2 \;-\; \frac{1}{2}\mu^2|\phi|^2 \;+\; \frac{1}{4}|\lambda|\Big(|\phi|^2\Big)^2. \qquad (8.5.2.80)$$

The expectation value for the scalar field is then

$$\langle\phi_0\rangle \;=\; \big(0,\; \frac{v}{\sqrt{2}}\big). \qquad (8.5.2.81)$$

for $v \;=\; \sqrt{-\mu^2/\lambda}$. At this point the generators for the theory on the broken vacuum are

$$\begin{aligned}
\langle\phi_0\rangle\sigma_1 \;&=\; \big(\frac{v}{\sqrt{2}},\; 0\big)\,, \\
\langle\phi_0\rangle\sigma_2 \;&=\; \big(i\frac{v}{\sqrt{2}},\; 0\big)\,, \qquad (6) \\
\langle\phi_0\rangle\sigma_3 \;&=\; \big(0,\; -\frac{v}{\sqrt{2}}\big). \qquad (8.5.2.82)
\end{aligned}$$

These hold similarly for the generators of the other $SU(2)$ sector of the theory. There is a formula for the hypercharge, due to Nishijima, that when applied directly, would lead to an electric charge,

$$Q\langle\phi_0\rangle \;=\; \frac{1}{2}\langle\phi_0\rangle(\boldsymbol{\sigma}_3 \;+\; \boldsymbol{\tau}_1) \;=\; \big(0, -\frac{v}{\sqrt{2}}, 0, \frac{v}{\sqrt{2}}\big). \qquad (8.5.2.83)$$

This would mean that there are two photons that carry a $\pm$ charge respectively. We are obviously treating the hypercharge incorrectly. It is then proposed that the equation for hypercharge be modified as

$$Q\langle\phi_0\rangle \;=\; \frac{1}{2}\langle\phi_0\rangle\Big(\mathbf{n}_2\cdot\boldsymbol{\tau}_3 \;+\; \mathbf{n}_1\cdot\boldsymbol{\sigma}_1\Big) \;=\; 0, \qquad (8.5.2.84)$$

where the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors on the doublet defined by the two eigenstates of the vacuum. This projection onto $\boldsymbol{\sigma}_1$ and $\boldsymbol{\tau}_3$ is an ad hoc change to the theory that is required since we are using a single Higgs field on both bundles on both $SU(2)$ connections. This condition, an artifact of using one Higgs field, will be relaxed later. Now the generators of the theory have a broken symmetry on the physical vacuum. Therefore the photon is defined according to the

σ_1 generator in one $SU(2)$ sector of the theory, while the charged neutral current of the weak interaction is defined on the τ_3 generator.

We now consider the role of the ϕ^4 scalar field with the basic Lagrangian containing the electroweak Lagrangians

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + |\mathcal{D}_\mu\phi|^2 - \frac{1}{2}\mu^2|\phi|^2 + \frac{1}{4}\lambda\Big(|\phi|^2\Big)^2. \tag{8.5.2.85}$$

Here $G^a_{\mu\nu}$ and $F^a_{\mu\nu}$ are elements of the field strength tensors for the two $SU(2)$ principal bundles. In further work the Dirac and Yukawa Lagrangians that couple the Higgs field to the leptons and quarks will be included. It will then be pointed out how this will modify the B^3 field. The ϕ^4 field may be written according to a small displacement in the vacuum energy

$$\phi' = \phi + \langle\phi_0\rangle \simeq \frac{(v + \xi + i\chi)}{\sqrt{2}} \tag{8.5.2.86}$$

The fields ξ and χ are orthogonal components in the complex phase plane for the oscillations due to the small displacement of the scalar field. The small displacement of the scalar field is then completely characterized. The scalar field Lagrangian then becomes

$$\mathcal{L}_\phi = \frac{1}{2}\Big(\partial_\mu\xi\partial^\mu\xi - 2\mu^2\xi^2\Big) + \frac{1}{2}v^2\Big(g'\mathbf{A}_\mu + g\mathbf{b}_\mu + \Big(\frac{1}{gv} + \frac{1}{g'v}\Big)\partial_\mu\chi\Big) \cdot\Big(g'\mathbf{A}^\mu + g\mathbf{b}^\mu + \Big(\frac{1}{gv} + \frac{1}{g'v}\Big)\partial^\mu\chi\Big) \tag{8.5.2.87}$$

The Lie algebraic indices are implied. The Higgs field is described by the harmonic oscillator equation where the field has the mass $M_H \simeq 1.0TeV/c^2$. On the physical vacuum the gauge fields are

$$g'\mathbf{A}_\mu + g\mathbf{b}_\mu \to g'\mathbf{A}_\mu + g\mathbf{b}_\mu + \frac{1}{gv}\partial_\mu\chi = g'\mathbf{A'}_\mu + g\mathbf{b'}_\mu, \tag{8.5.2.88}$$

which corresponds to a phase rotation induced by the transition of the vacuum to the physical vacuum. Let us now break the Lagrangian, now expanded about the minimum of the scalar potential, out into components:

$$\mathcal{L}_\phi = \frac{1}{2}\Big(\partial_\mu\xi\partial^\mu\xi - 2\mu^2\xi^2\Big) + \frac{1}{8}v^2 \times\Big(g'^2|\mathbf{b}^3|^3 + g'^2\Big(|\boldsymbol{W}^+|^2 + |\boldsymbol{W}^-|^2\Big) + g^2|\boldsymbol{A}^1|^2 + g^2|\boldsymbol{A}^3 + i\boldsymbol{A}^2|^2\Big), \tag{8.5.2.89}$$

where we have identified the charged weak gauge fields as

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\Big(\mathbf{b}^1_\mu \pm i\mathbf{b}^2_\mu\Big). \tag{8.5.2.90}$$

The mass of these two fields are then $gv/2$. From what is left we are forced to define the fields

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\Big(g\boldsymbol{A}^3_\mu + g'\mathbf{b}^3_\mu - g\boldsymbol{A}^1_\mu\Big) \tag{8.5.2.91a}$$

$$Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\Big(g'\boldsymbol{A}^3_\mu + g\mathbf{b}^3_\mu + g'\boldsymbol{A}^1_\mu\Big). \tag{8.5.2.91b}$$

In order to make this consistent with the $SU(2) \times U(1)$ electroweak interaction[4] theory we initially require that $A^3{}_\mu = 0$ everywhere on scales larger than at unification. If this were nonzero then Z_0 would have a larger mass or there would be an additional massive boson along with the Z_0 neutral boson. The first case is not been observed, and the second case is to be determined. This assumption, while ad hoc at this point, is made to restrict this gauge freedom and will be relaxed later when a more complete discussion of the 3-photon is given. This is condition is relaxed in the following letter. This leads to the standard result that the mass of the photon is zero and that the mass of the Z_0 particle is

$$M_{Z^0} = \sqrt{g^2 + g'^2}\frac{v}{2} = \sqrt{1 + \Big(\frac{g'}{g}\Big)^2}M_W. \tag{8.5.2.92}$$

The weak angles are defined trigonometrically by the terms $g/(g^2 + g'^2)$ and $g'/(g^2 + g'^2)$. This means that the field strength tensor $F^3_{\mu\nu}$ satisfies

$$F^3_{\mu\nu} = \partial_\nu A^3_\mu - \partial_\mu A^3_\nu - i\frac{e}{\hbar}\left[A^1_\nu, A^2_\mu\right]$$

$$= -i\frac{e}{\hbar}\left[A^1_\nu, A^2_\mu\right], \qquad (8.5.2.93)$$

and further implies that the third component of the magnetic field in the $SU(2)$ sector is

$$B^3_j = \epsilon_j{}^{\mu\nu} F^3{}_{\mu\nu}$$

$$= -i\frac{e}{\hbar}\left(\mathbf{A}^1 \times \mathbf{A}^2\right)_j \qquad (8.5.2.94)$$

This is the form of the B^3 magnetic field. This also implies that the E^3 electric field is then

$$E^3_j = \epsilon_j{}^{0\mu} F^3_{0\mu} = -i\frac{e}{\hbar}\left(\mathbf{A}^1 \times \mathbf{A}^2\right)_j. \qquad (8.5.2.94a)$$

This demonstrates that $E^3 = B^3$ in naturalized units.

The duality between these electric and magnetic field means that the Lagrangian vanishes. The vanishing of this Lagrangian on symmetry principles means that there can not be any dynamics determined. This would indicate that this particular model simply reproduces $U(1)$ electrodynamics, plus additional nonLagrangian symmetries. Within this picture it appears as if the $\boldsymbol{B}^{(3)} = 0$ and that it simply represents the occurrence of various nonLagrangian symmetries, but where there are no dynamics for the $\boldsymbol{B}^{(3)}$ field.

This result is a curious and troubling one for the prospect that there can be a classical $\boldsymbol{B}^{(3)}$ field that has real dynamics. This would

imply that nonabelian symmetry is determined at all by a Lagrangian of the form $(1/2)(E^{3^2} - B^{3^2})$, as this is automatically zero by duality. However, if this were the case we would still have nonAbelian symmetry as a nonLagrangian symmetry. This strongly supports the possibility that the electrodynamic vacuum will continue to exhibit nonAbelian symmetries, such as squeezed states, even if we impose $E^3 = B^3 = 0$.

However, it can be suggested that the $B^3 = E^3$ field duality is broken when we consider the Lagrangian for the 3-field with the massive A^3 field introduced as

$$\mathcal{L} = \frac{1}{2}F^3_{\mu\nu}F^{3\mu\nu} + \frac{1}{2}\mu A^3{}_\mu A^{3\mu} - (1/c)j^3{}_\mu A^{3\mu}$$

$$\mathcal{L} = \frac{1}{2}(E^{3^2} - B^{3^2}) + \frac{1}{2}\mu A^3{}_\mu A^{3\mu} - (1/c)j^3_\mu A^{3\mu}. \qquad (8.5.2.95)$$

The middle term is a Proca Lagrangian for a massive photon. Here the mass of this photon is assumed to be larger than the masses of the $W^\pm$ and W^0 bosons. The current $j^3{}_\mu$ is determined by the charged fermions with masses given by the Yukawa interactions with the Higgs field. These are yet to be explored. Now consider the term in the Euler-Lagrange equation

$$\frac{\partial L}{\partial D^\mu A^{3\nu}} = [A_\mu, A_\nu], \qquad (8.5.2.96)$$

with covariant derivatives that enter into the Euler-Lagrange equation as

$$D_\mu A^{3\nu} = \partial_\mu A^{3\nu} + i(e/\hbar)\epsilon^{3ab}[A^a_\mu, A^b_\nu] \qquad (8.5.2.97)$$

and the subsequent setting of $A^3 \to 0$. Then the full Euler-Lagrange equation

$$D^\mu \frac{\partial L}{\partial(D^\mu A^{3\nu})} - \frac{\partial L}{\partial A^{3\nu}} = 0, \qquad (8.5.2.98)$$

is then

$$\nabla \times \boldsymbol{B}^3 + \mu^2 \boldsymbol{A}^3 - \boldsymbol{j}^3 = \frac{\partial \boldsymbol{E}^3}{\partial t}, \qquad (8.5.2.99)$$

which is just a form of the Faraday-Maxwell equation. However, the Hodge-star dual of this equation, the Maxwell equation, does not contain the current term,

$$\nabla \times \boldsymbol{E}^3 + \mu^2 \boldsymbol{A}^3 = -\frac{\partial \boldsymbol{B}^3}{\partial t}. \qquad (8.5.2.100)$$

The nonvanishing A^3 field at high energy will then break the duality between the E^3 and B^3 fields.

