Chapter 10

SOME CONSEQUENCES OF FINITE PHOTON MASS IN ELECTROMAGNETIC THEORY

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Abstract

Some consequences of finite photon mass are reviewed in electromagnetic field theory, using as a conceptual framework the Einstein-de Broglie interpretation of wave particle dualism, recently supported by experiment. It is demonstrated that finite photon rest mass (m_0) embodied in the Proca field equation, is consistent with special relativity and with gauge transformation of the second kind provided $A_\mu A_\mu^* \rightarrow 0$, where A_μ is the (complex) potential four-vector. Finite m_0 is, however, inconsistent with the transverse, radiation, or Coulomb gauge, this being a subsidiary condition which is used routinely, but which is inconsistent with special relativity. Finite photon mass leads to longitudinal magnetic and electric fields in free space, fields which conserve the fundamental discrete symmetries of nature, which remain finite in the limit $m_0 \rightarrow 0$, and which are related to the corresponding transverse field components through Lie algebra. The limit $m_0 \rightarrow 0$ represents the transition from the Proca to the Maxwell formalism. The longitudinal fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ respectively are frequency and phase independent components of four-vectors B_μ and E_μ, so that electromagnetic energy density is represented by Poincaré invariants such as $E_\mu E_\mu$ and $B_\mu B_\mu$. Although $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ have no Planck energy (i.e. correspond to zero frequency) they are expected to produce a variety of novel spectral effects in the laboratory, effects which if observed, would provide evidence for finite photon mass.

1. Introduction and Brief Historical Perspective

The idea that the photon may have non-zero mass was developed by Louis de Broglie [1–5] over many years of investigation. His first massive photon equations [6] were proposed in 1934, shortly after the emergence of the Proca field equation in 1930 [7–10]. De Broglie's work in this area is recorded in numerous books and articles which are accessible [6] through Library of Congress listings [11, 12]. The development of these ideas in the Paris of the 1930's is summarized by Goldhaber and Nieto [6]. The de Broglie photon equation of 1934 (not to be confused with the de Broglie equation, the famous guiding theorem) is described by Goldhaber and Nieto as coming from the product of a Dirac particle space with a Dirac antiparticle space, from the outset they regard the photon as a particle with mass, and the anti-photon as an anti-particle with mass. The de Broglie photon equation is related to the Duffin-Kemmer-Petiau wave equation [6] for non-zero and spin-one particles in its fundamental (reducible) sixteen dimensional representation. The latter can be defined [6] as a symmetric product

space of two Dirac spaces (a composite of two Dirac particle spaces). The Duffin-Kemmer-Petiau equation provides the Klein-Gordon and Proca equations if it is assumed that the wave function transforms as the product of two Dirac wave functions. The Klein-Gordon equation is, in this context, the irreducible representation corresponding to a five dimensional pseudo-scalar equation obtained by specializing to a plane wave and diagonal matrix elements. The Proca equation is the irreducible representation corresponding to a ten dimensional spin one equation. The de Broglie photon equation on the other hand decomposes into a one dimensional pseudo-scalar irreducible representation; a five dimensional irreducible representation corresponding to a scalar Klein-Gordon equation; and a ten dimensional *axial vector* representation of the Proca equation. Duffin [13] has described the general mathematical properties of characteristic matrices of covariant, quantum relativistic systems.

It is clear, therefore, that the de Broglie photon equation of 1934 considers Dirac spaces for particles and anti-particles, giving the possibility of photons and anti-photons. The Duffin-Kemmer-Petiau equation on the other hand considers only Dirac spaces, and therefore only photons, which are their own anti-particles. If these equations are applied to photons (and anti-photons) in the classical limit $m_0 \rightarrow 0$, the Maxwell equations of the classical electromagnetic field must be recovered. In contemporary description [14], the charge conjugation operator $\hat{C}$ applied to the photon in de Broglie's equation must produce the anti-photon by definition, i.e. $\hat{C}$ is defined [14] as the operator that produces the anti-particle from the original particle, while having no effect on space-time properties such as helicity. Thus, $\hat{C}$ produces the Dirac anti-particle space from the Dirac particle space. This must mean that $\hat{C}$ has the effect, for example, of reversing the sign of all four components the potential four-vector A_μ of the electromagnetic field, and all components of electric and magnetic fields of a plane wave in vacuo. (In general the non-trivial topology [15] of the vacuum is also affected by $\hat{C}$, all particles of the Dirac sea [15], of which the vacuum is composed in contemporary thought, are by definition of $\hat{C}$, changed to anti-particles, and the effect of $\hat{C}$ on the vacuum is non-trivial.) It is clear that the effect of $\hat{C}$ in the de Broglie photon equation is to produce the anti-photon equation. In the interpretation of the Duffin-Kemmer-Petiau equation the photon is its own antiphoton. Therefore the fact that

$$\hat{C}(A_\mu) = -A_\mu \tag{1}$$

in this equation means that the photon is in an eigenstate of $\hat{C} = -1$, which in contemporary understanding is a consequence of the covariance of the U(1) (electromagnetic sector) field equations [16] under $\hat{C}$ and $\hat{C}\hat{P}\hat{T}$. Here $\hat{P}$ is the parity inversion operator and $\hat{T}$ the motion reversal operator. The Duffin-Kemmer-Petiau equation, being a physical law, i.e. a relativistically consistent equation of the electromagnetic field, must be invariant under a discrete symmetry operator such as $\hat{C}$. This implies that the negative charge parity of the photon must be conserved. If so, the distinction between photon and anti-photon becomes unmeasurable [14], which is consistent with the fact that the Duffin-Kemmer-Petiau equation is written in a symmetric product space of two Dirac spaces, i.e. is a composite of *particle* spaces, the particle being identified with the massive photon. In the de Broglie equation, which appears to this author to be equally valid, the anti-photon is produced by $\hat{C}$ from the photon and vice-versa, all space-time properties such as photon helicity being unaffected by definition of $\hat{C}$ [14].

In contemporary field theory, however, the notion that the photon is its own anti-particle is prevalent [14, 15], and we adopt this point of view as a matter of convention rather than as a logical necessity. In this framework the Duffin-Kemmer-Petiau equation is a description of the electromagnetic field considered as a massive gauge field, whose quantization produces well defined massive photons with three space-like polarizations, two transverse to the direction of propagation, one parallel to this direction, and therefore longitudinal. This equation must be invariant under local U(1) gauge transformations [16] in the contemporary description. We show in Sec. 3 that this leads to the limiting gauge condition

$$A_\mu A_\mu^* \to 0, \tag{2}$$

where A_μ is considered as a complex Dirac gauge [17, 18]. Equation (2) is valid for finite photon mass (m_0) in the limit of infinitesimally small photon radius, considered as a four-vector r_μ. The latter is orthogonal to the energy momentum vector, p_μ, of the photon in its rest frame [17, 18].

The usual contemporary description of the U(1) sector differs from this in that the photon mass is considered [15] to be identically zero. Goldhaber and Nieto [6] show that there is no evidence for this idea, nor can there be, since it implies that the range of electromagnetic radiation is infinite, and therefore unmeasurable experimentally. In contemporary unified theory, Huang [19] has discussed finite m_0 in the context of the Glashow-Weinberg-Salam (GWS) and SU(5) theories, showing that a non-zero m_0 leads to a finite electron lifetime, for example, and is a central theme in contemporary particle physics and grand unified field theories.

Since electromagnetic field theory is the U(1) sector of grand unified theory, it is essential that meaningful consideration be given to the concept of finite photon mass introduced in the 1934 de Broglie equation. Earlier considerations of finite m_0 date to Einstein's proposals [20] of 1916, and before that, the existence of non-zero m_0 had been proposed [6], in necessarily classical and non-relativistic terms, since Cavendish. This implies an overhaul of habitual notions in electrodynamics. As soon as we accept the possibility that $m_0 \neq 0$, the theory of gauge invariance is affected at a fundamental level because the Lagrangian [15] is supplemented by a mass term. If this is non-zero, invariance under local U(1) gauge transformation is lost, meaning that the action

changes under gauge transformation of the second kind. Since the mass term is

$$\mathcal{L}_m \propto \frac{1}{2} m_0 A_\mu A_\mu^*, \tag{3}$$

it follows immediately that for $m_0 \neq 0$, gauge invariance is satisfied under the limiting condition (2) [21]. The idea that $m_0 = 0$ identically [15], the conventional idea, means that $A_\mu A_\mu^*$ is allowed to take *any* value, i.e. there is gauge freedom. A startling consequence of Eq. (2) is that the everyday Coulomb, or transverse, gauge [22] becomes inconsistent with finite photon mass. If (in S.I. units), ϵ_0 being the permittivity in vacuo,

$$A_\mu = \epsilon_0 \left(\mathbf{A},\ i\frac{\phi}{c} \right), \tag{4}$$

as usual, Eq. (2) implies

$$\phi \to c|\mathbf{A}|, \tag{5}$$

which contradicts the condition of the Coulomb gauge because in that gauge [22],

$$\phi = 0, \quad \mathbf{A} \neq \mathbf{0}. \tag{6}$$

Equation (5) means that the difference between ϕ and $c|\mathbf{A}|$ is infinitesimally small, but Eq. (6) means that ϕ is identically zero while $\mathbf{A}$ is non-zero identically.

The Coulomb gauge is inconsistent with finite photon mass.

Equation (6) is *also inconsistent with special relativity* unless $\mathbf{A}$ is identically zero, in which case there is no electromagnetic field. This fundamental inconsistency in the transverse gauge is related to the habitual assumption in electrodynamics that the longitudinal components of Maxwell's equations in vacuo are "unphysical", presumably zero. The reason for this is that the longitudinal and time-like components of A_μ are discarded in the transverse (Coulomb gauge), so that Lorentz covariance is lost. The longitudinal and time-like components of A_μ cannot therefore appear in the definition of the electric and magnetic components of the plane wave in vacuo. This inconsistency is accepted customarily on the grounds that the four-vector A_μ is not directly observable or physically influential. In this view [15], ϕ and $\mathbf{A}$ form parts of a mathematical subsidiary condition to the Maxwell equations, and since these are produced by (6), the transverse gauge is accepted. This leads in turn to the habitual assertion that longitudinal solutions of Maxwell's equations in vacuo are "unphysical" and presumably therefore unrelated to the usual transverse solutions. However, it has been known experimentally for over thirty years that the Bohm-Aharonov effect [23] means that A_μ is *physically* meaningful, since $\mathbf{A}$, in the absence of a magnetic field, produces fringe shifts in electron diffraction by changing the electron's wave function. Since the space-like part of A_μ, i.e. $\boldsymbol{A}$, is a physically meaningful quantity, then all four components of A_μ are also physically meaningful if A_μ is to be accepted as a four-vector of special relativity. This in turn leads to the conclusion that longitudinal solutions of Maxwell's equations in vacuo cannot be discarded, i.e. the manifest covariance of the theory of electromagnetism must be maintained rigorously. The habit of discarding the time-like part of A_μ destroys the structure of Minkowski space-time, i.e. is geometrically unsound.

Recent work by the present author [24–30] has resulted in a Lie algebra which shows mathematically that *the longitudinal solutions of Maxwell's equations in vacuo are related to the transverse components*. This algebra is consistent, furthermore, with the Proca equation for finite m_0, and by implication, with the Duffin-Kemmer-Petiau and de Broglie equations for massive photons. The novel Lie algebra [24–30] remains valid, furthermore, in the Maxwellian field, where $m_0 \rightarrow 0$, and is therefore consistent with what is known about the U(1) sector of contemporary grand unified field theory. For example, the magnetic components of the Maxwellian field can be described by the following vectorial Lie algebra in the three space-like dimensions,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} - iB^{(0)}\boldsymbol{B}^{(3)}, \tag{7a}$$

$$\boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} = iB^{(0)}\boldsymbol{B}^{(1)*} - iB^{(0)}\boldsymbol{B}^{(2)}, \tag{7b}$$

$$\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} = iB^{(0)}\boldsymbol{B}^{(2)*} = iB^{(0)}\boldsymbol{B}^{(1)}. \tag{7c}$$

This algebra is derived in Sec. 4. Equations (7) are written in a circular basis [24–30] in which (1) and (2) denote the transverse polarizations, $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ being complex conjugate pairs. Importantly, there appears in Eqs. (7) a *longitudinal* component, labelled (3), which is real, so that $\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(3)*}$. $B^{(0)}$ in these equations is the scalar amplitude of magnetic flux density in vacuo of the electromagnetic plane wave. Furthermore, if $\boldsymbol{B}^{(3)} = \boldsymbol{0}$, then $\boldsymbol{B}^{(2)} = \boldsymbol{0}$ from Eq. (7b) and $\boldsymbol{B}^{(1)} = \boldsymbol{0}$ from Eq. (7c), and because it is the complex conjugate of $\boldsymbol{B}^{(2)}$.

We conclude that non-zero transverse components imply a non-zero longitudinal component, the latter being frequency independent.

This is an obvious departure from the conventional idea that $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, while $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} \neq \boldsymbol{0}$. More details are given in Sec. 4. Since electromagnetic theory is a Poincaré invariant local gauge theory, it must conserve $\hat{C}\hat{P}\hat{T}$ [14]. It is shown in Sec. 4 that Eqs. (7) conserve the seven [14] discrete symmetries: $\hat{C}$, $\hat{P}$, $\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{P}\hat{T}$, and $\hat{C}\hat{P}\hat{T}$, a result which is consistent with the fact that $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$ are solutions of Maxwell's equations in vacuo, equations of a field theory that conserves $\hat{C}\hat{P}\hat{T}$. Since the equations conserve $\hat{C}\hat{P}\hat{T}$, then no

mathematical solutions thereof can violate $\hat{C}\hat{P}\hat{T}$. The habitual proposal that

$$\boldsymbol{B}^{(3)} =? \boldsymbol{0}, \qquad \boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} \neq \boldsymbol{0}, \tag{8}$$

is therefore heterodox, in that it contradicts the $\hat{C}\hat{P}\hat{T}$ theorem and a Lie algebra such as that embodied in Eqs. (7). It is clear that the longitudinal solution, $\boldsymbol{B}^{(3)}$, must therefore conserve the seven discrete symmetries in vacuo. For example, applying $\hat{C}$ to $\boldsymbol{B}^{(3)}$ we obtain,

$$\hat{C}(\boldsymbol{B}^{(3)}) = -\boldsymbol{B}^{(3)}, \qquad \hat{C}(\lambda) = \lambda, \tag{9}$$

while the beam helicity (λ) is unchanged by definition. Equation (9) must be interpreted to mean that the negative charge parity of $\boldsymbol{B}^{(3)}$ *has been conserved*. In other words, there is no $\hat{C}$ violation [14] implied by the existence of $\boldsymbol{B}^{(3)}$, and there is, for the same reason, no $\hat{C}$ violation implied by the existence of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$.