There is rub to this construction. This Proca equation is really only applicable on a scale that approaches the high energy physics where the $\boldsymbol{A}^{(3)}$ boson has appreciable influence. This will be only at a range of $10^{-17} cm$. On the scale of atomic physics $10^{-8} cm$, where quantum optics is applicable, this influence will be negligable. In effect on a scale where the $\boldsymbol{A}^{(3)}$ does not exist, as it has decayed into pion pairs, the duality is established and there is no Lagrangian for the $\boldsymbol{B}^{(3)}$ field. This puts us back to square one, where we must consider nonabelian electrodynamics as effectively $U(1)$ electrodynamics plus additional nonLagrangian and nonHamiltonian symmetries.

It has been demonstrated that there is an $SU(2) \times SU(2)$ electroweak theory that gives rise to the $Z_0, W^{\pm}$ gauge vector bosons plus electromagnetism with the photon theory with the cyclic condition for the B^3 fields. What has not been worked out are the implications for quark and lepton masses by inclusion of Yukawa coupling Lagrangians. However, that sector of the theory has little bearing upon this examination of the electromagnetic theory, with $A^3 = 0$, that emerges from the $SU(2) \times SU(2)$ gauge theory. We now have a theory for electromagnetism on the physical vacuum that is

$$\mathcal{L} = -(1/4)F^{\mu\nu}F_{\mu\nu} - (1/4)G^{a\mu\nu}G^a_{\mu\nu} + \frac{1}{2}\Big((E^3)^2 - (B^3)^2\Big) +$$

$$M_0|Z_0|^2 + M_w|W^{\pm}|^2 + \frac{1}{2}\Big(|\partial\xi|^2 - 2\mu^2|\xi|^2\Big)$$

$$+\text{Dirac Lagrangians} + \text{Yukawa}\,\{\text{Fermi} - \text{Higgs}\} \qquad (8.5.2.101)$$

where $F_{\mu\nu}$ and $G^a_{\mu\nu}$ are the field tensor components for standard electromagnetism and the weak interaction, and the cyclic electric and magnetic fields define the Lagrangian in the third term. The occurrence of the massive Z_0 and $W^{\pm}$ particles obviously breaks the gauge symmetry of the $SU(2)$ weak interaction.

8.5.3 DUALITY AND CHIRAL BREAKING

Physically we have the fields E^3, B^3 which are easily demonstrated to be longitudinal fields. Longitudinal fields result from the breaking of gauge invariance. What is unique is that the B^3 and E^3 fields are massless; unlike most fields resulting from symmetry breaking that are massive. That these fields are divergences and are equal up to c, before the breaking of duality, suggests that there are clues to field duality and monopole physics [10,11,12,13]. So we have on the physical vacuum three fields, where two are equivalent and all three are massless. This mechanism of breaking the $E^3 = B^3$ duality needs to be further explored.

Electrodynamics may be more fundamentally nonabelian leads to the existence of B^3 field that gives rise to nonlinear optical effects. This means that in a nonlinear optical system there should be exhibited a Lorentz force on a moving charge that is not predicted by $U(1)$ electrodynamics. The coupling of a Fermi field to the B^3 field is just such an interaction, and empirical evidence for this coupling can be found in the inverse Faraday effect [6]. The $\boldsymbol{\sigma} \cdot \mathbf{A} \times \mathbf{A}^*$ term is given in Eq.(8.6) of Ref. 9. That this field exists, and apparently the E^3 field does not, should be determined more explicitly by an extended standard model.

Now let us perform a gauge transformation $A^1 \rightarrow UA^1U^{-1} + U\partial U^{-1}$. The electric field in the 3-sector is then transformed as

$$B^{3i} = \epsilon^{ijk} U \Big[A^1_j,\ A^2_k\Big] U^{-1}. \qquad (8.5.3.102)$$

This can be easily demonstrated by the antisymmetry of j and k with the commutators with A^1 or A^2. So fortunately the theory appears to be gauge invariant. So there is a general situation of proper gauge transformation if self-duality holds, or if it is broken. In effect if self-duality, and the broken duality due to the A^3 potential, is true for flat gauge connections then it is true for all gauge connections.

This duality is an artifact of excluding the masses predicted by the Higgs mechanism and their role in the decay processes of the charged and neutral weak currents, and the decay of $SU(2)$ massive photon A^3. Setting $A^3 = 0$ allowed us to ignore this problem to examine the basic electroweak issues. The explicit inclusion of massive Fermions in the decay of these fields will break field duality[10]. The condition that $A^3 = 0$ everywhere is relaxed and a 3-photon is defined. It can be demonstrated that the currents contain vector and axial vector components that obey the $SU(2) \times SU(2)_C$ algebra. On the physical vacuum fields acquire masses that violate the current conservation of the axial vector current. Within this context a better understanding of duality breaking can be derived.

8.5.4 DISCUSSIONS OF THE THEORY, ITS PROBLEMS, AND THEIR REMEDIES

So here we have constructed, in some ways rather artificially, an $SU(2) \times SU(2)$ gauge theory that is able to reproduce the standard model $U(1) \times SU(2)$ with the additional cyclic magnetic field given by equation 18. However, we are left with two uncomfortable conditions imposed on the theory to make this work. The first is that the electric charge is computed in an ad hoc fashion so that we do not have the massless photons $\mathbf{A}^1$ and $\mathbf{A}^2$ that carry a unit of opposite electric

charges. The second problem is that we have by hand eliminated the $\mathbf{A}^3$ vector potential. If this were nonzero we would have the following gauge potential

$$\omega'^3_\mu = \frac{g}{\sqrt{g^2 + g'^2}} A^3_\mu. \qquad (8.5.4.103)$$

This field would have a mass equal to $\sqrt{g^2 + g'^2}v/2$ and would then contribute a large decay signal at the same scattering transverse momenta where the Z_0 is seen.

The problem is that we have a theory with two $SU(2)$ algebras that both act on the same Fermi spinor fields. We further are using one Higgs field to compute the vacuum expectation values for both fields. The obvious thing to do is to first consider that each $SU(2)$ acts on a separate spinor fields doublets. Next the theory demands that we consider that there be two Higgs fields that compute separate physical vacuums for each $SU(2)$ sector independently. This means that the two Higgs fields will give 2×2 vacuum expectations, which may be considered to be diagonal. If two entries in each of these matrices are equal then we conclude that the resulting massive fermion in each of the two spinor doublets are the same field. Further, if the spinor in one doublet assumes a very large mass then at low energies this doublet will appear as a singlet and the gauge theory that acts on it will be $O(3)$, with the algebra of singlets

$$\mathbf{e_i} = \epsilon_{ijk}[\mathbf{e}_j,\ \mathbf{e}_k]. \qquad (8.5.4.104)$$

This will leave a theory on the physical vacuum that involves transformations on a singlet according to a broken $O(3)_b$ gauge theory, and transformations on a doublet according to a broken $SU(2)$ gauge theory. The broken $O(3)_b$ gauge theory reflects the occurrence of a very massive $\mathbf{A}^3$ photon, but massless $\mathbf{A}^1$ and $\mathbf{A}^2$ fields. This broken $O(3)_b$ gauge theory then reduces to electromagnetism with the cyclicity condition. The broken $SU(2)$ theory reflects the occurrence of massive charged and neutral weakly interacting bosons.

To take this theory further would be to embed it into an $SU(4)$ gauge theory. The gauge potentials are described by 4×4 traceless Hermitian matrices and the Dirac spinor has 16 components. The neutrality of the photon is then given by the sum over charges, which vanishes by the tracelessness of the theory. The Higgs field is described by a 4×4 matrix of entries

It is concluded within the above "toy model" that the B^3 field is consistent with an extended $SU(2) \times SU(2)$ model of electroweak interactions. A more complete formalism of the $SU(2) \times SU(2)$ theory with fermion masses will yield more general results. A direct measurement of B^3 should have a major impact on the future of unified field theory and superstring theories. The first such measurement was reported in Ref. 14, (see also Refs.6 and 7).

8.6.1 CHIRAL AND VECTOR FIELDS IN $SU(2) \times SU(2)$ ELECTROWEAK FIELD

The cyclic theory of electromagnetism has been demonstrated to be consistent with a $SU(2) \times SU(2)$ theory of electroweak unification[15]. It has been demonstrated that if we set $\mathbf{A}^3 = 0$ on the physical vacuum that a cyclic theory of electromagnetism is arrived at. This theory contains longitudinal $\mathbf{E}^3$ and $\mathbf{B}^3$ fields that are dual $\mathbf{E}^3 = \mathbf{B}^3$, but where this duality is broken by current interactions. By setting $\mathbf{A}^3 = 0$ the transverse 3-modes of the theory have been completely eliminated by this arbitrary restriction of this gauge freedom. The elimination of these transverse 3-modes guarantees that photons are entirely defined by the $\boldsymbol{\sigma}^{1,2}$ generators of the $SU(2)$ theory of electrodynamics. Since the field defined by the $\boldsymbol{\sigma}^3$ generators are longitudinal this means they are irrotational $\nabla \times \mathbf{E}^3 = \nabla \times \mathbf{B}^3 = 0$ and thus time independent. By Maxwell's equations this means that there are no electromagnetic waves or photons associated with this field.

8.6.2 AXIAL-VECTOR SU(2) × SU(2) FIELDS: A FIRST LOOK

To start we examine a putative model of a chiral-vector model at low energies to determine what sorts of processes may be involved with the broken symmetry of such a model. We start by naively considering a chiral-vector model to see what sorts of structure may emerge at low energy without explicit consideration of the Higgs mechanism. The field theory starts out as a twisted bundle of two chiral groups $SU(2) \times SU(2)$ and emerges as a theory that is an axial-vector theory at low energy. We consider initially the situation where the theory is an axial-vector theory at low energy. We then consider the situation where there is a breakdown of chiral symmetry. This is then used to set up the more complete situation that involves the breakdown of the chiral theory at high energy into an axial-vector theory at low energy.

In this letter we relax the condition that $\mathbf{A}^3 = 0$. This statement would physically mean that the current for this gauge boson is highly nonconserved with a very large mass so that the interaction scale is far smaller than the scale for the cyclic electromagnetic field. In relaxing this condition we will find that we still have a violation of current conservation.

With $\mathbf{A}^3 \neq 0$ we have the fields[15]

$$A'^1_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gA^3_\mu + g'b^3_\mu - gA^1_\mu)$$

$$Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gb^3_\mu + g'A^1_\mu). \qquad (8.6.2.105)$$

$$\omega^3_\mu = \frac{g'}{\sqrt{g^2 + g'^2}}A^3_\mu.$$

One purpose here is to examine the ω_μ^3 connection; which will have a chiral component. This at first implies that the $\mathbf{B}^3$ field is partly chiral, or that it is mixed with the chiral component of the other $SU(2)$ chiral field in some manner to remove its chirality.

The theory of $SU(2)$ electromagnetism, at high energy, is very similar to the theory of weak interactions in its formal structure. Further, it has implications for the theory of leptons. The electromagnetic interaction acts upon a doublet, where this doublet is most often treated as an element of a Fermi doublet of charged leptons and their neutrinos in the $SU(2)$ theory of weak interactions.

Following in analogy with the theory of weak interactions we let ψ be a doublet that describes an electron according to the 1 field and the 3 field. We start with the free particle Dirac Lagrangian and let the differential become gauge covariant,

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - gA_\mu^b \bar{\psi}\gamma^\mu \sigma_b \psi \\ &= \mathcal{L}_{free} + A_\mu^b J_b^\mu, \end{aligned} \tag{8.6.2.106}$$

where $\bar{\psi} = \psi^\dagger \gamma_4$. From here we decompose the current J_μ^b into vector and chiral components,

$$J_\mu^b = \psi^\dagger \gamma_4 \gamma_\mu (1 + \gamma_5)\sigma^3 \psi = V_\mu^b + \chi_\mu^b, \tag{8.6.2.107}$$

This is analogous to the current algebra for the weak and electromagnetic interactions between fermions. We have the two vector current operators[16]

$$V_\mu^a = \frac{i}{2}\bar{\psi}\gamma_\mu \sigma^a, \psi \tag{8.6.2.108}$$

and the two axial-vector current operators

$$\chi_\mu^b = \frac{i}{2}\bar{\psi}\gamma_\mu \gamma_5 \tau^b \psi. \tag{8.6.2.109}$$

Here $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4$, and τ^b are Pauli matrices. These define an algebra of equal time commutators:

$$\begin{aligned} [V_4^a, V_\mu^b] &= it^{abc} V_\mu^c, \\ [V_4^a, \chi_\mu^b] &= -it^{abc}\chi_\mu^b. \end{aligned} \tag{8.6.2.110}$$

If we set $\mu = 4$ we then have the algebra

$$[V_4^a, V_4^b] = it^{abc} V_4^c,$$

and

$$[V_4^a, \chi_4^b] = -it^{abc}\chi_4^c, \tag{8.6.2.111}$$

If we set

$$Q_\pm^a = \frac{1}{2}(V_4^a \pm \chi_4^a), \tag{8.6.2.112}$$

we then have the algebra

$$\begin{aligned} [Q_+^a, Q_+^b] &= it^{abc} Q_+^c \\ [Q_-^a, Q_-^b] &= it^{abc} Q_-^c \\ [Q_+^a, Q_-^b] &= 0 \end{aligned} \tag{8.6.2.113}$$

This can be seen to define the $SU(2) \times SU(2)$ algebra.

The action of the parity operator on V_4^b and χ_4^b due to the presence of γ_5 in the axial vector current.

$$\begin{aligned} PV_4^b P^\dagger &= V_4^b \\ P\chi_4^b P^\dagger &= -\chi_4^b. \end{aligned} \tag{8.6.2.114}$$

As such one $SU(2)$ differs from the other by the action of the parity operator and the total group is the chiral group $SU(2) \times SU(2)_P$.

We have at low energy half vector and half chiral vector theory $SU(2) \times SU(2)_P$. On the physical vacuum we have the vector gauge theory described by $\mathbf{A^1} = \mathbf{A^{2*}}$ and $\mathbf{B^3} = \nabla \times \mathbf{A}^3 + (ie/\hbar)\mathbf{A^1} \times \mathbf{A^2}$ and the theory of weak interactions with matrix elements of the form $\bar{\nu}\gamma_\mu(1 - \gamma_5)e$ and are thus half vector and chiral on the level of elements of the left and right handed components of doublets. We then demand that on the physical vacuum that we have a mixture of vector and chiral gauge connections within both the electromagnetic and weak interactions due to the breakdown of symmetry. This will mean that the gauge potential $\mathbf{A}^3$ will be massive and short ranged.

One occurrence is a violation of the conservation of the axial-vector current. We have that the 1 and 2 currents are conserved and invariant. On the high energy vacuum we expect that currents should obey

$$\partial^\mu J^b_\mu = 0, \tag{8.6.2.115}$$

where $b \in \{1, 2\}$, which are absolutely conserved currents. However, for the A^3_μ fields we have the nonconserved current equation[17]

$$\partial^\mu J^3_\mu = i m_\psi \psi^\dagger \gamma_4 \gamma_5 \sigma^3 \psi, \tag{8.6.2.116}$$

where inhomogeneous terms correspond to the quark-antiquark and lepton-antilepton pairs that are formed from the decay of these particles. This breaks the chiral symmetry of the theory. Then this current's action on the physical vacuum is such that when projected on a massive eigenstates for the 3-photon with transverse modes

$$\langle 0|\partial^\mu J^3_\mu|X_k\rangle = \left(\frac{m^2}{\sqrt{\omega(k)\omega(k')}}\right)\langle X_{k'}|X_k\rangle e^{ikx}. \tag{8.6.2.117}$$

The mass of the chiral $\{1, 2\}$ -bosons will then vanish, while the mass of the chiral 3-boson will be m. So rather than strictly setting $\mathbf{A}^3 = 0$, it is a separate chiral gauge field that obeys axial-vector nonconservation and only occurs at short ranges.

So now that we have an idea of what nature may look like on the physical vacuum, we need to examine how it is that we can have symmetry breaking and an $SU(2) \times SU(2)_P$ gauge theory that gives rise to some of the above requirements of $\mathbf{B}^3$ electromagnetism. A mixing of the two chiral $SU(2)$ bundles at low energy is what will produce vector gauge bosons for the electromagnetic interaction. It is apparent that we need to invoke the mixing of two chiral gauge bosons in such a manner as to produce a vector theory of electromagnetism at low energy with a broken chiral theory of weak interactions.

8.6.3 CHIRAL AND VECTOR GAUGE THEORIES FROM CHIRAL GAUGE THEORIES ON THE PHYSICAL VACUUM

The $SU(2) \times SU(2)$ theory should mimic the standard model with the addition of the $\mathbf{B^3} = (e/\hbar)\mathbf{A^1} \times \mathbf{A^2}$ field at low energies. This means that we demand that a field theory that is completely chiral at high energy becomes a field theory that is vector and chiral in separate sectors on the physical vacuum of low energies. This means that a field theory that is chiral at high energy will combine with the other chiral field in the twisted bundle to produce a vector field plus a broken chiral field at low energy. Generally this means that a field theory that has two chiral bundles at high energies can become vector and chiral within various independent fields that are decoupled on physical vacuum at low energies.

We consider a toy model where there are two fermion fields ψ and χ, where each of these field consists of the two component right and left handed fields R_ψ, L_ψ and R_χ, L_χ. These Fermi doublets have the masses m_1 and m_2. We then have the two gauge potentials A_μ and B_μ that interact respectively with the ψ and χ fields. In general with more Fermi fields this situation becomes more complex, where these two Fermi fields are degeneracies that spit into the multiplet of fermions known. In this situation there are four possible masses

for these fields on the physical vacuum. These masses occur from Yukawa couplings with the Higgs field on the physical vacuum. These will give Lagrangians terms of the form $Y_\phi R_\psi^\dagger \phi L_\chi + H.C.$ and $Y_\eta L_\psi^\dagger \eta R_\chi + H.C.$, where now we have a two component ϕ^4 field for the Higgs mechanism. These two components assume the minimal expectation values $\langle \phi_0 \rangle$ and $\langle \eta_0 \rangle$ on the physical vacuum. We then have the Lagrangian[18]

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu + igA_\mu) - m_1)\psi \tag{8.6.3.118}$$

$$+ \bar{\chi}(i\gamma^\mu(\partial_\mu + igB_\mu) - m_2)\chi - Y_\phi R_\psi^\dagger \phi L_\chi + H.C. - Y_\eta L_\psi^\dagger \eta R_\chi + H.C.,$$

that can be further broken into the left and right two component spinors

$$\mathcal{L} = R_\psi^\dagger i\sigma^\mu(\partial_\mu + igA_\mu)R_\psi + L_\psi^\dagger i\sigma^\mu(\partial_\mu + igA_\mu)L_\psi$$

$$+ R_\chi^\dagger i\sigma^\mu(\partial_\mu + igB_\mu)R_\chi + L_\chi^\dagger i\sigma^\mu(\partial_\mu + igB_\mu)L_\chi \tag{8.6.3.119}$$

$$- m_1 R_\psi^\dagger L_\psi - m_1 L_\psi^\dagger R_\psi - m_2 R_\chi^\dagger L_\chi - m_2 L_\chi^\dagger R_\chi$$

$$- Y_\phi R_\psi^\dagger \phi L_\chi + Y_\phi^* L_\chi^\dagger \phi^* R_\psi - Y_\eta L_\psi^\dagger \eta R_\chi + Y_\eta^* R_\chi^\dagger \eta^* L_\psi,$$

The gauge potentials A_μ and B_μ are 2×2 Hermitian traceless matrices and the Higgs fields ϕ and χ are also 2×2 matrices. These expectations are real valued, and so we then expect that the non-zero contributions of the Higgs field on the physical vacuum are given by the diagonal matrix entries [18]

$$\langle \phi \rangle = \begin{pmatrix} \langle \phi^1 \rangle & 0 \\ 0 & \langle \phi^2 \rangle \end{pmatrix} \quad \langle \chi \rangle = \begin{pmatrix} \langle \chi^1 \rangle & 0 \\ 0 & \langle \chi^2 \rangle \end{pmatrix} \tag{8.6.3.120}$$

In a recently submitted paper these issues were not discussed [15]. There this matrix is proportional to the identity matrix and the matrix nature of the Higgs field was conveniently ignored. This means that the $SU(2) \times SU(2)$ electroweak theory shares certain generic features with the $SU(2) \times U(1)$ theory. The values of the vacuum expectations are such that at high energy the left handed fields R_χ and the right handed doublet field L_ψ couple to the $SU(2)$ vector boson field B_μ, while at low energy the theory is one with a left handed

$SU(2)$ doublet R_ψ that interacts with the right handed doublet L_χ through the massive gauge fields A_μ. Then the mass terms from the Yukawa coupling Lagrangians will then give

$$m' = Y_\eta \langle \chi^1 \rangle >> m'' = Y_\eta \langle \chi^2 \rangle >> \tag{8.6.3.121}$$

$$m''' = Y_\phi \langle \phi^1 \rangle >> m'''' = Y_\phi \langle \phi^2 \rangle. \tag{8.6.3.122}$$

Further, if the $SU(2)$ theory for B_μ potentials are right handed chiral and the $SU(2)$ theory for A_μ potentials are left handed chiral then we see that a chiral theory at high energies can become a vector theory at low energies. The converse may also be true in another model.

In the switch between chirality and vectorality at different energies there is an element of broken gauge symmetry. So far we would have a theory of a broken gauge theory at low energy. However, there is a way to express this idea so that at low energy we have a gauge theory accompanied by a broken gauge symmetry. To illustrate this let us assume we have a simple Lagrangian that couples the left handed fields ψ_l to the right handed boson A_μ and the right handed fields ψ_r to the left handed boson B_μ

$$\mathcal{L} = \bar{\psi}_l(i\gamma^\mu(\partial_\mu + igA_\mu) - m_1)\psi_l \tag{8.6.3.123}$$

$$+ \bar{\psi}_r(i\gamma^\mu(\partial_\mu + igB_\mu) - m_2)\psi_r - Y_\phi \psi_l^\dagger \phi \psi_r - Y_\phi^* \psi_r^\dagger \eta \psi_l.$$

If the coupling constant Y_ϕ is comparable to the coupling constant g, then the Fermi expectation energies of the Fermions occur at the mean value for the Higgs field $\langle \phi_0 \rangle$. In this case the vacuum expectation of the vacuum is proportional to the identity matrix. This means that the masses acquired by the right chiral plus left chiral gauge bosons $A_\mu + B_\mu$ are zero, while the right chiral minus left chiral gauge bosons $A_\mu - B_\mu$ acquires masses approximately $Y_\phi \langle \phi_0 \rangle$. The theory at low energies is a theory with an unbroken vector gauge theory plus a broken chiral gauge theory[18]. It is also the case that we demand that the charges of the two chiral fields

$A^{1,2}$, $B^{1,2}$ that add are opposite so that the resulting vector gauge bosons are chargeless.

Just as we have gauge theories that can change their vector and chiral character so also do the doublets of the theory. In so doing this will give rise to the doublets of leptons and quarks plus doublets of very massive fermions. These massive fermions should be observable in the multi TeV range of energy.