From Eq. (7a) it follows that changing the beam helicity implies changing the sign of $\boldsymbol{B}^{(3)}$, because the sign of $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ is changed, while that of the scalar amplitude $B^{(0)}$ is unaffected by beam helicity. Equation (9) becomes

$$\hat{C}(-\boldsymbol{B}^{(3)}) = \boldsymbol{B}^{(3)}, \qquad \hat{C}(-\lambda) = -\lambda, \tag{10}$$

and the negative charge parity of $-\boldsymbol{B}^{(3)}$ has this time been conserved, while $-\lambda$ has remained unaffected by definition of $\hat{C}$ [14].

It may be objected [31] that since $\boldsymbol{B}^{(3)}$ depends on the sign of λ (through the cross product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$) that $\hat{C}$ should leave $\boldsymbol{B}^{(3)}$ unchanged because it leaves λ unchanged and vice-versa. The flaw in this assertion is revealed simply by writing

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}}{iB^{(0)}} = \frac{f(\lambda)}{iB^{(0)}}, \tag{11}$$

where $f(\lambda)$ means "function of λ". While $\hat{C}$ does not change $f(\lambda)$ by definition, it changes the sign of the scalar amplitude $B^{(0)}$ by definition, so that Eq. (11) (i.e. Eq. (7a)) is invariant to $\hat{C}$ and there is no $\hat{C}$ violation implied by the existence of $\boldsymbol{B}^{(3)}$. Equation (7a) is invariant to $\hat{C}$ because operating on each symbol of the equation by $\hat{C}$ produces the same equation. Similarly for Eqs. (7b) and (7c). This exercise can be repeated for the other discrete symmetries, revealing that Eqs. (7) are invariant to the seven discrete symmetries, and thus violate none of these symmetries.

Another fundamental consequence of $m_0 \neq 0$ is the implied existence [32] of electric and magnetic four-vectors, E_μ and B_μ, in vacuo, four-vectors which are associated with the electromagnetic plane wave. The Proca equation [15] for $m_0 \neq 0$ may be written as

$$\Box A_\mu = -\xi^2 A_\mu, \qquad \xi = \frac{m_0 c}{\hbar}, \tag{12}$$

where A_μ is in general complex [32] and $\hbar$ is the reduced Planck constant. Thus A_μ takes on the role of a *physically meaningful* eigenfunction, upon which the d'Alembertian $\Box$ operates in Minkowski space-time to produce the eigenvalue $-\xi^2$. The Maxwellian limit of Eq. (12) may be reached in different ways (Sec. 3), but in this limit,

$$\Box A_\mu = 0, \tag{13}$$

which is the d'Alembert equation [13] in vacuo. Equation (13) is no longer an eigenfunction equation, and in conventional electrodynamics [33], A_μ is habitually regarded as a subsidiary mathematical consequence, or condition, arising from the Maxwell equations in vacuo. Although ξ is very small ($\sim 10^{26}\, m^{-1}$) there is a critical difference, therefore, between the Proca and d'Alembert equations, in that A_μ is a physically meaningful eigenfunction in the former, and a mathematical subsidiary condition in the latter. Experimentally, the Bohm-Aharonov effect, first observed by Chambers [34], and repeated several times in independent laboratories, shows conclusively that A_μ is *physically meaningful*. This result is support for the Proca equation and finite photon mass. The latter is consistent, in other words, with the fact that A_μ directly influences the wave function of an electron, meaning that all four components of A_μ are physically meaningful, as in the Proca equation. The latter is relativistically consistent, but mathematically *inconsistent* with the Coulomb, or transverse gauge. It is well known [15] that the Proca equation implies mathematically that

$$\frac{\partial A_\mu}{\partial x_\mu} = 0, \tag{14}$$

which is the Lorentz condition [15], defining the Lorentz gauge. In general, $\phi \neq 0$ and $\boldsymbol{A} \neq 0$ in this gauge, which is consistent with the fact that A_μ is a physically meaningful four-vector. Moreover, for complex A_μ, it can be shown that the Proca equation is consistent with the Dirac gauge [32], from which Eq. (2) has been derived [30] in the limit of infinitesimally small photon radius. In this limit, as we have seen, the Proca equation is consistent with local gauge invariance of U(1) for finite photon mass. The latter is numerically so small [6] that the Proca equation is always, for all practical purposes, consistent with local gauge invariance. The gauge freedom associated with identically zero photon mass is however lost. For example, the Proca equation is mathematically

inconsistent with the transverse gauge, as we have seen, and produces longitudinal photon polarization piloted in vacuo by longitudinal electric and magnetic fields.

Thus A_μ is a physically meaningful, fully (i.e. manifestly) covariant four-vector for $m_0 \neq 0$. This interpretation is retained in the Maxwellian field under the condition (2), which is consistent with local gauge invariance in the U(1) sector of unified field theory.

The Lie algebra (7) (that of a sub-group of the Lorentz group [15]) is consistent with the representation of the magnetic component of the electromagnetic field in vacuo as a four-vector in space-time [24–30]. The derivation of this result is given in Sec. 5, and is consistent [32] with the Proca equation and with the Einstein-de Broglie theory of light. The four-vector thus defined is, in the circular basis

$$B_\mu = (\boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \boldsymbol{B}^{(3)}, iB^{(0)}), \tag{15}$$

and similarly

$$E_\mu = (\boldsymbol{E}^{(1)}, \boldsymbol{E}^{(2)}, \boldsymbol{E}^{(3)}, iE^{(0)}), \tag{16}$$

for the electric component of the plane wave in vacuo. It can be shown from Eqs. (15) and (16) that $B^{(0)}$ and $E^{(0)}$ are identified as time-like components of B_μ and E_μ respectively. In consequence of Eqs. (15) and (16), and of the physical reality of A_μ, there is a relation between B_μ, E_μ, and A_μ which is established in Sec. 5. For our present purposes, we note that $E_\mu E_\mu$ and $B_\mu B_\mu$ are Lorentz invariants and contribute to the electromagnetic energy density in vacuo, the expression for which becomes

$$U = \frac{1}{2}\left(\epsilon_0 E_\mu E_\mu + \frac{1}{\mu_0} B_\mu B_\mu\right), \tag{17}$$

in S.I. units where ϵ_0 is the vacuum permittivity and μ_0 the vacuum permeability. Using,

$$E_\mu E_\mu = \boldsymbol{E}\cdot\boldsymbol{E} - E^{(0)2}, \qquad B_\mu B_\mu = \boldsymbol{B}\cdot\boldsymbol{B} - B^{(0)2}. \tag{18}$$

It becomes clear that the usual electrodynamical definition of energy in terms of $\boldsymbol{E}\cdot\boldsymbol{E}$ and $\boldsymbol{B}\cdot\boldsymbol{B}$ only, is inconsistent with special relativity, and with the Lie algebra exemplified in Eqs. (7). The space-like products $\boldsymbol{E}\cdot\boldsymbol{E}$ and $\boldsymbol{B}\cdot\boldsymbol{B}$ are not Lorentz invariant if E_μ and B_μ are regarded as physically meaningful four-vectors in Minkowski space-time.

However, it is known that the Planck radiation law is obeyed precisely [6], seeming to imply the existence of only two (transverse) degrees of freedom;

whereas finite photon mass implies three space-like degrees of freedom (two transverse and one longitudinal). How is this paradox resolved? The answer is given by the internal structure of $E_\mu E_\mu$ and $B_\mu B_\mu$,

$$E_\mu E_\mu = E^{(1)2} + E^{(2)2} + E^{(3)2} - E^{(0)2}, \tag{19}$$

$$B_\mu B_\mu = B^{(1)2} + B^{(2)2} + B^{(3)2} - B^{(0)2}, \tag{20}$$

and by using the Lie algebra described in Eq. (7). Specifically, Eq. (7a) gives, in the Maxwellian limit,

$$\boldsymbol{B}^{(3)} = \frac{\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}}{iB^{(0)}} = B^{(0)}\boldsymbol{k}, \tag{21}$$

where $\boldsymbol{k}$ is a unit axial vector in the propagation axis of the plane wave in vacuo. Similarly, it is shown in Sec. 5 that

$$\boldsymbol{E}^{(3)} = E^{(0)}\boldsymbol{k} \tag{22}$$

where $\boldsymbol{k}$ is a unit polar vector in the propagation axis. From Eqs. (21) and (22),

$$E^{(3)2} - E^{(0)2} = 0, \qquad B^{(3)2} - B^{(0)2} = 0, \tag{23}$$

and in Eqs. (19) and (20),

$$E_\mu E_\mu = E^{(1)2} + E^{(2)2}, \qquad B_\mu B_\mu = B^{(1)2} + B^{(2)2}, \tag{24}$$

which is precisely the result indicated by Planck's law. For all practical purposes, the same conclusion holds in the Proca field, where (Sec. 4),

$$\boldsymbol{B}^{(3)} = B^{(0)} e^{-\xi z}\boldsymbol{k}. \tag{25}$$

Since $\xi \sim 10^{-26}\, m^{-1}$ for $m_0 \sim 10^{-68}\, kg$ [32], we recover Eq. (21) to an excellent approximation. Thus, Planck's law is consistent with finite m_0 because the longitudinal electric and magnetic fields $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ do not contribute to the electromagnetic energy density of the classical field. In other words, $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$, being independent of frequency in the Maxwellian limit (Eq. (21)) have no Planck energy, an energy that is by Planck's law proportional to frequency through h.

We arrive at the important conclusion that neither $i\boldsymbol{E}^{(3)}$ nor $\boldsymbol{B}^{(3)}$ can contribute to electromagnetic energy density in the classical Maxwellian field *despite the fact that each is non-zero in vacuo.* This is a key result, and goes a long way towards explaining why the influence of $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ has been undetected through measurements of light intensity.

In the Proca field on the other hand $\boldsymbol{B}^{(3)}$ has a small amount of Planck energy (m_0c^2) through the de Broglie guiding theorem [32],

$$h\nu_0 = m_0c^2, \tag{26}$$

the fundamental theorem of wave mechanics. The photon rest mass which appears in ξ of Eq. (25) is also both a frequency, ν_0, and an energy $h\nu_0$. Since $\xi = 0$ identically in the Maxwellian field, this energy disappears when that field is quantized.

The question of why longitudinal photons from the Proca equation do not contribute to the Planck radiation law has been addressed by Bass and Schrödinger [35], using a statistical argument which shows that the approach to equilibrium of longitudinal photons in a cavity is very slow in comparison with that of transverse photons, comparable with the age of the universe. For this reason [6], longitudinal photons have a negligible effect on the statistical thermodynamics, and thus on the Planck radiation law. Thus, the effect of longitudinal photons on radiation energy [6] is negligible, a conclusion which is consistent with our demonstration given already. It is critically important to note, however, that this conclusion does not extend to the magnetic and electric effects of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ on matter, discussed in detail elsewhere [24–30, 36]. Both are expected to produce, in the laboratory, specific spectroscopic effects such as a Zeeman shift due to $\boldsymbol{B}^{(3)}$ [25] in atomic spectra. These effects, if observed, can be interpreted as being consistent with finite photon mass. While both $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are well defined in the Maxwellian field, where m_0 can be regarded as being identically zero, there has accumulated [32] plentiful experimental evidence from several independent sources that is consistent with $m_0 \neq 0$. For further consistency, therefore, $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ should be interpreted within the context of the Proca field, where $m_0 \neq 0$.

On the question of absorption or emission amplitudes for longitudinal photons of frequency ν, Goldhaber and Nieto [6] show that these are suppressed in comparison with their transverse counterparts by a factor $m_0c^2/(h\nu)$. The corresponding rates and cross sections are suppressed by the square of this factor [6]. Thus, the quantum mechanical matrix element for ordinary transverse photons is given [6] by

$$T_f^{(X,Y)} = \langle f|\hat{J}_{X,Y}|i\rangle, \tag{27}$$

for a photon induced transition to an arbitrary state f, where i is the initial target state. The corresponding matrix element for a longitudinal photon is

$$T_f^{(Z)} = \left(\frac{m_0 c^2}{h\nu}\right) \langle f|J_Z|i\rangle , \tag{28}$$

where ν is its frequency. For $m_0 \sim 10^{-52}\,kg$ (as given by Goldhaber and Nieto [6]), $c \sim 3\times10^8\,m\,s^{-1}$; $h \sim 10^{-34}\,J\,s$; $\nu \sim 10^{14}\,s^{-1}$, and for comparable transverse and longitudinal matrix elements it is seen that

$$\frac{T_f^{(Z)}}{T_f^{(X,Y)}} \sim 10^{-15} . \tag{29}$$

This result shows why spectral absorption and emission of longitudinal photons of spin zero and frequency ν are never observed in the usual infra red, visible, and ultra violet regions of the electromagnetic spectrum.

However, as $\nu \to 0$, (i.e. as the frequency of the longitudinal photon goes to zero),

$$\frac{m_0 c^2}{h\nu} \to \frac{m_0 c^2}{h\nu_0} = 1 , \tag{30}$$

from the de Broglie guiding theorem, Eq.(26). This means that at frequencies comparable with

$$\nu_0 = \frac{m_0 c^2}{h} \sim 10^{-7}\,Hz , \tag{31}$$

(for $m_0 \sim 10^{-52}\,kg$)

$$T_f^{(Z)} = T_f^{(X,Y)} , \tag{32}$$

in Eq. (29). This result is consistent with Eq. (7a) of the vectorial Lie algebra developed in Sec. 4, because Eq. (7a) shows that $\boldsymbol{B}^{(3)}$ is independent of frequency in the Maxwellian field. Equation (25) shows clearly that in the Proca field, $\boldsymbol{B}^{(3)}$ is associated with a frequency of about 10^{-7} hertz, as in Eq. (31).

We conclude that significant spectral absorption and emission of a longitudinal photon piloted by $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ takes place at $\sim 10^{-7}$ hertz in the Proca field and at zero frequency in the Maxwellian field. This is another way of saying that the frequency of $\boldsymbol{B}^{(3)}$ from Eq. (7a) is zero, and that from Eq. (25) is 10^{-7} hertz for m_0 of about $10^{-52}\,kg$. This explains why the well known electric dipole transition selection rule appears in the infra red, visible, and ultra violet always to be the familiar [37]

$$\Delta l = \pm 1 , \tag{33}$$

and not

$$\Delta l = 0, \pm 1 , \tag{34}$$

because the transition $\Delta l = 0$ accompanying the absorption of a longitudinal photon is possible only at very low frequency (Proca field) or zero frequency (Maxwell field). (The transition $\Delta l = 0$ happens also to be forbidden by the Laporte selection rule.)

Again it is emphasized that this result does not mean that $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are intrinsically unobservable by spectroscopy, because, as we have detailed elsewhere [36], both produce spectral effects normally attributed to magnetostatic and electrostatic fields, such as the Zeeman effect due to the $\boldsymbol{B}^{(3)}$ of a circularly polarized laser pulse [25, 36]. These effects do not depend on absorption, and do not time average to zero because $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are independent of the phase of the laser. Indeed, well known experimental effects such as magnetization by a circularly polarized laser pulse, the inverse Faraday effect [38], can be expressed in terms of $\boldsymbol{B}^{(3)}$ at first and higher orders [26]. The inverse Faraday effect (Sec. 4) is consistent therefore with finite photon mass, although it was originally interpreted in terms of the product $\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)}$, known in nonlinear optics as the conjugate product [39]. From Eq. (7a), it is easily verified that the conjugate product in vacuo is directly proportional to $\boldsymbol{B}^{(3)}$. The existence of the conjugate product implies that of $\boldsymbol{B}^{(3)}$, and therefore the experimentally observed [38] inverse Faraday effect is experimental evidence for $\boldsymbol{B}^{(3)}$, and by implication, for finite photon mass as we have argued.

Finally in this introductory survey we address the effect of $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ on the Poynting theorem of electrodynamics, the law of conservation of electromagnetic energy density. Further details, following a recent paper by Farahi and Evans [27] are given in Sec. 7. In classical electrodynamics [27] the law of conservation of energy is expressed customarily through the continuity equation,

$$\nabla\cdot\boldsymbol{N} = -\frac{\partial U}{\partial t} , \tag{35}$$

where $\boldsymbol{N}$ is known as Poynting's vector,

$$\boldsymbol{N} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} . \tag{36}$$

The vector $\boldsymbol{N}$ is therefore longitudinal, and is interpreted as the flux of electromagnetic energy of a plane wave in vacuo, i.e. the electromagnetic power per unit area. This notion is meaningful by tautology only when the beam interacts with matter, otherwise there is nothing that can be observed. We have argued that $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ do not contribute to U, and their contribution, if any, to $\boldsymbol{N}$ will therefore be governed by

$$U_L = 0, \quad (37)$$

where U_L is the contribution of these fields to electromagnetic energy density. From Eq. (35), therefore,

$$\nabla \cdot \boldsymbol{N}_L = 0, \quad (38)$$

showing that any non-zero contribution to $\boldsymbol{N}$ from $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ must be divergentless. This result is akin to the Gauss theorem in differential form. Furthermore, terms such as $\boldsymbol{E}^{(3)} \times \boldsymbol{B}^{(3)}$ cannot contribute to $\boldsymbol{N}_L$ because $i\boldsymbol{E}^{(3)}$ is parallel to $\boldsymbol{B}^{(3)}$. Terms such as $\boldsymbol{E}^{(1)} \times \boldsymbol{B}^{(3)}$ and so on could provide a contribution in principle to $\boldsymbol{N}_L$.

A term such as

$$\boldsymbol{E}^{(2)} \times \boldsymbol{B}^{(3)} \; -? \; iB^{(0)}\boldsymbol{E}^{(2)}, \quad (39)$$

time averages to zero, and cannot contribute to the electromagnetic momentum density $\boldsymbol{N}$. It is shown later that Eq. (39) also violates $\hat{T}$ symmetry, and is not a valid law of electrodynamics for this reason.

It is concluded therefore that neither $i\boldsymbol{E}^{(3)}$ nor $\boldsymbol{B}^{(3)}$ contribute to the Poynting vector $\boldsymbol{N}$, and it becomes clear that the Lie algebra (7) and (39) represents a limiting procedure [40] which does not affect $\boldsymbol{N}$ as $m_0 \rightarrow 0$. This means that only transverse field components affect $\boldsymbol{N}$, *which is the usual conclusion of classical electrodynamics*. Again, the fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ elude direct observation through that of the flux of electromagnetic energy, Poynting's vector $\boldsymbol{N}$.

To conclude our brief introductory survey of the effects of finite photon mass we put these results in the context of the Einstein-de Broglie interpretation of wave-particle dualism, the cornerstone of wave mechanics, and thus of much of twentieth century natural philosophy. Our remarks lean heavily on a recent article by Vigier [32].

The school of thought of Einstein and de Broglie in this context interpret wave particle dualism to mean that light is constituted by real waves, be these Maxwellian or from the Proca field, which *physically co-exist* with photons in Minkowski space-time. This is a causal, realist, and stochastic interpretation of wave particle dualism which has recently received strong, if not definitive, experimental corroboration in the experiment [32] of Mizobuchi and Ohtake on single photons. This experiment, and other related experiments [32] have now succeeded in demonstrating that a single photon coexists with a physically

meaningful wave (the electromagnetic field). In the Copenhagen interpretation of dualism, proposed by Bohr and others, this is not possible [32]. Einstein concluded that in his view, light energy ($p_0 = \hbar\nu_0$) travels in particle form, the particle being identified with the photon. The particle is dual with an electromagnetic wave, to which there is a co-existent physical reality. The school of Einstein and de Broglie therefore interpret light as being constituted by spin one photons (bosons) which are controlled, or piloted, by physically meaningful fields. The wave drives the particle through the quantum potential [32]. Photons are the only elements of light that are *directly* observable, for example by transfer of momentum in the Compton effect or photoelectric effect. Fields are indirectly observable through interaction Hamiltonians. The photons as particles carry energy-momentum and angular momentum (spin). They behave in Minkowski space-time as relativistic particles with finite mass, and are therefore governed by the Proca equation. Electromagnetic energy is not carried by the Maxwellian field in Einstein's interpretation but by the photons which are piloted by these fields. Each photon carries an energy $h\nu$, a power $h\nu^2$, and acts in time ν^{-1}. Therefore, if ν is zero or very small, as for a longitudinal photon as we have seen, its energy and power vanish and it never acts. The longitudinal fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ in this interpretation do not therefore contribute to $h\nu$, nor do they contribute to the power of each individual photon. This is consistent with the fact that $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ do not contribute to the classical electromagnetic energy density U, which is the power per unit volume in classical form. The internal motion [32] of each photon is governed by the de Broglie guiding theorem, and photons oscillate in phase with the surrounding and oscillating electromagnetic field. Photons conserve energy-momentum and angular momentum when they interact with matter, for example with electrons in the Compton effect. The quantum potential is only indirectly observable through interference phenomena, as in the Bohm-Aharonov effect, which reveals the physical reality of the wave A_μ, the eigenfunction of the Proca equation. In this context the eigenfunction A_μ is usually written [32] as the complex wavefunction ψ_μ in Minkowski space-time, and the Proca equation becomes

$$\Box \psi_\mu = 2\xi^2 \psi_\mu . \quad (40)$$

In the limit $m_0 \rightarrow 0$ the field or wave component of light obeys Maxwell's equations in the classical limit, equations which give the novel Lie algebra (7), an algebra which shows ineluctably that the longitudinal field $\boldsymbol{B}^{(3)}$ has a physical reality. The fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ interact with matter, as in the experimentally observed [38] inverse Faraday effect, to produce measurable effects, even though the longitudinal photon is not absorbed, i.e. does not act because $\nu = 0$. This is of course an example of the fact that the wave and particle components in the dualism of de Broglie, the cornerstone of wave mechanics, each have a physical reality. In some cases the photon as particle is seen to act, i.e. to produce experimental effects as in Compton scattering, in other cases the field is observed to act, as in the inverse Faraday effect, even though the longitudinal photon has no Planck energy and no power, and cannot be absorbed. Wave and particle are present simultaneously, and both have physical reality.

2. Summary [32] of Experimental Evidence for Finite Photon Mass

The various types of experimental evidence for this conclusion has been reviewed by Vigier [32] whose summary is repeated here.

Hall *et al.* [41] have recently observed a direction dependent anisotropy of light in the direction of the apex of the 2.7 K background of microwave radiation in the universe. These data are consistent [32] with non-zero photon mass. Experiments on the existence of superluminal action at a distance [32] have been performed and are being repeated with increasing accuracy with the overall intention of investigating the centrally important idea of non-locality in the quantum potential. Other types of experiments are designed to investigate directly the Heisenberg uncertainty principle for single photons, this being central to the interpretation of wave particle dualism. In atomic self-interference, optical and neutron experiments, tests are being devised for the existence of particle trajectories (Einweg-Welcherweg). Other experiments are being designed to test the Copenhagen interpretation of dualism, put forward by Bohr and others, and in which light is constituted either by waves of *probability*, or by particles. The probability waves never coexist with the particles in space-time. The physical coexistence of wave and particle is therefore a central point of interpretation. It is possible in that of Einstein and de Broglie, impossible in the Copenhagen interpretation. There are no real waves in the latter school of thought, and experimental evidence for real waves would be in favor of the Einstein-de Broglie school even though the influence of the concomitant particle may not be observable. This, as we have argued, is the case for the $\boldsymbol{B}^{(3)}$ field, which produces observable effects [38] such as magnetization of matter, without the concomitant longitudinal photon, which, being associated with a zero frequency, has no Planck energy $\hbar\nu$, and no power, $\hbar\nu^2$. This type of evidence for the Einstein-de Broglie interpretation of dualism is also provided by an experiment [32] such as that of Bartlett and Corle [42] which measures the Maxwell displacement current in vacuo and without electrons.

Experimental tests for the existence of the quantum potential, which is responsible for the idea of photons being piloted by waves (electromagnetic fields) have been devised [32] using coherent intersecting laser beams. Experimental evidence from such sources, and form laser-induced fringe patterns [32] is available, and shows an observable enhancement of photon energy due to the quantum potential. In this context the optical equivalent of the Bohm-Aharonov effect, in which the tiny solenoid (iron whisker) of the conventional experiment is replaced by a narrow, circularly polarized, laser beam, would be a critical test of the existence of the vector potential associated with $\boldsymbol{B}^{(3)}$ [43]. Experiments such as that of de Martini *et al.*, discussed by Vigier [32], show that it is possible to pass continuously from Bose Einstein to Maxwell Boltzmann (or Poisson) statistics in an ensemble of photons. The passage from one type of statistics to the other can be explained [32] in terms of non-

locality in the quantum potential, which results in non-local action at a distance, currently a central question in quantum mechanics.

In astrophysics the consequence of non-zero photon mass are many and varied, and have been considered repeatedly throughout the twentieth century [32]. Foremost among these is that the Proca equation, as we have argued, produces longitudinal photons which do not affect the validity of the Planck radiation law. The present author has now shown that the longitudinal field $\boldsymbol{B}^{(3)}$ is present in both the Proca and Maxwellian formalisms, and is for all practical purposes identical in both. The longitudinal photons are therefore piloted by the longitudinal fields $i\boldsymbol{B}^{(3)}$ and $\boldsymbol{B}^{(3)}$ in the Einstein-de Broglie interpretation of wave particle dualism. Furthermore, the field $\boldsymbol{B}^{(3)}$ produces observable effects in matter, such as magnetization [38] due to $\boldsymbol{B}^{(3)}$ at first and higher orders in the inverse Faraday effect. The existence of these fields is furthermore consistent with non-zero photon mass, as we have argued here. In the Proca formalism the field $\boldsymbol{B}^{(3)}$ decreases exponentially, and over large enough distances, Z, on a cosmic scale (e.g. light from far distant galaxies) the decrease in $\boldsymbol{B}^{(3)}$ might become observable. This is of course a direct measurement of the photon mass through the parameter ξ.

The photon flux from the Proca equation also decreases exponentially, and the Coulomb potential is replaced by a Yukawa potential [32]. This phenomenon solves the Olbers paradox [32] and results in low velocity photons, i.e. those that travel at considerably less than the speed of light. The residual mass of these photons contribute to the mass of the universe, and may solve the missing mass problem of cosmology [32]. The factor such as that which appears in $\boldsymbol{B}^{(3)}$ from the Proca equation, implies a distance proportional red shift,

$$\frac{\Delta\nu}{\nu} \sim exp(-\xi Z). \tag{41}$$

This is the well known "tired light" of Hubble and Tolman [32]. This type of red shift could contribute significantly to the cosmological red shift, and explain numerous reports by astronomers of "anomalous" red shifts in objects such as quasars bound to galaxies by matter bridges. Photon mass is also consistent [32] with anomalies in: Q.S.O. sky distribution, double star motions, red shift discrepancies in galaxy clusters, anomalous variations in the Hubble constant, and quantized peaks in the N-log z plot [32].

Since all these phenomena are evidently detected through telescopes, i.e. by the use of electromagnetic radiation, the novel fields $i\boldsymbol{B}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are present in all of them in one form or another, as well as in laboratory scale optical experiments of many different kinds. The fields $i\boldsymbol{B}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are manifestations of finite photon mass, as we have argued, and are therefore fundamental in nature.

3. Gauge Invariance Under Local U(1) Transformation

Contemporary understanding of field theory implies invariance under local (U(1)) gauge transformation, this being a fundamental requirement of the Lagrangian formalism upon which the theory is based. Contemporary theory also requires that the Proca equation for finite photon mass be consistent with special relativity and that electromagnetic field theory be part of grand unified theory. The question of finite photon mass therefore becomes part of the general theoretical understanding of mass itself in elementary particles and fields, a central contemporary theme. A description of grand unified field theory such as SU(5) [44, 45] depends fundamentally on the ideas of local gauge invariance and gauge symmetry breaking, and it is therefore important to show that finite photon mass can be accommodated within this overall theoretical framework. The latter has been solidly effective in unifying the electromagnetic and weak fields, predicting new boson masses which have been verified experimentally. A consideration of the electromagnetic field in isolation of the weak and strong fields is therefore no longer complete unless a consistent understanding is attained within the context of unified and grand unified fields.