8.6.4 THE OCCURRENCE OF $O(3)_b$ ELECTRODYNAMICS ON THE PHYSICAL VACUUM

The two parts of the twisted bundle are copies of $SU(2)$ with a doublet fermion structures. However one of the fermions has the extremely large mass $m' = Y_\eta \langle \chi^1 \rangle$ that is presumed to be unstable and not observed at low energies. So one sector of the twisted bundle is left with the same abelian structure, but with a singlet fermion. This means that the SU(2) gauge theory becomes defined by the algebra over the basis elements $\hat{e}_i$, $i \in \{1, 2, 3\}$,

$$[\hat{e}_i, \hat{e}_j] = i\epsilon_{ijk}\hat{e}_k. \qquad (8.6.4.124)$$

We further need to examine the photon masses. We define the Higgs field by a small expansion around the vacuum expectations $\eta^1 = \xi^1 + \langle \eta_0^1 \rangle$ and $\eta^2 = \xi^2 + \langle \eta_0^2 \rangle$. The contraction of the generators σ^1 and σ^2 with the Higgs field matrix and right and left fields gives

$$\sigma^1 \cdot \eta R + \sigma^2 \cdot \eta L = 0, \qquad (8.6.4.125)$$

which gives that the charges of the $\mathbf{A}^1$ and $\mathbf{A}^2$ fields are zero. These fields on the low energy vacuum can be thought of as massless fields composed of two gauge bosons, with masses $\sqrt{m' + m''} >> M_Z$ and with opposite charges. These electrically charged fields can be thought of as $\mathbf{A}^{\pm} = \mathbf{A}^1 \pm \mathbf{A}^2$. These particles cancel each other and gives rise to massless vector photon gauges fields. The field $\mathbf{A}^3$

also has this mass. This massive field is also unstable and decays into particle pairs.

With the action of the more massive Higgs field we are left with the gauge theory $SU(2) \times O(3)$, where the first gauge group acts on doublets and the last gauge group acts on singlets. Further on a lower energy scale, or equivalently long enough time scales, the field $\mathbf{A}^3$ has decayed and vanished. At this scale the second gauge group is then represented by $O(3)_p$ meaning a partial group. This group describes Maxwell's equations along with the definition of the field $\mathbf{A}^1 \times \mathbf{A}^2$.

From this point we can then treat the action of the second Higgs field on this group in a manner described in [15]. If we set the second Higgs field to have zero vacuum expectation $\langle \phi^2 \rangle = 0$ then the symmetry breaking mechanism effectively collapses to this formalism which is similar to the standard $SU(2) \times U(1)$ model Higgs mechanism. We can the arrive at a vector electromagnetic gauge theory $O(3)_p$, p stands for partial, and a broken chiral $SU(2)$ weak interaction theory. The mass of the vector boson sector is in the $\mathbf{A}^3$ boson plus the $W^{\pm}$ and Z^0 particles.

8.6.4.1 THE $SU(4)$ MODEL

It is possible to consider the two $SU(2)$ group theories as being represented as the block diagonals of the larger $SU(4)$ gauge theory. The Lagrangian density for the system is then

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu + igA_\mu) - m_1)\psi - Y\bar{\psi}\phi\psi. \qquad (8.6.4.1.126)$$

The gauge potentials A_μ now have 4×4 traceless representations. The scalar field theory that describes the vacuum will now satisfy field equations that involve all 16 components of the gauge potential.

By selectively coupling these fields to the fermions it should be possible to formulate a theory that recovers a low energy theory that is the standard model with the $O(3)_p$ gauge theory of electromagnetism.

What has been presented is an outline of an $SU(2) \times SU(2)$ electroweak theory that can give rise to the nonabelian $O(3)_b$ theory of quantum electrodynamics on the physical vacuum. The details of the fermions and their masses has yet to be worked through, as well as the mass of the $\mathbf{A}^3$ boson. This vector boson as well as the additional fermions should be observable within the 10 Tev range of energy. This may be accessible by the CERN Large Hadron Collider within the next decade.

The principal purpose here has been to demonstrate what sort of electroweak interaction physics may be required for the existence of an $O(3)_b$theory of quantum electrodynamics on the low energy physical vacuum. This demonstrates that an extended standard model of electroweak interactions can support such a theory with the addition of new physics at high energy.

8.6.5 DUALITY IN GRAND UNIFIED FIELD THEORY, AND RECENT LEP1 DATA

Above is a construction that indicates that the electromagnetic and weak interactions may be dual field theories. If the above construction is experimentally verified then this would be the first empirical indication that the universe is indeed dual according to a theory along the lines of Olive-Montenen [11,12,13]. Within this theory there are coupling constants that have inverse relationships, or convergences at high energy, so that one field is weak and the other is strong at low energy. In this case the electromagnetic field is comparatively strong, but not when compared to the nuclear force, and the

other is very weak. It may be that both field theories have coupling constants that are both lowered and diverge at low energy within a Grand Unified Theory (GUT). The examination of this electroweak theory within such a construction has not been been done. Nonetheless, the experimental finding of the $\boldsymbol{A}^{(3)}$ would bring a tremendous change in our views on the foundations of physics.

It was recently suggested by Erler and Langacker [19] that an anomaly in Z decay widths points to the existence of Z' bosons. These are predicted to exist with a mass estimate of $812GeV^{+339}_{-152}$ within an $SO(10)$ GUT model and a Higgs mass posited at $145GeV^{+103}_{-61}$. This suggests that a massive neutral boson predicted by Grand Unified Theories has been detected. Further, variants of string theories predict the existence of a large number of these neutral massive bosons.

Analyses of the hadronic peak cross section data obtained at LEP 1[20] implies a small amount of missing invisible width in Z decays. These data imply an effective number of massless neutrinos, $N = 2.985 \pm 0.008$, which is below the prediction of 3 standard neutrinos by the standard model of electroweak interactions. The weak charge Q_W in atomic parity violation can be interpreted as a measurement of the S parameter. This indicates a new $Q_W = -72.06 \pm 0.44$ is found to be above the standard model prediction. This effect is interpreted as due to the occurrence of the Z' particle, which will be refered to as the Z_γ particle.

$SO(10)$ has the six roots α^i, $i = 1 \ldots 6$. The angle between the connected roots are all 120^o, where the roots α^3, α^4 are connected to each other and two other roots. The Dynkin diagram is illustrated

below:

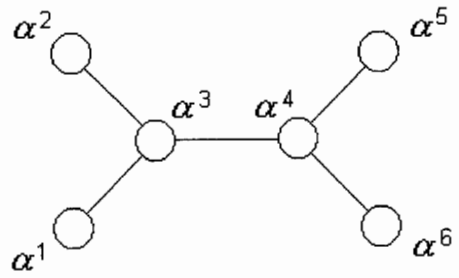

Figure 8.7

The decomposition of $SO(10) \rightarrow SU(5) \times U(1)$ is performed by removing the circles representing the roots $\alpha^{1,2,5,6}$ connected by a single branch. The remaining connected graph describes the $SU(5)$ group. However, by removing the circle α^4 connected by three branches forces $SO(10)$ to decompose into $SU(2) \times SU(2) \times SU(4)$. Here we have an $SU(2)$ and a mirror $SU(2)$ that describe opposite handed chiral gauge fields, plus an $SU(4)$ gauge field. The chiral fields are precisely the sort of electroweak structure proposed in reference [15]. Presumably since $SU(4)$ can be represented by a 4 that is $3 \oplus 1$ and $\bar{4}$ as $\bar{3} \oplus 1$, we can decompose this into $SU(3) \times U(1)$. Further, the neutrino short fall is a signature of the opposite chiralities of the two "mirrored" SU(2) gauge fields [15].

The $SU(2) \times SU(2) \rightarrow SU(2) \times O(3)_b$ predicts the occurrence of a massive photon. So it is possible that these data could corroborate the extended standard model that expands the electromagnetic sector of the theory. What we really understand empirically is QCD and electroweak standard model, and we may have some idea about quantum gravity for at least we do have general relativity and quantum mechanics. This leads to the strange situation that we have reasonable data on low TeV range physics and potential ideas about quantum gravity at $10^{19} GeV$, with a void of greater ignorance in between. However, these data and analyses suggest theoretical information about GUTs and cast some light on this energy region.

These experimental data do suggest that nonabelian electrodynamics is a valid theory, at least as an extended theory that predicts nonHamiltonian vacuum symmetries. It also suggests that at high energy electrodynamics and the weak interactions are dual field theories. This duality would then exist at energies that may be probed in the TeV range of energy. In order to completely verify that this is the case experiments at the TeV range need to be performed where the Z_γ and Higgs boson can be directly produced.

This leaves open the question about the nuclear interaction. It is tempting to conjecture that there is a dual field theory to the $SU(3)$ nuclear interaction or Quantum ChromoDynamics (QCD). It is easy to presume that such a construction would proceed in a manner outlined above with the chiral $SU(2) \times SU(2)$ electroweak field theory. This would then imply that there exists an additional weak field in nature. It the field theory is similar in construction, then there may exist some massive particle with weak coupling. It would then be tempting to pursue calculations to predict the existence of such particles. However, it must be stressed that this is rather speculative and has speculative implications for the foundations of physics.

It is tempting to think that there may be a generalized $SU(3) \times SU(3)$ type of theory for the strong interactions. As in the above $SO(10)$ theory we see that the nuclear interactions are embedded in an SU(4) theory. This would mean that there exist chiral colored gluons associated with QCD. This can most easily be seen if the $U(1)$ group associated with QCD according to $SU(4) = SU(3) \times U(1)$. The $U(1)$ group describes local phase changes according to

$$\psi \rightarrow e^{i\phi}\psi. \qquad (8.6.5.127)$$

We may assign this $U(1)$ group to a chiral transformation, similar to

a G parity operator, according to

$$\psi \rightarrow e^{i\phi\gamma_5}\psi. \qquad (8.6.5.128)$$

The Dirac Lagrangian would then assume the form

$$\mathcal{L} = \frac{1}{2}(\bar{\psi}(1 + \gamma_5)\gamma_\mu\partial^\mu\psi) \qquad (8.6.5.129)$$

where at high energies, before the Higgs field has assigned masses to the quarks through Yukawa couplings, the QCD sector would be chiral invariant. Once the quarks have masses then there is chiral breaking. One may then have a field where the dominant amplitudes favor vector gluons, but where there is a small chromo-chiral amplitude. This would also mean that quarks would exhibit a small chiral breaking. Further, if the coupling constants for the chiral component of the chromofield are very weak then we have in effect a duality within QCD.

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CHAPTER 9
POTENTIAL APPLICATIONS OF $O(3)_b$ QED

This short chapter is a discussion of some possible applications of quantum nonabelian electrodynamics. The $\boldsymbol{B}^{(3)}$ field is associated with quantum fluctuations and the loss of quantum information in QED. This has a number of potential applications for computational systems and with techniques in molecular biology. The intention here is to introduce some of these possibilities that might become real technologies in the twenty first century. Two possibilities are suggested. The first involves the use of biophysics and nonabelian electrodynamics in quantum computing. The second is a more modest suggestion of using the $\boldsymbol{B}^{(3)}$ field to detect the sequences of DNA.

9.1 COMPUTATION, BIOPHYSICS AND $B^{(3)}$ INDUCED ENTANGLED STATES

At the close of the twentieth century we are faced with two technologies that have profound impact on the course the world is taking. The first technology is computers and microprocessors. The second technology is the molecular biological revolution and biotechnology. The first technology is firmly in place in our world. This technology had its early beginnings with the enigma machine employed by British intelligence to break nazi codes. The transistor brought mainframe computers such as the IBM 360/370 series in the late 1960s and early 1970s. Later the microprocessor, with VLSI architecture, lead to the small computer. These trends continue to exponentially grow. The biotechnology revolution is in a more nascent stage of development. The most successful developments have been in the production of single proteins by genetic recombinant techniques. These gene products are used mostly for medical purposes, such as insulin, and for agriculture, which has sparked considerable controversy. These tech-

niques involve recombinant genetic techniques applied to plant germ cells, where genes are inserted into plants to make them resistant to diseases and our own agricultural chemicals.

These two technological trajectories are likely to continue through at least several decades in the twenty first century as the dominant trends in advanced industry. In fact it is likely that these two trends will in some places merge into single tracks. This makes sense in that DNA, and the protein product that DNA encodes, is essentially an information tape that could in principle be used to store information that is ultimately read by a computer. Further, the behavior of proteins is very complex in that it involves the mutual interaction between many charged amino acid residues. There is the potential for interactions that could be used to process information according to information states assigned to various conformational shapes a protein may exhibit.

Quantum computers are a curious possibility for the future. There are a class of problems, such as the travelling saleman problem, that are extremely difficult to solve. The reason for their diffuclty is that if there are n variables or inputs into the problem the space and/or time required for an algorithm to solve them increases as 2^n. This means that for n sufficiently large the problem becomes increasingly unwieldy in computer space or in time of computation. These are refered to as nonpolynomial (NP) complete problems. This means that computers that currently exist are limited in their ability to solve these types of problems for data sets that are at all large.

Quantum computers provide a method for solving these type of problems. Essentially, if such a problem can be run quantum mechanically then the system will process the problem as a quantum superposition of states. This means that a processing element, or bit, in a quantum computer can exist as a superposed set of possible states, a q-bit. Since such a processing element then processes a set given by $\sum_n e^{iE_n t}$ this means that the actual computer space and time required is a logarithm of the space and time required for a classical computer. In effect a quantum computer is a parallel processor that executes each of these individual processes in a superposition of states. In general this is accomplished by putting an array of two states atoms or ions into entangled states.

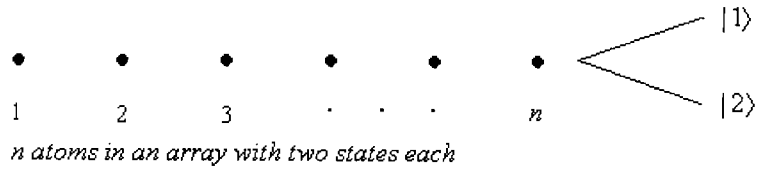


Figure 9.1

This would mean that a whole class or problems that are in general inaccessible to current computing power would become computable within reasonable space and time constraints.

The current approach to quantum computers involves ion traps. These are devices that are able to trap ions or atoms in local electromagnetic wells with lasers that manipulate the quantum states of the ions or atoms. The ion trap scheme was advanced by Cirac and Zoller [1] involved establishing entangled atomic states by the exchange of phonons. These phonons are extablished by the mutual electrostatic interaction between the ion charges. An atomic scheme was advanced by Pellizzari et. al. [2]where atoms communicate by the exchange of photons in an optical cavity. The laser pulses the atoms or ions with photons that induce a $\pi/2$ spin change on the two states system, here thought of as generically a spin $\frac{1}{2}$ system, that will result in entangled states.

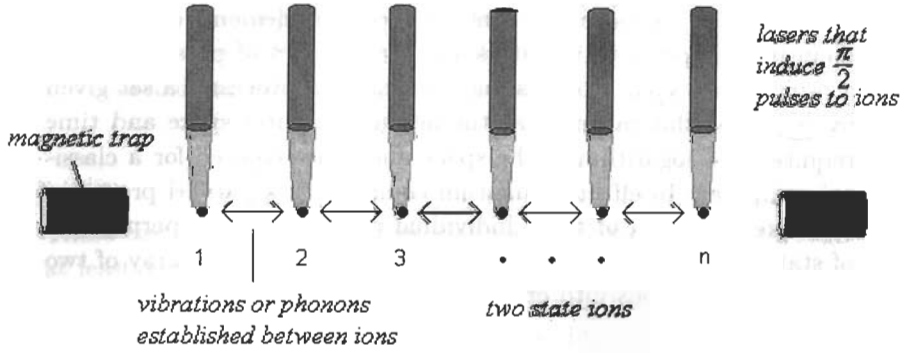


Figure 9.2

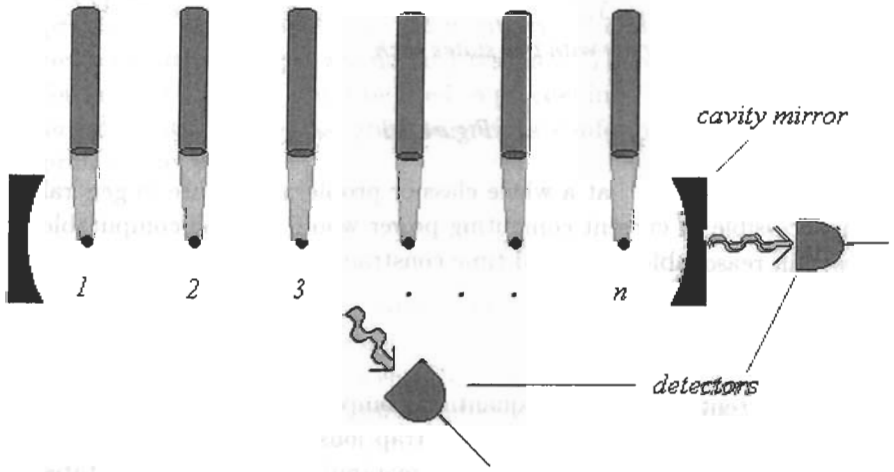


Figure 9.3

A quantum computer is composed of quantum bits, q-bits, that are two level atoms woth the quantum states $|1\rangle$ and $|2\rangle$. This is similar to a standard computer, but in the case of a standard or classical computer a bit exists in one state exclusive of the other and with a quantum computer a q-bit can exist in a superposition of the two states $|\psi\rangle = C_1|1\rangle + C_2|2\rangle$. In general this superposition can include and arbitrary number of basis elements. These q-bits each define a set of two state Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, ... $\mathcal{H}_n$. A state vector defined by this entanglement will then have 2^n possible states. This

means that an array of 3 atoms or ions, 3 q-bits, will then define the wave function

$$|\psi\rangle = C_{111}|1,1,1\rangle + C_{112}|1,1,2\rangle + ... + C_{222}|2,2,2\rangle. \quad (9.1.1)$$

This system is then a quantum parallel processor with quantum wave interference between the various computational paths. The atoms or ions for these q-bits exist in an array and are placed in an entanglement of states by the $\pi/2$ laser pulses.

The operation of a quantum computer involves three basic steps: preparation of states, computation, and read out of the output,

$$|\psi_{(in)}\rangle = \sum_n C_n^{(in)}|n\rangle : \textit{preparation of input},$$

$$|\psi_{final}\rangle = e^{iHt}|\psi_{in}\rangle : \textit{computation by Schrodinger equation},$$

$$|\psi_{(out)}\rangle = \sum_n C_n^{(out)}|n\rangle : \textit{read out of input}. \quad (9.1.2)$$

The preparation of states involves the establishment of entangled states. The nature of the entangled states determines the form of the parallel computation to be performed. The evolution of the computation is governed by a Hamiltonian that governs a time reversible computation, where reversibility is established by the addition of registers that record each step of a computation, of a Turning machine or Von Neumann type of computational system. Reversibility is important, for the erasure of information involves irreversibility and entropy and is not an aspect of quantum mechanics. The read out must be performed by measurements of the quantum system. This results in an irreversible change in the system. The addition of quantum erasers, pumped optical cavity systems that can reestablish quantum coherence, would be required if output is required during the execution of the quantum computer.

Of central concern with quantum computers is the problem of decoherence. This problem stems from two sources. The first is thermal

noise in the environment. Zurek has demonstrated that thermal noise has the effect of increasing the volume of a system in phase space. Since information is contained on the energy surface that bounds this volume, the increased volume expands this energy surface means that this information is increasingly mixed in with random bits of information. The result is that the system will continually lose information. The second source is the existence of the vacuum in the universe. The quantum computer has a finite number of states. This means that the system's information is completely recoverable since there will be a Poincare reccurrence of states. However the vacuum has an infinite number of states, defined by the set of modes $\{|0_k\rangle\}$, for $k \in \{0, \ldots \infty\}$ and continuous, which means that a spontaneous emission of a photon by the computer that cancels one of these vacuum modes will not have a Poincare reccurrence cycle that is finite in time. The vacuum is effectively a system of random quantum fluctuations and contributes to the noise that a quantum computing system would encounter. Since each of the q-bits has a certain probability of exhibiting decoherence during any time interval this means that a quantum computer will have an exponential sensitivity to noise and spontaneous emission. This problem is a major arena of research at this time.

The nonlocality of quantum mechanics is most often seen in the entanglement of states. These entanglements give rise to spacelike connections between states. The simplest example is seen with a pair of spin 1/2 particles that result from the decay of a spin 0 particle. The two particles fly apart in opposite directions. Assume, if one particle passes through a region with a magnetic field both particles will be subjected to the spin precession. A measure of the spin position of both particles will reveal that the spins are still in opposite directions. This will obtain even when both particles are measured simultaneously so that signal can be communicated between the two detectors at the time of measurement.

The precise nature of entangled states can be seen with an example from quantum optics. Basically an entangled state is the following. Say we have N two state atoms. Each atom has the states $|1\rangle$ and $|2\rangle$ that are relevant to the problem. We put all the atoms in their ground states, so the state of the n atoms in combination are

$$|1\rangle_1|1\rangle_2 \ldots |1\rangle_n. \tag{9.1.3}$$

Now we apply a $\pi/2$ pulse to the first atom and we get the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_1 + |2\rangle_2)|1\rangle_2 \ldots |1\rangle_n. \tag{9.1.4}$$

Now we apply the following transformation between the first and second atomic states

$$\begin{aligned} |1\rangle_1|1\rangle_2 &\rightarrow |1\rangle_1|1\rangle_2 \\ |1\rangle_1|2\rangle_2 &\rightarrow |1\rangle_1|2\rangle_2 \\ |2\rangle_1|1\rangle_2 &\rightarrow |1\rangle_1|1\rangle_2 \\ |2\rangle_1|2\rangle_2 &\rightarrow |2\rangle_1|1\rangle_2. \end{aligned} \tag{9.1.5}$$

This can be more generally seen as the following map

$$|a\rangle|b\rangle \rightarrow |a\rangle|a + b\rangle, \tag{9.1.6}$$

where a is a control state for this control-NOT operation. Now we apply this to our wave function $|\psi\rangle$ where atom 1 is the control state for the second state

$$|\psi\rangle \rightarrow \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2 + |2\rangle_1|2\rangle_2)|1\rangle_1 \ldots |1\rangle_n. \tag{9.1.7}$$

Now just repeat the process where the first atomic state is the control state

$$|\psi\rangle \rightarrow \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2 \ldots |1\rangle_n + |2\rangle_1|2\rangle_2 \ldots |2\rangle_n). \tag{9.1.8}$$

This is the final entangled, or maximally entangled state, for the system.

The problem with decoherence can be illustrated with this entangled state. It is easy to see that if there is a jump that occurs due to stochastic noise that this maximally entangled state will become

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2 \ldots no\, i^{th} \ldots state\, |1\rangle_n + |2\rangle_1|2\rangle_2 \ldots no\, i^{th}\, state\, |n\rangle)|i\rangle, \tag{9.1.9}$$

which is no longer a maximally entangled state. Entangled states are vulnerable to environmental effects such as spontaneous emission. It is then apparent that if an entangled, or partially entangled, state in the operation of a quantum computer emits a photon and is further disentangled that the quantum computer would be effectively solving an entirely different problem than the initial problem. Further, the information pertaining to the intended problem is lost.