Following the development by Ryder [15] the complex scalar field (ϕ) in electromagnetism is described by the Euler Lagrange equation of motion,

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x_\mu}\right)}\right), \tag{42}$$

where $\mathcal{L}$ is the Lagrangian and x_μ the four-vector,

$$x_\mu \equiv (X,\ Y,\ Z,\ ict), \tag{43}$$

in Minkowski space-time. If the complex scalar field has two real components ϕ_1 and ϕ_2 then

$$\phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}, \qquad \phi^* = \frac{(\phi_1 - i\phi_2)}{\sqrt{2}}, \tag{44}$$

and the Lagrangian required to produce a real action [15] must be

$$\mathcal{L} = \frac{\partial \phi}{\partial x_\mu}\frac{\partial \phi^*}{\partial x_\mu} - m^2c^4\phi\phi^*, \tag{45}$$

where, in S.I. units, m is a mass associated with the fields (44). (This should not be confused with photon rest mass, m_0.) It is seen that the mass enters the Lagrangian $\mathcal{L}$ by premultiplying a quadratic term $\phi\phi^*$. This is a prominent feature of contemporary unified field theory [15], where boson mass, for example, is identified in this way. A gauge transformation can be understood geometrically as a rotation of the vector,

$$\boldsymbol{\phi} = \phi_1\mathbf{i} + \phi_2\mathbf{j}, \tag{46}$$

in the internal space (1,2) through an angle Λ. The action must not change as a consequence, because this would violate the fundamental principle of least action, which in classical mechanics states that motion is determined by minimizing action, S. The Lagrangian (45) must therefore be invariant to

$$\phi \to e^{-i\Lambda}\phi, \qquad \phi^* \to e^{i\Lambda}\phi^*, \tag{47}$$

which is equivalent to a rotation of the vector $\boldsymbol{\phi}$ through an angle Λ.

This is a simple example of gauge invariance, by which every contemporary theory of fields is guided. It is usual in the development of the theory [15] to show that *the gauge transformation of the first kind* defined by Eq. (47) results in

$$\frac{\partial J_\mu}{\partial x_\mu} = 0, \tag{48}$$

where J_μ is a four-vector identified as the generalized current associated with a complex scalar field and defined by

$$J_\mu \equiv i\left(\phi^*\frac{\partial \phi}{\partial x_\mu} - \phi\frac{\partial \phi^*}{\partial x_\mu}\right). \tag{49}$$

The generalized charge is defined as the integral

$$Q = \int J_0 dV = i\int\left(\phi^*\frac{\partial \phi}{\partial t} - \phi\frac{\partial \phi^*}{\partial t}\right)dV, \tag{50}$$

over a volume V in three dimensional space.

Invariance of a complex scalar field under gauge transformation of the first kind produces a conserved current, defined by Eqs. (48) and (49), and a conserved charge, defined by Eq. (50). This is an expression of Noether's theorem [15] which summarizes the conservation laws of physics under this type of essentially geometrical transformation in an internal space of the complex field. For example, from Noether's theorem it can be deduced that the laws of physics are invariant to the origins of space and time, a fundamental requirement for any analytical development or description. By application of the $\hat{C}$ operator (see introduction) to Eqs. (49) and (50),

$$\hat{C}(J_\mu) = J_\mu, \quad \hat{C}(Q) = Q, \tag{51}$$

and it becomes clear that neither J_μ nor Q can be identified with electric current or charge, which are both $\hat{C}$ negative. The quantities J_μ and Q arise from the way in which the complex scalar field has been defined, and because of the fundamental requirements of Noether's theorem. The application of $\hat{C}$ in Eq. (51) therefore emphasizes the difference between the complex scalar field and the electromagnetic field. We proceed to show that the latter *arises as a requirement of special relativity applied to the gauge transformation of the first kind.*

It follows therefore that all aspects of electromagnetic field theory must be consistent with special relativity. Furthermore, as we shall see, electromagnetism enters the theory of the complex scalar field through the potential four-vector A_μ scaled, or multiplied, by the unit of electric charge e (the charge on the electron). In consequence, the potential four-vector A_μ is *physically meaningful* in the contemporary gauge theory of electromagnetism, and is regarded from the outset as being fully, or manifestly, covariant. All four of its components, in other words, are regarded as physically meaningful. The habitual use of the Coulomb gauge is obviously inconsistent with this method, because the Coulomb gauge (see introduction) destroys the manifest covariance of the electromagnetic field by arbitrarily discarding the time-like and longitudinal space-like component of A_μ. The Coulomb gauge is known as the transverse gauge precisely because it has only two, transverse, polarizations.

The gauge transformation of the first kind (47) is inconsistent with special relativity because it implies instantaneous action at a distance [15]. The rotation angle Λ is the same at all points in space-time, i.e. all points along the vector $\boldsymbol{\phi}$ must rotate simultaneously in space-time. In special relativity c, the speed of light, is a universal constant and cannot be exceeded, so that there must be a minimum time delay between action at one point in space-time and another. Electromagnetism clearly has something to do with this time delay because the latter is due to the speed of light in vacuo. Electromagnetism introduces "curvature" into the vector $\boldsymbol{\phi}$. (In general relativity, mass itself becomes curvature of space-time, so photon mass must be similarly interpretable.) Special relativity therefore requires that the internal space angle Λ become a function of the four-vector x_μ, and so, it is written as $\Lambda(x_\mu)$. Realizing this leads to the central concept of contemporary field theory, which is *local gauge transformation [15], also known as gauge transformation of the second kind.* We are led to conclude that photon mass must be consistent with local gauge transformation requirements of the U(1) sector of grand unified field theory. If this were not the case, then m_0 would be inconsistent with special relativity and grand unified field theory.

In the following development we derive a condition (Eq. (2) of the introduction) which retains the internal consistency of the theory for finite m_0.

This is a departure from the usual development [15], which asserts that $m_0 = 0$ identically, and thereby retains gauge freedom, as discussed in the introduction. Clearly, the ad hoc choice $m_0 = 0$ is fundamentally at odds with the Proca equation and with collected *experimental* evidence [32] consistent with $m_0 \neq 0$.

For $\Lambda < 0$,

$$\phi \to \phi - i\Lambda\phi, \tag{52}$$

but if Λ is a function of x_μ, however, as required by special relativity, then action can no longer be invariant under the gauge transformation (52). We have

$$\delta\mathcal{L} = J_\mu \frac{\partial\Lambda}{\partial x_\mu} \neq 0, \tag{53}$$

if the original Lagrangian is defined by Eq. (45).

The electromagnetic field is introduced to *force* $\delta\mathcal{L} = 0$ in Eq. (53) through the intermediacy of the extra Lagrangian term,

$$\mathcal{L}_1 \equiv -eJ_\mu A_\mu, \tag{54}$$

such that eA_μ has the same units as $\partial/\partial x_\mu$, the four-derivative of space-time. Applying the $\hat{C}$ operator to $\mathcal{L}_1$ in Eq. (54),

$$\hat{C}(\mathcal{L}_1) = \mathcal{L}_1, \tag{55}$$

revealing that eA_μ must be positive to $\hat{C}$, where e is a scalar and A_μ a four-vector. At this stage of the development it is *asserted* that under gauge transformation of the second kind [15],

$$A_\mu \to A_\mu + \frac{1}{e}\frac{\partial\Lambda}{\partial x_\mu}. \tag{56}$$

The introduction of $\mathcal{L}_1$ implies the further need to introduce [15]

$$\mathcal{L}_2 = e^2 A_\mu A_\mu \phi^* \phi, \tag{57}$$

in order to force the condition

$$\delta\mathcal{L} + \delta\mathcal{L}_1 + \delta\mathcal{L}_2 = 0, \tag{58}$$

so that the total Lagrangian is invariant under local gauge transformation. The latter has therefore produced the result

$$\mathcal{L}_{new} = \mathcal{L}_{old} + eJ_\mu A_\mu - e^2 A_\mu A_\mu \phi^* \phi \equiv \mathcal{L}_{old} + \mathcal{L}_I , \quad (59)$$

where $\mathcal{L}_1$ is due to interaction with the four-vector A_μ through the scaling constant e. The $\mathcal{L}_{new}$ term can be written

$$\mathcal{L}_{new} = -(D_\mu \phi)(D_\mu \phi)^* + m^2 c^4 \phi\phi^* , \quad (60)$$

where

$$D_\mu \equiv \frac{\partial}{\partial x_\mu} + ieA_\mu , \quad (61)$$

is the *covariant derivative*, i.e. a derivative which transforms covariantly under local gauge transformation, i.e. D_μ transforms in the same way as ϕ.

The *minimal coupling* interaction term $\mathcal{L}_I$ is therefore obtained by replacing the ordinary derivative $\partial/\partial x_\mu$ by D_μ, and *local gauge symmetry* dictates the field dynamics through $\mathcal{L}_I$ minimal coupling. For this reason A_μ is a *gauge field*, and produces its own Lagrangian, $\mathcal{L}_3$, which must also be invariant under local gauge transformation, i.e. invariant under Eq. (56). We now see that the latter is a consequence of the covariant derivative D_μ.

The four-curl of A_μ, defined by

$$F_{\mu\nu} \equiv \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} , \quad (62)$$

is invariant to (56), and provides the Lagrangian,

$$\mathcal{L}_3 = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} . \quad (63)$$

The standard theory [15] identifies $F_{\mu\nu}$ with the electromagnetic four-tensor; e with the charge on the electron; and A_μ with the electromagnetic potential four-vector. From this, the inhomogenous Maxwell equations correspond to varying A_μ in the appropriate Euler Lagrange equation to produce

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = -eI_\mu \equiv j_\mu , \quad (64)$$

where

$$I_\mu = i(\phi^* D_\mu \phi - \phi D_\mu \phi^*) , \quad (65)$$

and j_μ is identified as the usual current four-vector of electromagnetism. It is clear from this development that $\hat{C}$ can act only on a scalar quantity, the scaling constant e, the elementary electronic charge. Thus, as in the introduction, we obtain

$$\hat{C}(B^{(0)}) = -B^{(0)} \quad (66)$$

for the *scalar* amplitude of a magnetic flux density. *All space-time properties are invariant to* $\hat{C}$ *by definition of this operator [14]*, and so all types of unit vector are invariant to $\hat{C}$. It follows that A_μ is a four-vector whose absolute scalar magnitude is $\hat{C}$ negative.

We now address the role of finite electromagnetic field mass in local gauge transformation. This is necessarily within a classical framework, we have not yet addressed the effects of field quantization.

The conventional development [15], having arrived at Eqs. (54) and (58), asserts that the vector A_μ cannot be associated with electromagnetic field mass, m_0, because the latter would necessarily be identified through a Lagrangian of the type,

$$\mathcal{L}_m = \frac{1}{2} m_0^2 c^4 A_\mu A_\mu , \quad (67)$$

in S.I. units, and not in reduced units [15], where c is habitually set to unity. Since $\mathcal{L}_m$ is not invariant under Eq. (56), it is asserted that $m_0 = 0$ identically, so that the electromagnetic field has "no mass". As surveyed in the introduction, this is inconsistent with collected experimental data [32], and as is actually well known [15], leads to considerable physical *obscurity* in electromagnetic field quantization. The Coulomb gauge is one of these obscurities, and attempts at quantization [15] in the Coulomb gauge are meaningless. Nevertheless, it is habitually accepted in contemporary electrodynamics, although standard tables [6] no longer list the photon mass as identically zero. The literature is therefore frequently self contradictory.

It is clear that for $m_0 \neq 0$, the Lagrangian $\mathcal{L}_m$ vanishes if

$$A_\mu A_\mu = 0 , \quad (68)$$

a condition which is also consistent with

$$\delta \mathcal{L}_2 = 0 , \quad (69)$$

because

$$\mathcal{L}_2 = e^2 A_\mu A_\mu \phi^* \phi = 0 , \quad (70)$$

and

$$\delta\mathfrak{L} + \delta\mathfrak{L}_1 = -\delta\mathfrak{L}_2 = 0, \tag{71}$$

automatically. Therefore for $A_\mu A_\mu = 0$ finite photon mass is consistent with local gauge invariance, and the latter does not mean that photon mass is identically zero. The condition $A_\mu A_\mu = 0$ is also consistent with the Lorentz condition $\partial A_\mu / \partial x_\mu = 0$, which is implied automatically by the Proca equation [15, 32]. As described in the introduction, $A_\mu A_\mu = 0$ is inconsistent with the Coulomb gauge. In view of the available experimental evidence [32] for finite m_0, we abandon the Coulomb gauge and quantization procedures [15] based thereon, and adopt $A_\mu A_\mu = 0$ instead of $m_0 = 0$ in $\mathfrak{L}_m$ and $\mathfrak{L}_2$. This implies that A_μ must always be fully covariant and physically meaningful in special relativity, where $A_\mu A_\mu$ and $\partial A_\mu / \partial x_\mu$ are invariant to Lorentz transformation in Minkowski space-time. Furthermore, if we write the Proca equation as [15]

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = -\xi^2 A_\nu, \tag{72}$$

and multiply both sides by A_ν,

$$A_\nu \frac{\partial F_{\mu\nu}}{\partial x_\mu} = -\xi^2 A_\nu A_\nu = 0, \tag{73}$$

it becomes clear that if $A_\mu A_\mu = 0$, the Proca equation reduces to the Maxwell equation [15],

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0, \quad A_\nu \neq 0, \tag{74}$$

even though the mass m_0 is not zero.

We conclude that the Maxwell equation in vacuo, Eq. (74), is a "subsidiary" of the Proca equation described by the requirement $A_\mu A_\mu = 0$ for m_0 in general $\neq 0$.

This result, although mathematically self-consistent, is not physically acceptable however, because we know that finite m_0 is the very factor that distinguishes the Maxwell equation from the Proca equation. This is obvious from writing

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0, \qquad (Maxwell),$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = -\xi^2 A_\nu, \quad A_\nu \neq 0, \qquad (Proca), \tag{75}$$

whereby the Maxwell equation is seen in this light to be the limiting form of the Proca equation for $m_0 = 0$ identically. Therefore $A_\mu A_\mu = 0$ *identically* implies $m_0 = 0$ identically, even though m_0 is not *specifically* set to zero in Eq. (67). We must therefore accept that $A_\mu A_\mu$ is not identically zero if m_0 is to remain finite, in line with collected experimental evidence [32]. We must therefore look for a way in which this conclusion can be made to be consistent with local gauge invariance of U(1). This is possible through the use of, for example, the Dirac gauge [32],

$$r_\mu r_\mu = k^2 = constant, \tag{76}$$

where r_μ is the photon radius four-vector defined through the complex wave four-vector ψ_μ [32],

$$\psi_\mu \equiv r_\mu \exp\left(\frac{(P + iS)}{\hbar}\right). \tag{77}$$

The Proca equation in the form (12) can be derived from the Lagrangian [32],

$$\mathfrak{L}_5 \equiv -\frac{1}{4} F^*_{\mu\nu} F_{\mu\nu} - \xi^2 \psi^*_\mu \psi_\mu + \lambda(r_\mu r_\mu + k^2), \tag{78}$$

with

$$F_{\mu\nu} = \frac{\partial \psi_\nu}{\partial x_\mu} - \frac{\partial \psi_\mu}{\partial x_\nu}, \tag{79}$$

which yields, by varying ψ^*_μ,

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 2\xi^2 \psi_\mu. \tag{80}$$

By contraction

$$\frac{\partial \psi_\mu}{\partial x_\mu} = \frac{\partial \psi_\mu^*}{\partial x_\mu} = 0, \tag{81}$$

so that

$$\Box \psi_\mu = 2\xi^2 \psi_\mu . \tag{82}$$

Converting this notation into that of the present paper, and assuming that A_μ is in general complex, the Dirac gauge corresponds to

$$A_\mu A_\mu^* = constant, \tag{83}$$

and the Lagrangian $\mathcal{L}_5$ is consistent both with finite photon mass and local gauge invariance of U(1). The condition $A_\mu A_\mu^* = 0$ is the limit of Eq. (83) for $constant = 0$.

What does this mean physically? The magnitude of the photon radius is about $10^{-22} m$ [45], and it follows that

$$r_\mu r_\mu^* \sim 10^{-44} m^2 \sim 0, \tag{84}$$

is an excellent approximation. Therefore the condition $A_\mu A_\mu = 0$ for real A_μ is equivalent to vanishingly small photon radius, a condition which for all practical purposes is the Dirac gauge [32]. Finite photon mass is therefore consistent with local gauge invariance in the Dirac gauge, a limiting form of which is $A_\mu A_\mu \to 0$.

4. Lie Algebra of the Electromagnetic Field in Vacuo

In conventional, classical electrodynamics it is customary to consider only the transverse, oscillating, solutions of Maxwell's equations [46]. For the magnetic flux density in vacuo these are,

$$\boldsymbol{B}^{(1)} = \frac{B_0}{\sqrt{2}}(\,i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \tag{85a}$$

$$\boldsymbol{B}^{(2)} = \frac{B_0}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i\phi}, \tag{85b}$$

where $\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}$ is the phase of the travelling electromagnetic plane wave. In Eqs. (85a) and (85b), $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the X and Y axes of the laboratory frame (X, Y, Z), mutually orthogonal to the propagation axis, Z, of the wave. Here ω is the angular frequency in radians per second at an instant of time t, $\boldsymbol{\kappa}$ the wavevector in inverse meters at a position $\boldsymbol{r}$ in the laboratory frame. The complex conjugate of $\boldsymbol{B}^{(1)}$, i.e. $\boldsymbol{B}^{(2)}$, is also a physical solution of Maxwell's equations in vacuo.

Corresponding to Eqs. (85a) and (85b) there are oscillating, transverse, electric fields, usually written as

$$\boldsymbol{E}^{(1)} = \frac{E_0}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \tag{86a}$$

$$\boldsymbol{E}^{(2)} = \frac{E_0}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-i\phi}, \tag{86b}$$

in the same notation. In Eqs.(85a) and (85b), B_0 is a scalar magnetic flux density amplitude and E_0 a scalar amplitude of electric field strength. By direct substitution, it can be shown that Eqs. (85) and (86) obey Maxwell's equations in vacuo, and in the standard development of electrodynamics these transverse solutions are usually the only ones considered. Some texts mention briefly the possibility of longitudinal solutions, but the latter are not linked directly to the transverse waves. Two of the Maxwell equations in vacuo,

$$\nabla\cdot\boldsymbol{E} = 0, \tag{87a}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{87b}$$

would be violated by any longitudinal solution which depends on the phase ϕ, because a field of that kind cannot be divergentless. In other words these fields would not obey the Gauss theorem in differential form.

On the other hand, it is well known in the literature [15] that the Proca equation leads to three well defined space-like polarizations, and upon quantization, leads to a well defined particle with mass, identified with the photon [15]. On the other hand, attempts at quantization of the d'Alembert equation are beset with obscurity. For reasons described already, we reject quantization in the Coulomb gauge. In the Lorentz gauge, quantization of the d'Alembert equation can proceed only through the use of a gauge fixing term, and through the Gupta Bleuler condition [15]. Quantization of the Proca equation, being a natural wave equation, proceeds straightforwardly, producing two transverse and one *longitudinal* space-like polarizations for the photon in vacuo. The Proca and d'Alembert equations are identical, however, for all practical purposes [32], because the photon mass is very small, and we are therefore led to expect a well defined longitudinal space-like polarization in the *Maxwellian* field as well as the Proca field. Clearly, this polarization emerges classically through equations (7) of the introduction, i.e. through a Lie algebra recently proposed by the present author [24–30].

Furthermore, the existence of the longitudinal, classical Maxwellian fields $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ in vacuo must not violate the Gauss theorem or Planck radiation law (see introduction). Well known photon selection rules, experimentally well supported, must not be affected by $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ as discussed in the introduction.

The first indication [24] of the well defined nature of $\boldsymbol{B}^{(3)}$ in vacuo appeared through its relation to the well known conjugate product of nonlinear optics [47], the cross product $\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)}$. This is also referred to in nonlinear optics as the vectorial part of the light intensity tensor, I_{ij} [48], and therefore has a well known physical interpretation. The quantity I_{ij} is habitually used as the basis for calculations in electro- and magneto-optics [49, 50]. From Eqs. (86a) and (86b), the conjugate product in vacuo is easily seen to be a pure imaginary, longitudinal quantity

$$\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)} = iE_0^2\boldsymbol{k}, \tag{88}$$

which is independent by definition of the phase of the plane wave. Furthermore, the unit vector $\boldsymbol{k}$ on the right hand side of Eq. (88) must be axial by definition, because it is formed from a vector cross product of two polar vectors. For this reason, the conjugate product is positive to parity inversion, $\hat{P}$, and for this reason in turn cannot produce a static electric field. It changes sign with the sense of circular polarity, and vanishes for this reason in a linearly polarized electromagnetic field, because the latter is made up of 50% right and 50% left circularly polarized components. The conjugate product also vanishes in a standing wave, i.e. is non-zero only in a travelling wave which has a degree of circular polarization. It has been shown [24–30] that the $\hat{T}$ (motion

reversal) symmetry of the conjugate product is *negative*. Finally its $\hat{C}$ (charge conjugation) symmetry is positive.

Using the well known fundamental relation from the Maxwell equations in vacuo,

$$E_0 = cB_0, \tag{89}$$

the field $\boldsymbol{B}^{(3)}$ emerges simply [24],

$$iE_0^2\boldsymbol{k} = iE_0cB_0\boldsymbol{k} \equiv iE_0c\boldsymbol{B}^{(3)}. \tag{90}$$

Therefore in Eq.(88)

$$\boldsymbol{B}^{(3)} \equiv B_0\boldsymbol{k} = \frac{\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)}}{(iE_0c)}. \tag{91}$$

It is fundamentally important to note that the field $\boldsymbol{B}^{(3)}$ has all the known attributes of magnetic flux density (tesla). It is an axial vector, positive to $\hat{P}$, and negative to $\hat{T}$. It is also negative to $\hat{C}$ as required [51] because although the conjugate product is positive to $\hat{C}$, it is divided by the scalar iE_0c to produce $\boldsymbol{B}^{(3)}$ in Eq. (91). The $\hat{C}$ symmetry of E_0 (Sec. 3 and introduction) is negative by definition of $\hat{C}$ as an operator which leaves all space-time quantities unchanged but which reverses the sign of (scalar) electric charge. In particle physics $\hat{C}$ produces the anti-particle from the original particle as described in the introduction. In the classical Maxwellian field however, the particle-antiparticle interpretation is inappropriate because quantization has not occurred. $\hat{C}$ is therefore the operator that reverses the sign of the unit of electric charge e. This is not a quantized quantity, i.e. is well defined in classical physics.

A similarly simple analysis leads to the result

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \quad B^{(0)} \equiv B_0, \tag{92}$$

which is Eq. (7a). Note that Eqs. (91) and (92) are identical in vacuo, clearly, they both define $\boldsymbol{B}^{(3)}$, and both conserve the seven discrete symmetries [51] of nature: $\hat{C}$, $\hat{P}$, $\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{P}\hat{T}$ and $\hat{C}\hat{P}\hat{T}$. This is straightforwardly verified by operating on each symbol of Eqs. (91) or (92) by each of these operators in turn. In each case the original equation is recovered unchanged, and is therefore *invariant* to the symmetry operator thus applied. The equation is therefore said to *conserve* the symmetry described by the operator being applied (for example, $\hat{C}$, $\hat{P}$, $\hat{T}$ etc.) If this were not so, a symmetry would have been *violated*. It is clear, as discussed in the introduction, that $\boldsymbol{B}^{(3)}$ violates none of the seven known discrete symmetries. In order to come correctly to this conclusion, it is essential to realize that the scalar amplitudes E_0 and B_0 are *negative* to $\hat{C}$. The $\hat{C}$ symmetry of $\boldsymbol{B}^{(3)}$ is therefore negative, and opposite in sign to that of the conjugate product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$. We emphasize these points because of the recent incorrect assertion [31] that $\boldsymbol{B}^{(3)}$ violates $\hat{C}$ and $\hat{C}\hat{P}\hat{T}$.

Further simple calculations of this type, based on vector cross products, leads to the Lie algebra described in Eqs. (7), showing that there is a cyclical, or closed, structure which ties together the three magnetic fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$. *If any one of these fields be zero, then Eqs. (7) imply that the other two must vanish in vacuo.* The importance of this result, simple as it is, cannot be overemphasized, because it is habitually neglected in classical electrodynamics. The latter *cannot* therefore be a complete description of nature.

Equation (91), furthermore, rewrites the conjugate product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$, a well accepted, experimentally verified [52] concept, in terms of $\boldsymbol{B}^{(3)}$, *which is therefore physically meaningful in the classical Maxwellian field.* It is also of great significance that $\boldsymbol{B}^{(3)}$ (and $i\boldsymbol{E}^{(3)}$) is a natural consequence of the Proca field, implying that $\boldsymbol{B}^{(3)}$ is consistent with finite photon mass. Since the conjugate product has been used [53–55] to interpret the experimentally observed inverse Faraday effect [52], then the latter is *experimental* evidence for $\boldsymbol{B}^{(3)}$, and is also experimentally consistent with finite photon mass. Evidence for the latter can therefore be accumulated [56] in the laboratory as well as through astronomy and cosmology [32]. These points are emphasized by the equation

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = icE_0\boldsymbol{B}^{(3)}, \tag{93}$$

in which the conjugate product on the left hand side has been rewritten in terms of $\boldsymbol{B}^{(3)}$ premultiplied by a scalar iE_0c. Obviously, experimental evidence for the quantity on the left hand side of this equation is also evidence for the quantity on the right hand side. The latter is simply the non-zero scalar iE_0c multiplied by the longitudinal magnetic flux density $\boldsymbol{B}^{(3)}$. If the latter were "unphysical" (presumably zero) as habitually asserted in electrodynamics [57], then the inverse Faraday effect would not be observable, in direct contradiction with well defined and well accepted experimental data [52]. This is one example of many of the inherent contradictions of classical electrodynamics [57] if longitudinal solutions in vacuo are discarded. These data also show *experimentally* that $\boldsymbol{B}^{(3)}$ conserves the seven discrete symmetries of nature, because $\boldsymbol{B}^{(3)}$ has been observed experimentally, through the inverse Faraday effect, to magnetize material matter.

That $\boldsymbol{B}^{(3)}$ is a solution of the four Maxwell equations in vacuo is demonstrated as follows, bearing in mind that its concomitant electric field is the longitudinal $i\boldsymbol{E}^{(3)}$,

$$\nabla \cdot \boldsymbol{B}^{(3)} = 0, \quad \nabla \cdot \boldsymbol{E}^{(3)} = 0, \quad \nabla \times \boldsymbol{B}^{(3)} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} = 0, \quad \nabla \times \boldsymbol{E}^{(3)} = -\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = 0. \tag{94}$$

These results follow from the fact that neither $\boldsymbol{B}^{(3)}$ nor $i\boldsymbol{E}^{(3)}$ [26] depend on the phase ϕ of the electromagnetic plane wave in vacuo, and are both divergentless and time independent. The curls of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ both vanish for this reason. It is concluded that $\boldsymbol{B}^{(3)}$, but *not its concomitant* $i\boldsymbol{E}^{(3)}$, is physically meaningful in the classical Maxwellian field, which in this light is a limiting form of the Proca equation as m_0 approaches zero identically. Therefore $\boldsymbol{B}^{(3)}$ must produce observable effects when it interacts with matter, an example being the already well observed and accepted inverse Faraday effect [52], which is magnetization by a circularly polarized laser pulse. The usual [52–55] interpretation of the inverse Faraday effect in terms of the conjugate product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$ is, through Eq. (93), an interpretation in terms of $\boldsymbol{B}^{(3)}$. Consistently, therefore, $\boldsymbol{B}^{(3)}$, being a magnetic field, magnetizes material.

There are available in the literature other types of experimental indications for the existence of $\boldsymbol{B}^{(3)}$. For example the well known "light shifts" [58] of atomic spectroscopy. These are observable shifts in atomic absorption frequencies, usually in the visible range, due to intense light, and interpreted in terms of an "effective magnetic field" [58]. The latter is the conjugate product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$, which is simply $iE_0c\boldsymbol{B}^{(3)}$. It is critically important to note that $\boldsymbol{B}^{(3)}$ is a *physical* magnetic field (i.e. flux density in vacuo), and not an "effective" magnetic field. The terminology in this subject is therefore confusing, and it is not always clear whether the light being used to produce the observed shifts is circularly polarized, or whether it is being absorbed. (In contrast, the data obtained [52] in the inverse Faraday effect measurements were clearly obtained from a circularly polarized giant ruby laser pulse, light which was not being absorbed.) Despite these uncertainties in the light shift literature, it is safe to conclude that when there is any amount of circular polarization in the light beam or laser being used, they are being caused by

$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$, and therefore by the product of $\boldsymbol{B}^{(3)}$ with iE_0c. Light shifts are therefore evidence for the existence of $\boldsymbol{B}^{(3)}$, because if the latter were zero, then so would $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$ be zero.