The situation that is proposed is one involving electron spin resonance (ESR), or nuclear magnetic resonance (NMR), with charged amino acid residues. The basic side chains lysine, arginine and histidine are alkaline amino acids with a positive charge. The acidic side chains aspartic acid and glutamic acid have a negative charge. In either case, and excess electron for an acidic chain or an excess nuclear charge, that may be effectively an electron hole, for the basic side chain. These amino acids are common in alpha helical proteins. The alpha helical structure is one that is a secondary structure where the protein exists in a helix, in much the same way DNA does, and where the helix executes on turn 2π for every 3.6 amino acid residues. Charged amino acid residues are common in alpha helices where the charges on these residues form hydrogen bonds that stabilizes the structure of the helix. Often nonploar amino acids, except proline due a bond between a trimethyl group with the carbon and nitrogen in the amine group that bends this in a fixed angle, are also found in alpha helical chains. These protein subunits naturally find themselves inside cell membranes, due to their hydrophobic nature, that act as anchors, and as regions that bind to various sequences of DNA. It is the charge nature of these proteins that they are proposed n experiments for ESR based quantum computers.

A brief discussion is in order on the nature of how proteins are constructed in a cell. The usual laws of physics apply, mechanics and thermodymanics in the formation of a protein. Here we examine the ratchet like machinery that is involved with the formation of a polypeptide. At each step in the formation of a protein an amino acid is linked to the nascent polypeptide chain through ATP hydrolysis. This iterative mechanism involves fluctuations that are modeled according to the Feynman ratchet.

We compare the formation of a protein to the mechanism of a ratchet form of a tire jack. One pushed on a lever to lift the car and a ratchet that held the car at that height as you reset the lever. The ratchet was a device that permitted work on the system to be conducted in one direction. The work in lifting the car is then stored as gravitational potential energy. This device is one that can be imagined scaled to the nanoscale. This device would then be subject to the random motion of gas or liquid molecules. As such, it is possible that a molecule of gas could strike the lever thus lifting the jack and the added height kept fixed by the ratchet. As a result, this jack could just sit in an ambient environment and simply lift some small object. We then have a curious situation where the potential energy of some small mass is increased with no heat flow. This is then an apparent violation of the second law of thermodynamics. Feynman discussed this problem in his Lectures on Physics [3] and demonstrated that the apparent violation is resolved by considering the fact that the ratchet is itself an object with a finite temperature, and as such the ratchet is also subject to fluctuations. It is then concluded that the ratchet is unable to lift the mass up or even store some potential energy for an indefinite period of time. It is possible for the Brownian motion of air molecules to impart net energy onto the ratchet if the ratchet is at a substantially lower temperature than the air. Such a system will work with the normal Carnot efficiency. As such the nanoscale jack would need to have some input of external energy to move the system from thermodynamic equilibrium.

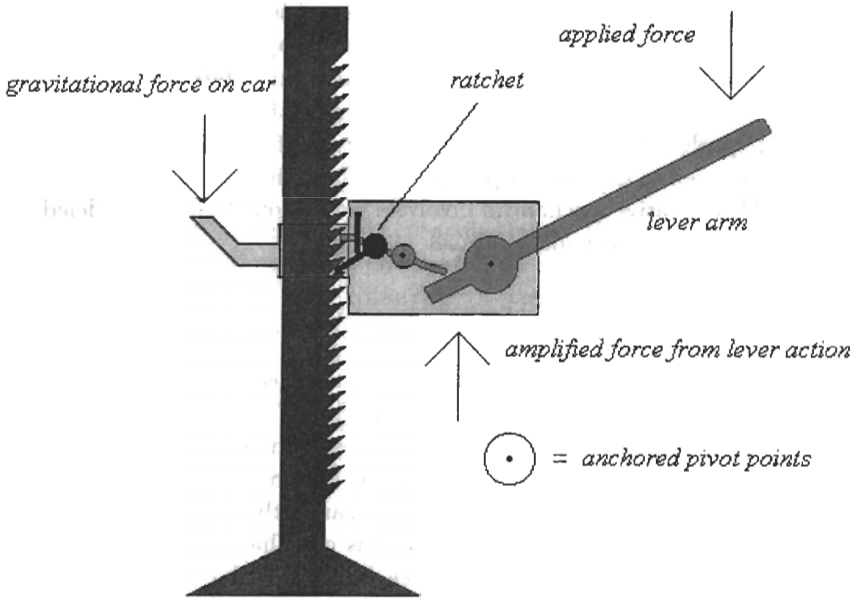


Figure 9.4

There are molecular forms of the tire jack. In this regime, 10-100nm, one is at a scale that is significantly larger than quantum fluctuations, but where statistical fluctuations due to Brownian motion predominate. Further, the Reynolds number = mass/friction becomes very large. This means that a bacillus swimming through water faces the same sort of resistance a swimmer would in a pool of honey or molasses. On this scale the predominant mechanisms are then viscosity and the diffusive effects from Brownian motion of water molecules.

The role of actin and myosin in muscle cells is very analogous to the tire jack. Muscle tissue is made of elongated multinucleated cells, segmented into sarcomeres, that are threaded with alternating threads of of actin and myosin fibers. The actin fibers are attached to what are called Z disks and the myosin fibers lie in between the actin fibrils.

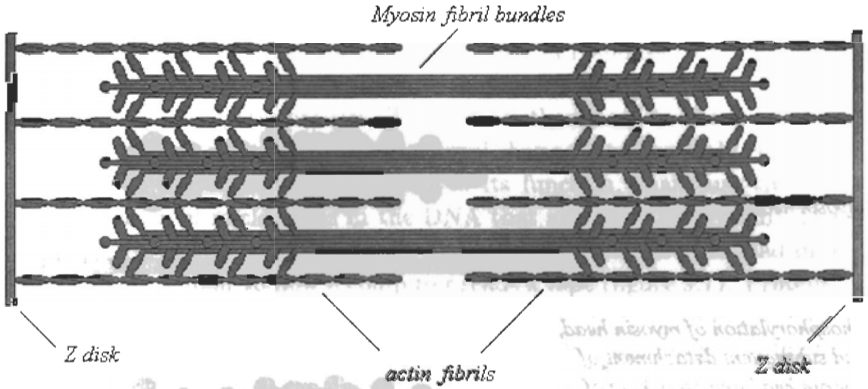


Figure 9.5

The myosin heads then, through a process of phosphorylation, attach to the actin fibrils and move inward, thus exerting an inward force on the Z disks. This results in a contraction of the overall sarcomere. The myosin heads move along the chain of actin molecules in much the same manner that a ratchet works (figure 9.6). At the start of each cycle each myosin head is attached to an actin. With the binding of ATP each head detaches and is cocked. The phosphorylation of the myosin head with $ATP \rightarrow ADP + P_i$ results in a power stroke. The process is similar to the ratchet device in that at the start of the cycle the myosin head is locked into a position. If there is an opposing force on the sarcomere at the start of the cycle, the sarcomere is held rigid. When an organism dies the myosin heads are left in this position and is the cause for rigormortis. The myosin-actin mechanism of muscle is an example of Feynman's resolution of the Brownian propelled ratchet. If the myosin head were not subject to thermal fluctuations then it might be possible for muscular action without the need of energetic ATP that gives 7.3kcal/mole of free energy[4].

Another example of the ratchet mechanism in biology is the process by which nRNA is translated into a polypeptitide chain. This mechanism is more complex for unlike the actin-myosin system where

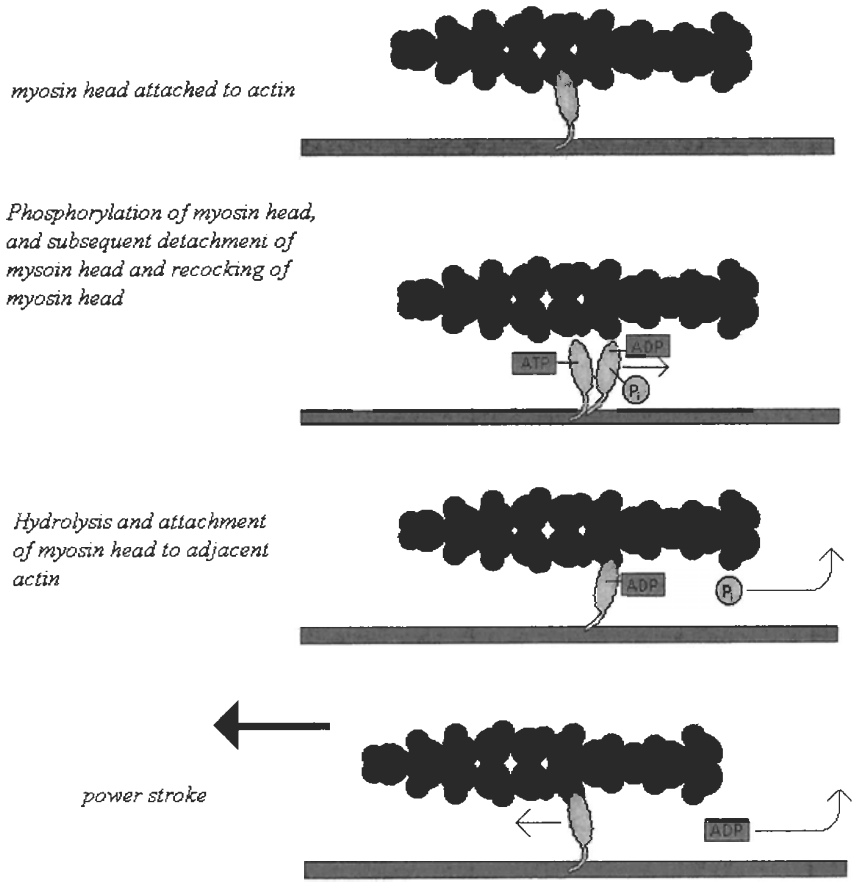


Figure 9.6

the actin and myosin fibers are fixed to the sarcomere, the polypeptide chain that is produces is free to fold in response to the internal stresses within the chain and the mechanical fluctuations that ensue from the energetic process. The process then leads to the formation of a protein with a three dimensional shape that effectively contains different information, according to its function, than just the linear chain of nucleotides in the DNA that encoded the protein. The polypeptide is produced in the ribosome, where mRNA is read in a manner similar to how a computer reads a tape (figure 9.7). Proteins are linked in polypeptide bonds that connect the amine and carboxyl ends of amino acids together. The process is diagrammed in figure 9.8.

This process involves the use of energy at four steps. The first is that that $ATP \rightarrow AMP + P_i$ is required in the formation of an adenylated amino acid with a phosphate bond. Secondly this in turn transfers the amono acid to the tRNA, where energetically the process is left with an AMP along with the aminoacyl-tRNA. This subsequent energetic process involves an aminoacyl tRNA synthetase that is specific for each amino acid. The third energetic process involves the formation of the peptide bond through ATP hydrolysis. The final step is the translocation of the t-RNAs with elongation factors and the energetic pathway induced by $GTP \rightarrow GDP + P_i$. As such the formation of a polypeptide is an energetically intensive process. The first energetic pathway, while biologically important is not of specific interest here, for it is not explicitely coupled to any nascent polypeptide or the ribosome. The second process may be of importance, but since it is coupled to the ribosome this step may not result in vibrational modes that effect the polypeptide. This assumption is based on the reasoning that the ribosome is a massive molecule and any energy fluctuation is either not likely to induce significant bulk motion and that modes induced inside the ribosome are likely to be distributed according to the virial theorem throughout the ribosome. As such these modes will be "diluted" within this macromolecle and will weakly couple to the nascent polypeptide. As

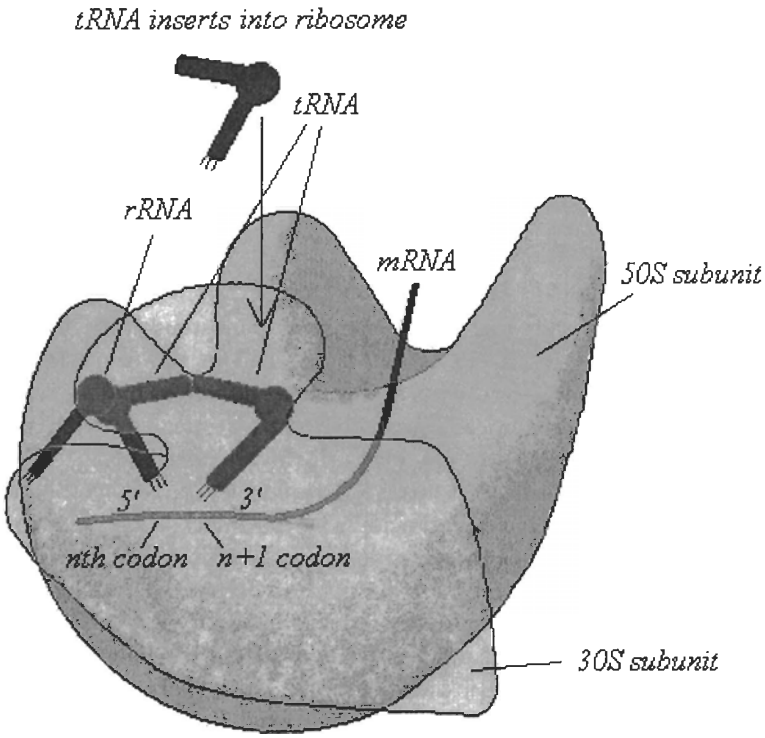


Figure 9.7

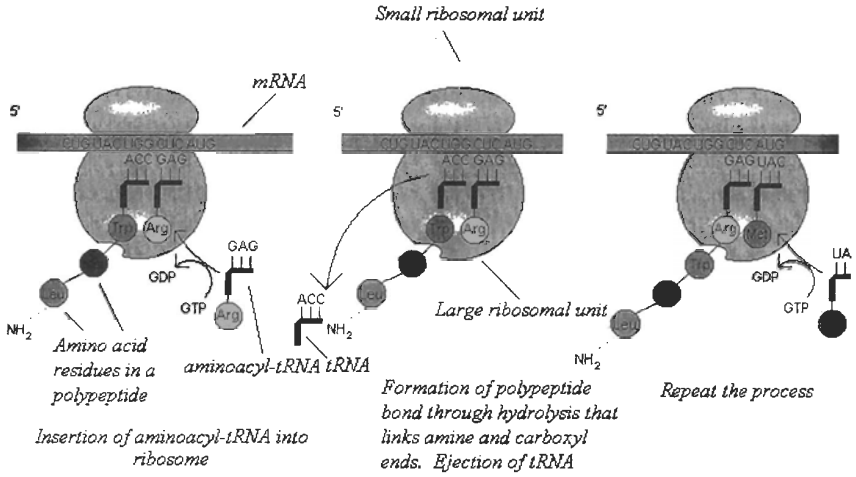


Figure 9.8

such this energetic pathway is not considered. The process that is of direct importance is ATP hydrolysis. It is here that the energy fluctuation will be strongly coupled to the nascent polypeptide. This will then act as the energy involved with the mechanics of protein folding. The final process that is of importance is the translocation of t-RNA, for this effectively moves the nascent protein out of the ribosome. As such the second and fourth steps will induce modes on the polypeptide chain, where these modes may induce conformational changes or put the nascent protein into various "folding states." Both of these steps also act to release a ratchet.

It has been demonstrated that polymerases that replicate DNA exert a force on the DNA that it is measurable throught the application of laser tweezers. Here a bead is attached to the DNA and the optical molasses that occurs from two opposing laser beams can be used to detect this minute force. The force is exerted in pulses as NTP's expend their energies and are incorporated into the replicated DNA chain. In the case of the formation of peptide bonds we see that in the middle diagram that the occurrence of the peptide bond requires ATP. Without ATP the process is "stuck" at the first frame. It is then reasonable to think that the occurrence of the aminoacyl-tRNA in the A binding area of the ribosome, the A region is where the first tRNA binds and the P region is where the second tRNA binds, acts much as the ratchet in a car jack. Without the energetically favorable dephosphporylation that occurs with $ATP \rightarrow ADP$, this system is stuck. Hence this situation implies that there is some nkT of energy required to change this binding of the aminoacyl-tRNA in the A binding region in order the to change this system. The ATP hydrolysis involves the release of a Gibbs free energy mkT where $m > n$. As such the process of polypeptide formation and the "Virginia wheeling" of tRNAs then ejects the left tRNA is energetically possible. External energy has been applied to remove the ratchet from its lock and keep the system in motion.

The ATP hydrolysis of aminoacyl-tRNA into tRNA that binds the amino acid onto the nascent protein involves the use of Gibbs free energy to forge the peptide bond. Translocation of the t-RNAs also require the release of energy from GTP. We will at this point treat both of these steps as being one from the standpoint of stochastic mechanics. Both steps trip the ratchet and they further deliver their work onto the nascent polypeptide. This process will then involve a quantity of Gibbs free energy $\delta G = \delta E - T\delta S$, where E is the energy available to to work, T is the temperature, and S is the entropy. In effect the work available to the system changes the state of the ratchet and overcomes the potential barrier associated with it in the locked state. As such the Gibbs free energy gives a chemical potential that induces a definate direction for the progression of the system.

Suppose that we have a ribosome that is producing an polypeptide chain with charged amino acid residues. The mRNA is engineered to produce a particular alpha helix. If we wish for these charged residues to interact with each other two problems must be overcome. The first is that the polypeptide chain must be prevented from coiling up with the formation of hydrogen bonds between the acidic and alkaline residues. These bonds will neutralize the charges. There are two ways of preventing this. We could have a polypeptide chain with only acidic or alkaline amino acid residues. The second problem is that these amino acid residues will attract water molecules and form bonds. Again this is a problem, but these bonds can be broken by a microwave that is tuned to their frequencies after the polypeptide is crystalized. The result is a chain of amino acid residues that will act just as the ion traps currently employed in experimental arrangements of quantum computers. An obvious advantage is that there is no need for expensive arrangements to trap ions, for in this case the charges are held in place by the peptide bonds between the amine and carboxyl groups in the polypeptide chain. An other advantage is that these polypeptide chains can be arrayed in lattices and arrangements.

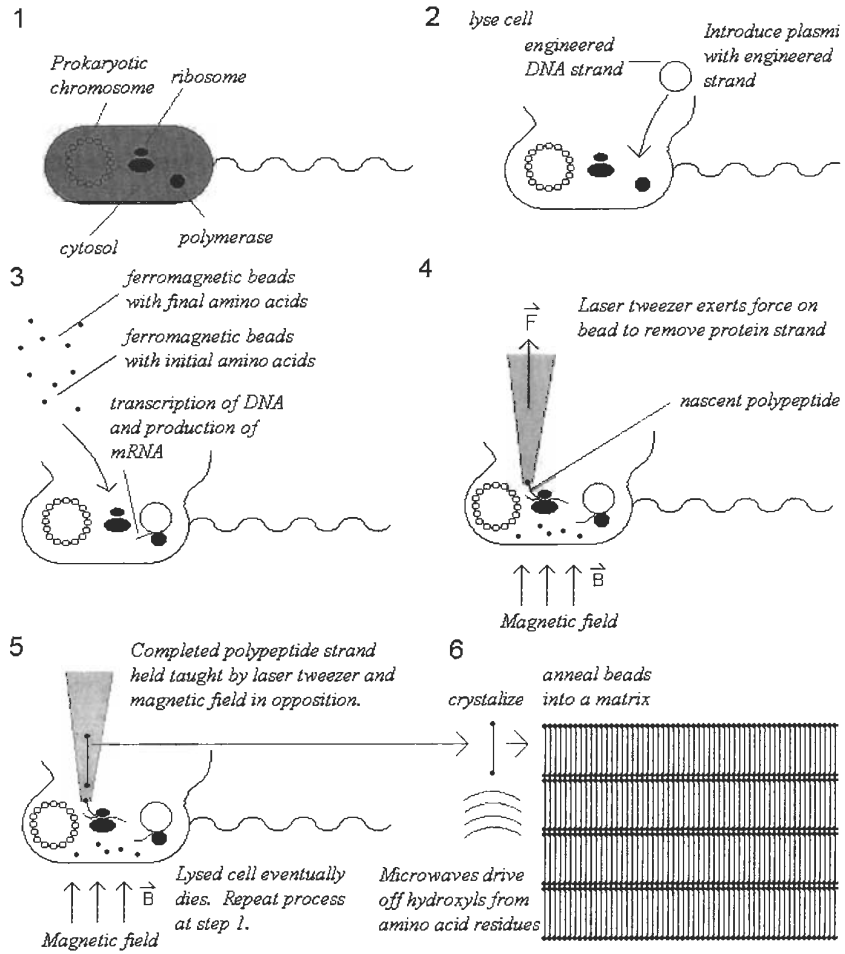
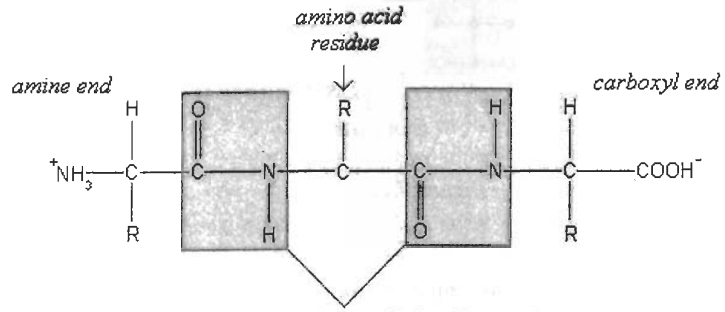


Figure 9.9

Before discussing the nature of quantum computers and the NMR/ESR role with the $B^{(3)}$ the following is proposed as the method used for producing a polypeptide quantum computer. A prokaryotic cell such as Escherichia coli is a system with polymerases for replicating DNA and for translating it into mRNA. The mRNA is then translated by ribosomes to produce polypeptides. A method for producing polypeptides for quantum computing is to lyse an E-coli and introduce a plasmid with the a gene for the repeated sequence of a charged amino acids. This peptide is designed and its template is its designed DNA. The molecular machinery of the cell will produce the mRNA that is complementary to the DNA sequence and the ribosome will in turn produce the polypeptide. The amino acids that form the amine and carboxyl ends of the polypeptide chain are attached to optically sensitive beads and magnetic beads respectively. The laser tweezer and the magnetic field applied to the final polypeptide keep the chain taught enough to prevent the formation of hydrogen bonds between the amino acid residues, as the laser tweezer lifts the product from the lysed cell. The protein is crystallized and this is in turn exposed to an electromagnetic field that is able to break the bonds that hold OH^- and H^+ ions from the alkyline and acidic amino acids respectively. The beads that hold the amine and carboxyl ends of this protein are annealed to a substrate to form a matrix. This forms the basic circuitry of a quantum computer chip. A sketch of the process is given in figure 9.9. Proteins are composed of amino acids linked by peptide bonds. Figure 9.10 illustrates a single peptide bond between two amino acids that form a dipeptide.



Proteins are formed by peptide bonds. This bond is a rigid planar structure with no rotational freedom

Figure 9.10

In order to produce an electron spin resonance effect one needs an external magnetic field and an electromagnetic field that acts to flip the spin of the various residues. The spin states are defined by the external magnetic field as these spin states are aligned in the direction of this field. The timing of the electromagnetic pulse is what determines if the spin is completely flipped or if it enters into a state that is a superposition of the spin up and spin down states. The contolled-not gate can be seen with the correlations between the two spin states of the electron on the amino acid residue. The coupled spins of the electrons will energetically favor spin antiallignment. If we initially start out with two electrons in aligned states and a pulse places one electron into a superposition of the spin up and spin down state then after a time when a second $\pi/2$ pulse is placed on this electron it will be antialligned in its final state. A repeat of this experiment on an electron that is antialligned with the second one will reveal that this electron after the two $\pi/2$ pulses will end up back in the antialligned state. This is the basically the controlled-not gate.