Recently, more evidence for $\boldsymbol{B}^{(3)}$ has emerged using contemporary NMR spectroscopy, in which circularly polarized laser light has been observed experimentally [59] to produce small but useful and measurable shifts in NMR frequencies in liquids, following a theory by the present author [60]. The observed shifts of a few hertz are far too large to be explicable on the basis of perturbation theory applied [61] to shielding coefficients. Although there appear to be several contributing mechanisms, because the shift pattern is site specific, their magnitude is qualitative evidence for the influence of the magnetic field $\boldsymbol{B}^{(3)}$ at the resonating nucleus. It appears from the available data, however, that only a small fraction of $\boldsymbol{B}^{(3)}$ reaches the nucleus, because the observed shifts give no indication of a straightforward intensity dependence [59] and are too small to be compatible with the applied $\boldsymbol{B}^{(3)}$ in vacuo. This is in contrast with light shifts in atomic spectroscopy [58] and with the inverse Faraday effect [52], where a simple intensity dependence has been observed. The effect of $\boldsymbol{B}^{(3)}$ at the nuclear level is very different therefore from its effect at the electronic level. The reason for this is not yet understood. The technique of optical NMR (or light enhanced NMR spectroscopy) is potentially very useful, however, for structural analysis of molecules, because the observed shifts from $\boldsymbol{B}^{(3)}$ and other sources appear to be site specific. The present author has indicated [56] several other spectroscopic tests for $\boldsymbol{B}^{(3)}$, such as the optical Faraday effect, optical Zeeman effect, and so on which can be used to accumulate experimental evidence consistent with finite photon mass.

Circularly polarized light is also capable of producing an azimuth rotation in a linearly polarized probe (the optical Faraday effect [62]) and recently experimental data for this phenomenon have been obtained by Frey *et al.* [63] in a magnetic semiconductor known to give a giant Zeeman effect. It is interesting to see that a plot of the observed azimuth rotation versus the *square root* [64] of laser intensity falls on a straight line for six experimental points. This indicates that the observed azimuth rotation (in this case a self rotation of the pump laser) is proportional to $\boldsymbol{B}^{(3)}$. This result is not unequivocal, however, because the pump laser used by Frey *et al.* [63] was not circularly polarized, and the data plotted as a function of $\boldsymbol{B}^{(3)}$ did not go through the origin [64]. However, the fact that intensity dependent azimuth rotation was observed means that the initially linearly polarized laser acquired a degree of circular polarization while traversing a permanent magnetic field from a powerful laboratory magnet, and that $\boldsymbol{B}^{(3)}$ in the experiment of Frey *et al.* [63] was not zero as the laser passed through the magnet. These authors were not looking specifically for the effect of $\boldsymbol{B}^{(3)}$ however, but rather for a nonlinear optical Faraday effect using a conventional magnet and pump laser.

These examples show, in summary, that available data support $\boldsymbol{B}^{(3)}$, quantitatively or qualitatively, and are therefore consistent with finite photon mass. A much more systematic investigation of $\boldsymbol{B}^{(3)}$ is needed, however, because

this field is fundamentally important to classical and quantum electrodynamics. This brief survey is enough to show that the existence of $\boldsymbol{B}^{(3)}$ is supported experimentally, and that the habitual assertion of classical electrodynamics, $\boldsymbol{B}^{(3)} =? 0$, $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} \neq 0$, is inconsistent with experimental data and with the novel Lie algebra (7) to be developed in this section.

The existence of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ in vacuo is consistent, as we have argued, with finite photon mass, m_0 in the Proca equation. This is our central theme, and to support it directly we now derive $\boldsymbol{B}^{(3)}$ from the Proca equation in vacuo,

$$\Box A_\mu = -\xi^2 A_\mu, \tag{95}$$

in which ξ is non-zero only if m_0 is non-zero. The order of magnitude of ξ is so small, however, that Eq. (95) must closely approximate the d'Alembert equation in vacuo, although the two equations signify very different things. We therefore expect Eq. (95) to produce $\boldsymbol{B}^{(3)}$ for all practical purposes. In a non-relativistic approximation [15] we write the Proca equation in terms of a Laplacian instead of a d'Alembertian,

$$\nabla^2 \boldsymbol{A} = \xi^2 \boldsymbol{A}, \tag{96}$$

an approximation that is obtained in the galilean limit, which is recoverable [33] analytically by asserting that the term in $1/c$ in the d'Alembertian can be neglected. Using the well known

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{97}$$

it can be seen that the equation

$$\nabla^2 \boldsymbol{B} = \xi^2 \boldsymbol{B} \tag{98}$$

is the same as Eq. (96), because

$$\nabla^2 (\nabla \times \boldsymbol{A}) = \xi^2 \nabla \times \boldsymbol{A}, \tag{99}$$

i.e.,

$$\nabla \times \nabla^2 \boldsymbol{A} = \nabla \times \xi^2 \boldsymbol{A}, \tag{100}$$

meaning that the curl of the quantities on both sides is the same.

The solution of Eq. (98) is

$$\boldsymbol{B}^{(3)} = B^{(0)} e^{-\xi z} \boldsymbol{k}, \tag{101}$$

and since $\xi \sim 10^{-26} m^{-1}$ for a photon mass of $10^{-68} kg$ [32] this is identical for all practical purposes with the result (91) from the classical Maxwellian field. Since Eq.(101) is time independent, it is also a solution of the original Proca equation (95), without the need to approximate the d'Alembertian with the Lagrangian. The effect of photon mass is therefore to transform $\boldsymbol{B}^{(3)}$ of the classical Maxwellian field into (101) of the Proca field, i.e. to introduce an extremely slow, distance dependent, exponential decay into $\boldsymbol{B}^{(3)}$. As discussed in the introduction, this exponential decay is the same as that responsible for "tired light", in the terminology of Hubbard and Tolman [32], and thus for the distance dependent red shifts, $\Delta\nu/\nu$, observed by astronomers [32] on many occasions. The magnetic field $\boldsymbol{B}^{(3)}$ and electric field $i\boldsymbol{E}^{(3)}$ arriving from far distant radiation sources in the universe are therefore "tired fields", associated with a frequency change $\Delta\nu/\nu$. The very fact that they are associated with a frequency change implies the presence of finite photon rest mass. If m_0 were zero identically, then there would be no anomalous red shifts of this type.

4.1. Commutator Algebra in the Circular Basis: Quantization

It is shown in this section that the Lie algebra (7) can be expressed in terms of commutators of matrices, allowing a direct route to the quantization of the three Maxwellian fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$. This route to quantization of the electromagnetic field depends on the existence of the vectorial Lie algebra (7), and, assumes from the outset that $\boldsymbol{B}^{(3)}$ is non-zero and physically meaningful, and is for all practical purposes consistent with a finite field mass which is interpretable after quantization as photon mass. The quantization procedure depends on the existence of commutation relations between $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ which are identical, within a factor $\hbar$, with the well known commutators of angular momentum in quantum mechanics. These are in turn identical, within a factor $\hbar$, to the commutator relations among the generators of rotation in O(3) of three dimensional space [15]. This novel route to quantization avoids the obscurities of electromagnetic field quantization in the Lorentz gauge for $m_0 = 0$ [15].

It is convenient to develop this commutator Lie algebra in a circular, rather than Cartesian, basis, defined by

$$\boldsymbol{e}^{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}), \quad \boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j}), \quad \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)} = i\boldsymbol{k}, \tag{102}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are Cartesian unit vectors defined by

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \tag{103}$$

The unit vectors in the circular basis form the following cyclical Lie algebra,

$$\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*} = i\boldsymbol{e}^{(3)}, \\ \boldsymbol{e}^{(2)} \times \boldsymbol{e}^{(3)} &= i\boldsymbol{e}^{(1)*} = i\boldsymbol{e}^{(2)}, \\ \boldsymbol{e}^{(3)} \times \boldsymbol{e}^{(1)} &= i\boldsymbol{e}^{(2)*} = i\boldsymbol{e}^{(1)}, \end{aligned} \tag{104}$$

where * denotes "complex conjugate". Tautologically, if $\boldsymbol{e}^{(3)} = \boldsymbol{0}$, then $\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*} = \boldsymbol{0}$ and if $\boldsymbol{e}^{(3)} \neq \boldsymbol{0}$, then $\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*} \neq \boldsymbol{0}$. This structure is the same as that of Eq. (7), revealing that the latter, for the classical Maxwellian field in vacuo, is essentially *geometrical* in nature. The traditional assertion of electrodynamics: $\boldsymbol{B}^{(3)} =? 0$; $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = 0$, is therefore *geometrically incorrect in three dimensional space*.

To extend these calculations to quantum mechanics and space-time, it is essential to use commutator algebra [15]. Equations (7) can be put in commutative form by using the result from tensor analysis [65] that an axial vector is equivalent to a rank two antisymmetric polar tensor,

$$B_k = \frac{1}{2}\epsilon_{ijk}\hat{B}_{ij}, \tag{105}$$

where ϵ_{ijk} is the rank three, totally antisymmetric, unit tensor (the Levi Civita symbol). The rank two tensor representation, $\hat{B}_{ij}$, of the magnetic field axial vector B_k is mathematically equivalent, but has the key advantage of being accessible to commutator algebra. In other words, $\hat{B}_{ij}$ is a matrix, so that the vectorial Lie algebra (7) can be expressed entirely equivalently in *commutator form*. This in turn provides a direct means of expressing $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ in terms of O(3) rotation generators [15] and thereby in quantum mechanical form. In so doing, these magnetic fields are related directly to quantum mechanical angular momentum operators, and have the same commutator properties. This was originally shown by the present author [24] using an independent method. These methods show that the photon has an elementary longitudinal flux quantum, $\boldsymbol{B}^{(3)}$, which is an operator of quantum mechanics.

The classical magnetic fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ in vacuo are all axial vectors by definition, and it follows that the unit vector components of these vectors must also be axial in nature. In matrix form, they are, using tensor analysis of the type illustrated in Eq. (105),

$$\hat{i} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \hat{j} \equiv \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{k} \equiv \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{106}$$

It follows that the matrix representation of the unit vectors in the circular basis is,

$$\hat{e}_1 \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{bmatrix}, \quad \hat{e}_2 \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{bmatrix}, \quad \hat{e}_3 \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{107}$$

and that these form a commutator Lie algebra which is mathematically equivalent to the vectorial Lie algebra (104),

$$[\hat{e}_1, \hat{e}_2] = -i\hat{e}_3^* = -i\hat{e}_3, \quad [\hat{e}_2, \hat{e}_3] = -i\hat{e}_1^* = -i\hat{e}_2, \quad [\hat{e}_3, \hat{e}_1] = -i\hat{e}_2^* = -i\hat{e}_1. \tag{108}$$

These are our basic geometrical commutators in the circular basis convenient for the electromagnetic plane wave in vacuo. Equations (104) and (108) therefore represent a closed, cyclical algebra, in which all three space-like components are meaningful. If it is arbitrarily asserted that one of these components is zero, the geometrical structure is destroyed, and the algebra rendered meaningless.

Our geometrical basis (108) can now be used to build a matrix representation of the three space-like magnetic components of the electromagnetic plane wave in vacuo,

$$\hat{B}^{(1)} = iB^{(0)}\hat{e}^{(1)}e^{i\Phi}, \quad \hat{B}^{(2)} = -iB^{(0)}\hat{e}^{(2)}e^{i\Phi}, \quad \hat{B}^{(3)} = B^{(0)}\hat{e}^{(3)}, \tag{109}$$

from which emerges the commutative Lie algebra equivalent to the vectorial Lie algebra (7),

$$\begin{aligned} [\hat{B}^{(1)}, \hat{B}^{(2)}] &= -iB^{(0)}\hat{B}^{(3)*} = -iB^{(0)}\hat{B}^{(3)}, \\ [\hat{B}^{(2)}, \hat{B}^{(3)}] &= -iB^{(0)}\hat{B}^{(1)*} = -iB^{(0)}\hat{B}^{(2)}, \\ [\hat{B}^{(3)}, \hat{B}^{(1)}] &= -iB^{(0)}\hat{B}^{(2)*} = -iB^{(0)}\hat{B}^{(1)}. \end{aligned} \tag{110}$$

This algebra can be immediately expressed in terms of the well known [15] rotation generators of O(3) in three dimensional space. These rotation generators are complex matrices,

$$\begin{aligned} \hat{J}^{(1)} &= \frac{\hat{e}^{(1)}}{i} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{bmatrix}, \\ \hat{J}^{(2)} &= -\frac{\hat{e}^{(2)}}{i} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{bmatrix}, \\ \hat{J}^{(3)} &= \frac{\hat{e}^{(3)}}{i} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{111}$$

The magnetic field matrices and rotation generators are linked by

$$\hat{B}^{(1)} = -B^{(0)}\hat{J}^{(1)}e^{i\Phi}, \quad \hat{B}^{(2)} = -B^{(0)}\hat{J}^{(2)}e^{i\Phi}, \quad \hat{B}^{(3)} = iB^{(0)}\hat{J}^{(3)}. \tag{112}$$

This is a result which is of key importance in recognizing that the commutative algebra of the magnetic fields (110) *is part of the Lie algebra [15] of the Lorentz group of Minkowski space-time*. This suggests that the magnetic field is in general a property of space-time, i.e. that it has a time-like component, and can be expressed as a four-vector B_μ as discussed in the introduction, and developed further in Sec. 5. It is already clear from equations such as (112) that the longitudinal component in space, $\hat{B}^{(3)}$, must be physically meaningful, because it is directly proportional to the fundamental rotation generator $\hat{J}^{(3)}$. The latter is a geometrical property of space, and from Eq. (111) is non-zero.

Furthermore, the generalization of the rotation generators of O(3) from space to space-time is well known [15] to be,

$$\hat{\mathcal{J}}^{(1)} = \hat{\mathcal{J}}^{(2)*} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\mathcal{J}}^{(2)} = \hat{\mathcal{J}}^{(1)*} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (113)$$

$$\hat{\mathcal{J}}^{(3)} = -\hat{\mathcal{J}}^{(3)*} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

i.e., the rotation generators of O(3) become four by four matrices of the Lorentz group. It follows directly from Eq. (112) that the magnetic fields in the Lorentz group are also described by four by four matrices of type (113), i.e. *that the magnetic components of the electromagnetic plane wave in vacuo are well defined properties of space-time.*

Similarly, it is shown in Sec. 5 that the electric components are related to boost generators of the Lorentz group. These are also four by four matrices in Minkowski space-time.