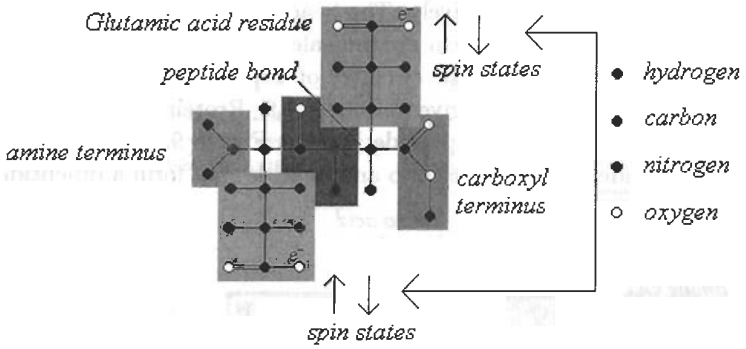


Figure 9.11

With the $B^{(3)}$ field the construction of a quantum processing unit or chip is unencumbered by the requirement for large magnets. The $B^{(3)}$ field is produced by an electromagnetic wave in a medium that

induces nonlinear and nonabelian behavior. This field is comparatively strong and localized. Due to the $1/\omega^2$ effect in the RFR effect longer wavelength radiation is required. A 10Mhz radio wave can induce a 14.5 mega tesla field. For the application discussed here an infrared laser would probably suffice to generate a field. A far infrared laser could produce a field approaching a tesla.

Applications towards quantum computers will doubtless be researched as the empirical reality of the $B^{(3)}$ field becomes apparent. The discussions of the magnetic resonance effect from the $B^{(3)}$ indicates that it should be possible to induce the NMR effect on two spins when the two spins are coupled by a Hamiltonian of the form $\sigma \cdot \sigma'$. We have the standard Hamiltonians for the two spins $H = H_0 + H_{B^3_{int}} + H_{B^{(3)}}$, where the $B^{(3)}$ Hamiltonians are

$$H_{B^3_{int}} = \frac{e^2}{4m\hbar\epsilon_0 V}\sum_k \left(\omega_k I a_k^\dagger a_k + \sum_q \omega_q \sigma^{(3)}(a_q^\dagger a_{k-q} + a_q a_{k-q}^\dagger)\right)$$

$$H_{B^{(3)}} = \frac{e}{2\omega_q \epsilon_0 V}\sum_{k,k',q}(a_{k+q}^\dagger a_k a_{k'-q}^\dagger a_{k'}). \qquad (9.1.10)$$

This coupling between the spins will then permit the existence of entangled states and the existence of a controlled-NOT state.

The largest difficulty with any quantum computer is decoherence. In this simulation it is assumed that the cavity is a perfect Q cavity. However, reality is not so forgiving. The problem is that spontaneous emission of a photon from a q-bit can occur in 10^{-6} seconds. A High Q cavity can reduce this significantly, and the application of a quantum eraser can restore quantum entanglements lost through decoherence. A quantum eraser involves the application of a detector that measures a photon emitted by an excited atom, and then directs a photon to a half meter that stimulates that atom and the atom the state was entangled with. The result is that the entangled state can

be restored. However, the practical issues are not entirely solved. The quantum eraser allows one to measure a quantum state without destroying the quantum entanglements of that state, and can thus be used to restore the entanglement of a state due to the quantum noise of the environment.

The future appears to be open to new forms of computation. While the problems with quantum computing are not entirely solved, there is a prospect that quantum computers may exist in the early twenty first century and offer general solutions to NP complete problems.

9.2 $B^{(3)}$ FIELD AND THE SEQUENCING OF DNA

NMR basically looks at the energy gap associated with a spin flip. This usually involves looking at the spin flip of hydrogen. Now if $\boldsymbol{B}^{(3)}$ occurs in a nonlinear quantum optical system this might be a way of mapping out the occurrence of hydrogens in a molecular chain. By this it may be possible to sequence a gene by directly mapping the places for hydrogens.

The idea would be to use far infrared lasers that will have nonlinear and NMR behavior. This would allow for the mapping of hydrogens in a DNA chain. This would allow for quick sequencing of DNA samples for the purpose of genome identification. Current methods, while better than they were 10 years ago, are still fairly laborious. Recombinant genetics costs have gone from \$5.00 per base pair to \$.50 per per base pair in the last 10 years. A nonchemical sequencing technique would undoubtedly drive this cost down further.

Most NMR involves finding the spin flip of a proton in a hydrogen atom. The problem with NMR is that it is rather course grained. Now suppose a DNA sample is placed on an array of micro-Josephson junctions. An IR laser scans the sample so that the charged induced phase shift gives rise to a $\boldsymbol{B}^{(3)}$ field. Then the rfr effect occurs with the hydrogens in the DNA. The interaction of the $\boldsymbol{B}^{(3)}$ field with the spin will induce flux across the Josephson junction. This would be a quantum NMR device where we would detect the change in a quanta of magnetic flux $\simeq e/\hbar$. We can then form a map of the occurrence of hydrogens and map the H-bonds between the complementary strands of DNA. As the spin coupling is different for the two types of purine to pyrimadine bonds are different, a map may be made the occurrence of hydrogens in the base pairs and identify the purines and pyrimidines.

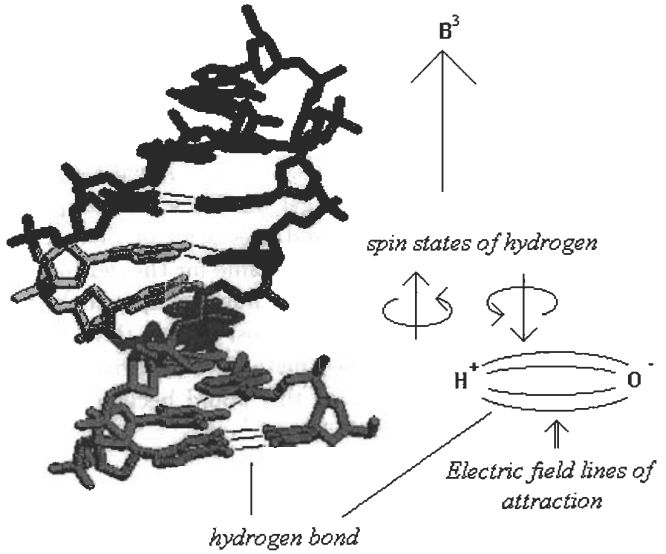


Figure 9.12

This would be faster and simpler than current methods. It involves taking a sample of single strand DNA (split off by heating), putting it in the presence of a polymerase and nucleosides where about 5% are dideoxynucleic acids. This adulteration of the 3' end at the ribose

prevents the phosphodiester bond between the 5' and 3' end of the nucleoside with the complementary DNA strand by the polymerase. The result is that one gets a mixture of complementary DNA with varying lengths. Heating then splits the complementary strands and this is run through an electrophoresis gel. The short strand move fast and the long ones move slow through the gel under the influence of the electrical field. By making 4 batches, each run on the gel side by side like a race track, with adenine thymidine, guanine and cytosine as the dideoxynucleoside in each lane. Then one can then read off on the blot the sequence of a gene. This is a rather complex, laborous and time consuming. It also involves the use of ethidium bromide for identification of bands. E-Br is a very dangerous substance for it is extremely mutanagenic and carcinogenic. Perkin Elmer produces an automated sequencer that makes this work certainly easier. It interfaces a MAC and the results are read off on the computer screen.

In the end economics makes the difference. The cost per base pair sequencing is currently between .25−.50, where it was ten times that in the 80s. One cost is augarose gel, and a serious sequencing is done on a pretty large gel frame and uses a lot of this. The other problem is that this takes considerable time for the gel to run as all those negatively charged DNA strands have to squirm through the gel under an electric potential. So the cost here is time, but as it is said time is money. This proposed method could drive the cost per base pair much further down then this would become an accepted way of gene sequencing.

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APPENDIX 1
APPLICATIONS OF STOKES' THEOREM

In this appendix three applications of Stokes' Theorem are given which distinguish between a static magnetic field; the magnetic component of an electromagnetic field; and a topological magnetic field.

1. The Static Magnetic Field: Integration around a Circle

The relevant Stokes Theorem in this case is the abelian:

$$\oint \boldsymbol{A} \cdot d\boldsymbol{l} \;=\; \int\int \boldsymbol{B} \cdot d\mathcal{A} \;=\; \pi R^2 B_\perp, \qquad (A1.1)$$

where $\boldsymbol{B} \;=\; \nabla \times \boldsymbol{A}$. The area integral is around a circle of radius R, and the line integral is over the circumference. The vector potential can be expressed as:

$$\boldsymbol{A} \;=\; \frac{1}{2} B(X\boldsymbol{j} \;-\; Y\boldsymbol{i}), \qquad (A1.2)$$

in cartesian coordinates $(X,\; Y)$, related to circular polar coordinates by:

$$X \;=\; R cos\theta;\; dX \;=\; -R sin\theta d\theta$$

$$Y \;=\; R sin\theta;\; dY \;=\; R cos\theta d\theta. \qquad (A1.3)$$

It is readily checked that the line integral is equal to the area integral:

$$\oint \boldsymbol{A} \cdot d\boldsymbol{l} \;=\; \frac{1}{2} B_\perp R^2 \int_0^{2\pi} (sin^2\theta \;+\; cos^2\theta) d\theta$$

$$= \pi B_\perp R^2, \qquad (A1.4)$$

as given by the abelian Stokes Theorem.

2. Magnetic Component of an Abelian Electromagnetic Field

The relation between the magnetic component of an abelian electromagnetic field and the vector potential is again $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where:

$$\boldsymbol{A} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}$$

$$\boldsymbol{B} = \frac{B^{(0)}}{\sqrt{2}}(\boldsymbol{j} - i\boldsymbol{i} + \boldsymbol{k})e^{i\phi}$$

are plane waves. Here ϕ is the abelian phase factor $\phi = \omega t - \boldsymbol{k} \cdot \boldsymbol{r}$ as usual. If an integration is attempted around a circle, with:

$$\mathcal{A} = |\mathcal{A}|\boldsymbol{k}, \qquad (A1.6)$$

it is seen immediately that the result is zero. As shown in chapter three, the line integral also vanishes. So the topological phase is undefined in abelian electrodynamics because $\boldsymbol{B}^{(3)}$ is undefined.

3. The Topological Magnetic Field $\boldsymbol{B}^{(3)}$

The topological magnetic field is defined in $O(3)$ electrodynamics as (see chapter three):

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \qquad (A1.7)$$

and is not defined as the curl of a vector potential. It gives the topological phase factor:

$$\gamma = g \int\int \boldsymbol{B}^{(3)} \cdot d\mathcal{A} = 2\pi \oint \boldsymbol{k} \cdot d\boldsymbol{R} \qquad (A1.8)$$

which is achromatic. The line integral equal to this achromatic phase factor is along the propagation axis, because as seen in example (2), the transverse contributions vanish. The Stokes Theorem needed for this result is non-Abelian and the B field is related to the topological charge, or magnetic monopole, through:

$$g_m = \frac{1}{V} \int\int \boldsymbol{B}^{(3)} \cdot d\mathcal{A}, \qquad (A1.9)$$

where V is a volume. The $\boldsymbol{B}^{(3)}$ field essentially derives a form of the Berry phase for one photon, and is related to the Pancharatnam phase. The phase factor γ for one photon is the topological charge ± 1, and is detected in interferometry by changing the state of polarization of the beam in one arm. Experimentally there is no difference in interferometry between a phase change induced by changing the length of one arm (changing the optical path length), and a phase difference induced by changing polarization in one arm. This is what the non-Abelian Stokes Theorem (A8) has to say. However, the topological phase is experimentally distinct from the dynamical phase in several ways, as summarized in chapter three.

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