For our present purposes, we note that the rotation generators in space form a commutator algebra of the following type in the circular basis,

$$\begin{aligned} [\hat{\mathcal{J}}^{(1)}\ \hat{\mathcal{J}}^{(2)}] &= -\hat{\mathcal{J}}^{(3)*} = \hat{\mathcal{J}}^{(3)}, \\ [\hat{\mathcal{J}}^{(2)}, \hat{\mathcal{J}}^{(3)}] &= -\hat{\mathcal{J}}^{(1)*} = -\hat{\mathcal{J}}^{(2)}, \\ [\hat{\mathcal{J}}^{(3)}, \hat{\mathcal{J}}^{(1)}] &= -\hat{\mathcal{J}}^{(2)*} = -\hat{\mathcal{J}}^{(1)}, \end{aligned} \qquad (114)$$

which becomes

$$[\hat{\mathcal{J}}_X, \hat{\mathcal{J}}_Y] = i\hat{\mathcal{J}}_Z, \qquad [\hat{\mathcal{J}}_Y, \hat{\mathcal{J}}_Z] = i\hat{\mathcal{J}}_X, \qquad [\hat{\mathcal{J}}_Z, \hat{\mathcal{J}}_X] = i\hat{\mathcal{J}}_Y, \qquad (115)$$

in the Cartesian basis, and which is, within a factor $\hbar$ *identical with the commutator algebra of angular momentum operators in quantum mechanics.* This provides a simple route to quantization of the magnetic fields of the electromagnetic plane wave in vacuo, giving the result

$$\hat{B}^{(1)} = -B^{(0)}\frac{\hat{\mathcal{J}}^{(1)}}{\hbar}e^{i\phi}, \qquad \hat{B}^{(2)} = -B^{(0)}\frac{\hat{\mathcal{J}}^{(2)}}{\hbar}e^{-i\phi}, \qquad \hat{B}^{(3)} = iB^{(0)}\frac{\hat{\mathcal{J}}^{(3)}}{\hbar}, \qquad (116)$$

where $\hat{B}^{(i)}$ are now operators in quantum mechanics. In particular, the longitudinal operator $\hat{B}^{(3)}$ is the elementary quantum of magnetic flux density in the propagation axis, in other words each photon in this quantized representation carries this quantum of magnetic field. This result is in precise agreement with the original derivation [24] of the present author, which used creation and annihilation operators, which in turn defined the Stokes operators introduced recently by Tanaś and Kielich [66].

We reach the general result that magnetization by light, and related phenomena such as the optical Zeeman and Faraday effects [24–30] are described in quantum optics through the operator $\hat{B}^{(3)}$, which is independent of the phase of the plane wave. As shown elsewhere, the effect of the operator is different in general from the effect of the classical field $\boldsymbol{B}^{(3)}$. For $\hat{B}^{(3)}$, the interaction Hamiltonian must be evaluated in a different way from the semi-classical Hamiltonian involving $\boldsymbol{B}^{(3)}$, because in the latter case, $\boldsymbol{B}^{(3)}$ can be taken outside the Dirac integral (Dirac brackets), and in the former it cannot, because it is an operator. This leads to the expectation [25] that optical Zeeman spectra in atoms due to $\hat{B}^{(3)}$, for example, are different theoretically from those due to $\boldsymbol{B}^{(3)}$.

The existence of the operators $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ is consistent with the Proca equation, which can be quantized straightforwardly to give a particle like interpretation of the electromagnetic field in three space-like dimensions. This particle is identified with the photon, and in the Proca equation, this has finite mass, m_0. As we have seen, the existence of this mass, however small, is critically important in removing obscurities of quantization of the classical electromagnetic field. The above simple method of quantization of the magnetic field components is free from obscurities such as these, obscurities which arise [15] from the habitual assertions that the electromagnetic field in vacuo can have only transverse components, and that the mass of the photon is identically zero. Therefore, the most important implication of the Lie algebra (7), or in commutator form (110), is that this algebra ties together the transverse and longitudinal space-like components of the magnetic fields in such a way that it becomes impossible to assert that the longitudinal component is unphysical.

4.2. Algebra Involving Electric Fields

If we attempt to extend the methods of the previous subsection to $i\boldsymbol{E}^{(3)}$ we obtain an algebra which is not cyclic (contrast Eqs. (7)),

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = icE_0 \boldsymbol{B}^{(3)*}, \tag{117a}$$

$$\boldsymbol{E}^{(2)} \times \boldsymbol{E}^{(3)} =? -cE_0 \boldsymbol{B}^{(1)*}, \tag{117b}$$

$$\boldsymbol{E}^{(3)} \times \boldsymbol{E}^{(1)} =? \; cE_0 \boldsymbol{B}^{(2)*}. \tag{117c}$$

Equation (117a) is identical with Eq. (7a), and can be expressed as a commutator,

$$(\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)})_k = -\frac{c^2}{2} \epsilon_{ijk} [\hat{B}^{(1)}, \hat{B}^{(2)}]_{ij}, \tag{118}$$

but the other two cannot be reduced to a cyclic form as for Eqs.(7b) and (7c). Equation (117a) conserves $\hat{C}$, $\hat{P}$, and $\hat{T}$ but Eqs. (117b) and (117c), although algebraically correct, violate $\hat{T}$ symmetry. The reason for this is that any attempt to construct a real $\boldsymbol{E}^{(3)}$ from phase independent cross products of transverse solutions of Maxwell's equations in vacuo violates $\hat{T}$, the cross products $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$, $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$, $\boldsymbol{E}^{(1)} \times \boldsymbol{B}^{(2)}$ and so on being $\hat{T}$ negative. An electric field must be $\hat{T}$ positive. Multiplying both sides of Eq. (117b), for example, by $\boldsymbol{E}^{(1)}$, and rearranging the vector triple product $\boldsymbol{E}^{(1)} \times (\boldsymbol{E}^{(2)} \times \boldsymbol{E}^{(3)})$ as $\boldsymbol{E}^{(2)}(\boldsymbol{E}^{(1)} \cdot \boldsymbol{E}^{(3)}) - \boldsymbol{E}^{(3)}(\boldsymbol{E}^{(1)} \cdot \boldsymbol{E}^{(2)})$; using the results $\boldsymbol{E}^{(1)} \cdot \boldsymbol{E}^{(3)} = 0$ and $\boldsymbol{E}^{(1)} \cdot \boldsymbol{E}^{(2)} = E_0^2$ we arrive at the putative physical law

$$\boldsymbol{E}^{(3)} =? \left(\frac{\mu_0}{B_0} \right) \boldsymbol{N}. \tag{119}$$

Since a real $\boldsymbol{E}^{(3)}$ is $\hat{T}$ positive and $\boldsymbol{N}$ is $\hat{T}$ negative, this equation clearly violates $\hat{T}$. Similarly, Eq. (117c) violates $\hat{T}$.

The imaginary $i\boldsymbol{E}^{(3)}$ on the other hand, is non-zero *but unphysical*. It has the same symmetry as $\boldsymbol{B}^{(3)}$ as in Eqs. (7). Symmetry analysis in electrodynamics is as important as algebraic analysis, therefore, and only after a careful application of $\hat{T}$ does it become clear that two out of the three Eqs. (117) are not valid physical laws. Equation (119) conserves $\hat{P}$ and $\hat{C}$, but violates $\hat{T}$. It follows from the $\hat{C}\hat{P}\hat{T}$ theorem (Introduction), that Eq. (119) also violates the fundamental symmetry $\hat{C}\hat{P}\hat{T}$, since $\hat{C}\hat{P}$ is conserved and $\hat{T}$ is violated.

Similarly, if an attempt is made to construct a formal algebra of "type three",

$$\boldsymbol{E}^{(1)} \times \boldsymbol{B}^{(2)} =? \; B^{(0)} \boldsymbol{E}^{(3)*}, \tag{120a}$$

$$\boldsymbol{E}^{(2)} \times \boldsymbol{B}^{(3)} =? \; iB^{(0)} \boldsymbol{E}^{(1)*}, \tag{120b}$$

$$\boldsymbol{E}^{(3)} \times \boldsymbol{B}^{(1)} =? -B^{(0)} \boldsymbol{E}^{(2)*}, \tag{120c}$$

it is not cyclic, unlike Eqs.(7), and all three Eqs.(120) violate $\hat{T}$ and $\hat{C}\hat{P}\hat{T}$. These findings follow from simple vector triple products such as

$$\boldsymbol{B}^{(1)} \times (\boldsymbol{E}^{(2)} \times \boldsymbol{B}^{(3)}) = \boldsymbol{E}^{(2)}(\boldsymbol{B}^{(1)} \cdot \boldsymbol{B}^{(3)}) - \boldsymbol{B}^{(3)}(\boldsymbol{B}^{(1)} \cdot \boldsymbol{E}^{(2)}). \tag{121}$$

In electrodynamics, care must always be used to apply the symmetry operators properly, particularly in relation to cross products such as the conjugate product of non linear optics, i.e. the antisymmetric part of the light intensity tensor. This is $\hat{T}$ *negative*, something which is not obvious from inspection. In particular, it is incorrect simply to apply $\hat{T}$ to each symbol of $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$. This would lead to the incorrect conclusion that the conjugate product is $\hat{T}$ positive. The reason is that [60] $\hat{T}(\boldsymbol{E}_L^{(1)}) = \boldsymbol{E}_R^{(2)}$, $\hat{T}(\boldsymbol{E}_L^{(2)}) = \boldsymbol{E}_R^{(1)}$, where L and R denote left and right circularly polarized respectively. The correct way of applying T and P to electrodynamic field cross products is given elsewhere [60].

Finally, the vectorial Lie algebra of type four:

$$\boldsymbol{E}^{(1)} \times \boldsymbol{B}^{(1)} = 0, \quad \boldsymbol{E}^{(2)} \times \boldsymbol{B}^{(2)} = 0, \quad \boldsymbol{E}^{(3)} \times \boldsymbol{B}^{(3)} = 0, \tag{122}$$

completes the cyclical relations between the three magnetic and three electric components of the electromagnetic plane wave in vacuo. In three dimensional space, type four is superfluous, for example $i\boldsymbol{E}^{(3)}$ is parallel to $\boldsymbol{B}^{(3)}$, but in Minkowski space-time, type four is part of the complete commutator algebra of electric and magnetic fields defined in the Lorentz group.

It is seen that in three dimensional space, $\boldsymbol{B}$ fields, and cross products of $\boldsymbol{E}$ fields, can be put in matrix form, but individual $\boldsymbol{E}$ fields cannot, suggesting that three dimensional space is not completely adequate for a description of electromagnetism, and that electric and magnetic fields are fully covariant four-vectors with a time-like as well as three space-like components. In the next section, this line of reasoning is developed, magnetic fields are related to rotation generators in space-time, and electric fields to boost generators.

The only valid Lie algebra is that of Eqs. (7), defining $\boldsymbol{B}^{(3)}$.

5. Electric and Magnetic Field Four-Vectors in Space-Time

Traditional electrodynamics, both in its classical and quantum forms, is satisfied with a representation of electric and magnetic fields in Euclidean space. The potential four-vector, however, is defined in Minkowski space-time, and electric and magnetic field vectors derived there from by taking the four-curl of A_μ to give the well known electromagnetic four-tensor $F_{\mu\nu}$, which is by definition traceless and antisymmetric. We have seen in Sec. 4 that the electromagnetic field enters gauge theory in a fully relativistic description based on A_μ and $F_{\mu\nu}$. In the introduction however, we have argued for the existence of E_μ and B_μ, electric and magnetic field four-vectors which are fully covariant in Minkowski space-time. The primary reason for this was that the Lorentz invariants $E_\mu E_\mu$ and $B_\mu B_\mu$ produced a satisfactory description of the Planck radiation law, in that only the transverse space-like components contribute to electromagnetic energy density. In view of the Lie algebra developed in this article, and in view of the experimental evidence reviewed herein, there is no doubt that there exists a longitudinal solution $\boldsymbol{B}^{(3)}$ of Maxwell's equations in vacuo, a solution which is tied inextricably (Sec. 4) to the well accepted transverse solutions of the same equations. Longitudinal fields such as $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are also consistent with finite photon mass, and we have demonstrated one simple method of obtaining $\boldsymbol{B}^{(3)}$ from the Proca equation. Acceptance of finite photon mass, which is consistent [32] with a broad range of experimental data from many different sources, leads, as we have seen, to the abandonment of the Coulomb gauge. Section 3 has shown that finite photon mass is consistent with the Dirac gauge in a limiting form.

The Proca equation is a wave equation with a physically meaningful, fully covariant, four-potential A_μ acting as eigenfunction. This is consistent with the fact that electromagnetism enters contemporary gauge theory through A_μ scaled by the charge on the electron, e (Sec. 3). The Bohm-Aharonov effect (introduction) shows experimentally [34] and theoretically [23] that A_μ is physically meaningful, and is not just a mathematical subsidiary condition of first order differential equations, the Maxwell equations in vacuo. These factors lead us to expect, therefore, that the well known four-tensor $F_{\mu\nu}$ of electromagnetism is also fully covariant, i.e. *that all its sixteen components are physically meaningful*. It is well known that the off-diagonal elements of the tensor $F_{\mu\nu}$ are physically meaningful electric and magnetic fields in vacuo. In S.I. units,

$$F_{\mu\nu} = \epsilon_0 \begin{bmatrix} 0 & -c\boldsymbol{B}^{(3)} & c\boldsymbol{B}^{(2)} & iE^{(1)} \\ c\boldsymbol{B}^{(3)} & 0 & -c\boldsymbol{B}^{(1)} & -iE^{(2)} \\ -c\boldsymbol{B}^{(2)} & c\boldsymbol{B}^{(1)} & 0 & -iE^{(3)} \\ iE^{(1)} & iE^{(2)} & iE^{(3)} & 0 \end{bmatrix}. \tag{123}$$

We note that the diagonals vanish by definition of the four-curl,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}. \tag{124}$$

The longitudinal fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are non-zero in general in the off-diagonal elements of $F_{\mu\nu}$. To discard them *a priori* (i.e. without analysis), as is often the custom in electrodynamics, is not consistent with the fully covariant nature of the four-curl of A_μ, and of A_μ itself. The Bohm-Aharonov effect shows that A_μ is physically meaningful. Using Eq. (93) it can be seen that $\boldsymbol{B}^{(3)}$ is experimentally observable in the inverse Faraday effect. Symmetry analysis of Eqs.(7) show that $\boldsymbol{B}^{(3)}$ is well defined in terms of the well known $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ and is therefore physically meaningful. If $\boldsymbol{B}^{(3)}$ is assumed to be zero, then Eqs. (7) show that $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are also zero, i.e. the electromagnetic plane wave vanishes in vacuo if $\boldsymbol{B}^{(3)}$ is zero. On the other hand, the real $\boldsymbol{E}^{(3)}$ is not connected to transverse solutions of Maxwell's equations because of $\hat{T}$ violation as argued already.

These findings are verified by solving the Proca equation (12) written out in the form,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \xi^2\phi, \tag{125a}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \xi^2\boldsymbol{A}. \tag{125b}$$

Since $\boldsymbol{B}^{(3)}$ from Eq. (7) is independent of time, longitudinal, and for all practical purposes identical in the Proca and Maxwell descriptions of electrodynamics, it must be obtainable from a time independent solution of Eq. (125b). For all practical purposes, the Maxwellian

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{126a}$$

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{126b}$$

can be used to find time independent $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ from the Proca equation. Equation (126a) shows that if $\boldsymbol{B}^{(3)}$ is time independent, then so is $\boldsymbol{A}^{(3)}$, its concomittant vector potential. So

$$\frac{\partial^2 \boldsymbol{A}^{(3)}}{\partial t^2} = 0, \tag{127}$$

in Eq. (125b) and

$$\boldsymbol{E}^{(3)} = -\nabla\phi^{(3)} \tag{128}$$

in Eq. (126b). Using the gauge condition (2), *necessitated by finite photon mass*, $\phi^{(3)}$ is the magnitude of $c\boldsymbol{A}^{(3)}$ and must be time independent if $\boldsymbol{A}^{(3)}$ is so. Therefore,

$$\frac{\partial^2\phi^{(3)}}{\partial t^2} = 0, \tag{129}$$

in Eq. (125a). In order to obtain from Eq. (126a) a $\boldsymbol{B}^{(3)}$ in the Z axis, the required solution to Eqs. (125b) and (127) is

$$\boldsymbol{A}^{(3)} = \frac{B_0}{2}(-X\boldsymbol{i} + Y\boldsymbol{j})e^{-\xi Z}, \tag{130}$$

where $R^2 = X^2 + Y^2$ is a constant. Therefore,

$$\phi^{(3)} = c|\boldsymbol{A}^{(3)}| = \frac{1}{2}E_0 R e^{-\xi Z}, \tag{131}$$

and

$$\boldsymbol{E}^{(3)} \equiv E^{(0)}\boldsymbol{k} = -\frac{\partial\phi^{(3)}}{\partial Z}\boldsymbol{k} = \frac{1}{2}\xi E_0 R e^{-\xi Z}\boldsymbol{k}. \tag{132}$$

$$\boldsymbol{B}^{(3)} \equiv B^{(0)}\boldsymbol{k} = B_0 e^{-\xi Z}\boldsymbol{k}. \tag{133}$$

Since $\xi \sim 10^{-26} m^{-1}$, Eqs. (132) and (133) show that in the Proca description of electrodynamics, $|\boldsymbol{B}^{(3)}| \gg |\boldsymbol{E}^{(3)}|$ in free space. In Maxwell's description, $\xi \to 0$, and the real $\boldsymbol{E}^{(3)}$ *vanishes* but $\boldsymbol{B}^{(3)}$ remains finite. The field $\boldsymbol{B}^{(3)}$ is divergentless, solenoidal and time independent, and is a solution of Maxwell's equations in free space because it is a solution of Eqs. (125) under the conditions (126).

This finding is based on the gauge condition (2), which is a consequence of finite photon mass, so that experimental evidence for $|\boldsymbol{B}^{(3)}| \gg |\boldsymbol{E}^{(3)}|$ is consistent with finite photon mass.

Significantly, the existence of $\boldsymbol{B}^{(3)}$ is supported experimentally in the inverse Faraday effect and in other experimental evidence surveyed in this article. In contrast, there appears to be no experimental evidence for the real $\boldsymbol{E}^{(3)}$. If the real $\boldsymbol{E}^{(3)}$ were non-zero, circularly polarized laser light would polarize materials such as solids and liquids, and there appears to be no trace of such an effect experimentally. The only well known experimental evidence for electric polarization is optical rectification, which occurs in a linearly polarized beam and only in chiral material, and is a small second order effect. Equation (132) shows that $E^{(0)} \ll E_0$ in the Proca description and that $E^{(0)}$ is zero in the Maxwell description. Equation (133) shows that $B^{(0)} = B_0$ in the Maxwell description. The time-like component $E^{(0)}$ and the longitudinal component $i\boldsymbol{E}^{(3)}$ of the four-vector E_μ therefore vanish in the Maxwell description. These two components are non-zero in general in the Proca description, but are very small in magnitude and for all practical purposes unobservable with contemporary apparatus. Experimentally, it is found that longitudinal components such as $\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ do not contribute to electromagnetic energy density. This is consistent with $\boldsymbol{E}^{(3)} = 0$ in the Maxwell description. It has been shown by Evans and Farahi [27] that if the condition of Appendix C is obeyed, the classical electromagnetic energy density is unchanged. *This is precisely what has been found from the Proca equation, i.e. if* $\boldsymbol{B}^{(3)}$ *is real, then* $i\boldsymbol{E}^{(3)}$ *has no real part.*

This is also consistent with the dual transform of special relativity, which states that Maxwell's equations are invariant to $\boldsymbol{B}^{(3)} \to -i\boldsymbol{E}^{(3)}/c$. Thus if $B^{(3)}$ is purely real, and a solution of Maxwell's equation, its dual is a purely *imaginary* electric field.

The four-vector descriptions B_μ and E_μ in free space are consistent with the experimental finding that neither $\boldsymbol{B}^{(3)}$ nor $i\boldsymbol{E}^{(3)}$ contribute to electromagnetic energy density. To be consistent with the fact that $E^{(0)}$ is not zero in the Proca description, we describe energy density through Eq. (17), leading, as we have seen, to Eqs.(22) and (23). These imply that the magnitudes of their longitudinal components of E_μ and B_μ must be equal to the magnitudes of their respective time-like components if $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ do not contribute to electromagnetic energy density in vacuo. In appendix B it is shown that the concept of E_μ and B_μ in vacuo is formally consistent with the Lorentz transformation, so that these quantities are valid four-vectors.

By realizing the significance of $E^{(0)}$ and $B^{(0)}$ as time-like components of E_μ and B_μ it follows that Eqs. (7) is an expression of the fact that the space-like and time-like components of $_\mu$ and E_μ are tied together inextricably. For E_μ, however, because the longitudinal component is zero in Maxwell's description and very small in Proca's. Several basically important consequences follow, some of which are summarized as follows.

1. In special relativity, the classical electromagnetic energy density must be expressed through the Lorentz invariant products $E_\mu E_\mu$ and $B_\mu B_\mu$, and not just through the space-like parts $\boldsymbol{E}\cdot\boldsymbol{E}$ and $\boldsymbol{B}\cdot\boldsymbol{B}$ as is customary in the Poynting theorem. As shown in the introduction, this leads to a description of the Planck radiation law which is consistent with the novel Lie algebra of Sec. 4, which is in turn consistent with finite photon mass and the Proca equation. The existence of E_μ and B_μ is also consistent with grand unified field theory, such as SU(5), within whose framework photon mass must be incorporated, as shown by Huang [19].

2. The existence of E_μ and B_μ is fundamentally inconsistent with the Coulomb gauge, in which $A^{(0)}$ is zero, implying from our novel Eq. (129) that $E^{(0)}$ should be zero. This is obviously inconsistent with the fact that $E^{(0)}$

is the scalar amplitude of the electric field strength (volt m^{-1}) of the electromagnetic plane wave in vacuo. This result reflects the fact that the Coulomb gauge is inconsistent also with a fully covariant A_μ, and inconsistent with the appearance of $i\boldsymbol{E}^{(3)}$, $\boldsymbol{B}^{(3)}$, $E^{(0)}$, and $B^{(0)}$ in the four-tensor $F_{\mu\nu}$. The Coulomb gauge must therefore be abandoned if further obscurity is to be avoided.

3. A_μ is fully consistent with the existence of E_μ and B_μ, which are linked to A_μ through $F_{\mu\nu}$, the four-curl of A_μ.

4. In order that E_μ and B_μ be consistent with the geometry of Minkowski space-time, the following relations must be obeyed,

$$E_\mu E_\mu = constant, \quad B_\mu B_\mu = constant, \tag{134}$$

$$\frac{\partial E_\mu}{\partial x_\mu} = 0, \quad \frac{\partial B_\mu}{\partial x_\mu} = 0. \tag{135}$$

The first of these follows from the fact that $E_\mu E_\mu$ and $B_\mu B_\mu$ are electromagnetic energy density terms, and are constant by conservation of energy. The second of these geometrical laws mean that the Gauss theorem in differential form appears in Minkowski space-time as the equations,

$$\nabla\cdot\boldsymbol{E} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t} = 0, \tag{136}$$

$$\nabla\cdot\boldsymbol{B} + \frac{1}{c}\frac{\partial B^{(0)}}{\partial t} = 0. \tag{137}$$

Since $E^{(0)}$ and $B^{(0)}$ are usually independent of time (e.g. in a constant intensity beam), this makes no difference to the usual in vacuo statements in Euclidean space of the Gauss theorem in differential form,

$$\nabla\cdot\boldsymbol{E} = 0, \quad \nabla\cdot\boldsymbol{B} = 0. \tag{138}$$

However, if for some reason $E^{(0)}$ and $B^{(0)}$ are changing with time, the statement of the Gauss theorem in space-time becomes different from that in Euclidean space.

5. The statement in vacuo of the Maxwell equations,

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0, \quad \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\mu} = 0, \tag{139}$$

is unchanged by the existence of E_μ and B_μ, because, as we have seen, the time-like components of $E^{(0)}$ and $B^{(0)}$ enter into $F_{\mu\nu}$ in such a way as to be consistent with its traceless, antisymmetric nature. This is essentially a tautology, because $F_{\mu\nu}$ is by *definition* a four-curl of Minkowski space-time, and must be traceless for this reason. The Maxwell equations in vacuo are consistent with E_μ and B_μ in vacuo.

6. The Proca equation is also consistent with E_μ and B_μ, because these can be derived from A_μ, the eigenfunction in space-time of the Proca equation. Therefore E_μ and B_μ are consistent with finite photon mass, i.e. with non-zero ξ in the Proca equation, and are also fully consistent with the novel Lie algebra of Sec. 4. In particular, the longitudinal equations in vacuo,

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}, \quad \boldsymbol{E}^{(3)} = E^{(0)}\boldsymbol{k}, \tag{140}$$

are identified as linking together the space-like (3) component and the time-like component (0). This is consistent with the Planck radiation law because

$$B^{(3)2} - B^{(0)2} = E^{(3)2} - E^{(0)2} = 0, \tag{141}$$

and the longitudinal components $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ cannot contribute to classical electromagnetic energy density. Obviously, they cannot be observed in this way, and this is also consistent with the fact that $\boldsymbol{B}^{(3)}$ can be defined (and was originally identified [24]) through the conjugate product $\boldsymbol{E}^{(1)}\times\boldsymbol{E}^{(2)}$, which is the *vectorial* part of the light intensity tensor [24–30]. Only the trace (scalar part) of I_{ij} is observable as light intensity in watts m^{-2}.

In conclusion of this part of Sec. 5, it seems clear that the existence of E_μ and B_μ makes a profound difference to classical electrodynamics, and therefore to quantum electrodynamics. In particular, the habitual use of the Coulomb gauge is internally inconsistent, there exists a well defined Lie algebra which proves this point beyond reasonable doubt. The notion that longitudinal solutions of Maxwell's equations are unphysical is profoundly misleading and inconsistent both with theory and with data such as those from the inverse Faraday effect, light shifts, optical NMR, the optical Faraday effect, and probably also with a range of data from other sources, such as Compton scattering. Finally, E_μ and B_μ are well defined within $F_{\mu\nu}$, the four-curl of A_μ, which is the eigenfunction of the Proca equation for finite photon mass.

5.1 Representation of E_μ and B_μ in the Lorentz Group and Relation To Rotation and Boost Generators in Minkowski Space-time

It has been shown that E_μ and B_μ of the classical Maxwellian field are four-vectors in space-time, and because of the relations,

$$E^{(1)2} + E^{(2)2} + E^{(3)2} = E^{(0)2}, \qquad B^{(1)2} + B^{(2)2} + B^{(3)2} = B^{(0)2}, \tag{142}$$

they are four-vectors *in the light-like condition*, in which the magnitude of the space-like part of a four-vector becomes equal to that of its time-like component. The light-like condition for the four-vector x_μ, for example, corresponds to

$$X^2 + Y^2 + Z^2 = c^2 t^2, \tag{143}$$

from which it can be seen that it corresponds to movement at the speed of light c. In the mathematical limit of the Maxwellian field, photon mass vanishes identically, and the electromagnetic plane wave in vacuo travels at the speed of light, identified with the universal constant c, of special relativity. Since E_μ and B_μ are defined in Eq. (142) in this limit, it is natural that they are in the light-like condition. Again, we see that the Coulomb gauge is inconsistent with this representation, because the time-like parts of E_μ and B_μ are set to zero. The four-vectors E_μ and B_μ cannot be defined in the Coulomb gauge, and manifest covariance is lost, meaning that the basic pseudo-Euclidean geometry of the problem is violated. In a rough analogy, this would be equivalent to trying to define a triangle with only two sides. Loss of manifest covariance in A_μ is another consequence of the Coulomb gauge, as discussed in the introduction, and can be traced to the arbitrary assertion that photon mass is identically zero, allowing (Sec. 3) physically meaningless gauge freedom.

By reinstating finite photon mass, using the Proca wave equation, this type of gauge freedom is lost, the Lorentz condition becomes a mathematical necessity, and A_μ is a physically meaningful, fully covariant eigenfunction in space-time. For finite photon mass, however, E_μ and B_μ are no longer light-like, because photons, being massive particles, no longer travel in the observer frame at the speed c. The latter is no longer the speed of light, but remains the universal constant of the theory of special relativity ($3 \times 10^8 \, ms^{-1}$). The phrase "tired light", used [32] by Hubble and Tolman, means just this, it does not imply that c is no longer a universal constant. It is essential also to realize that finite photon mass in the Proca equation implies rigorously that the Coulomb gauge can no longer be used. The Lorentz condition is a direct mathematical consequence of the Proca equation.

With this preamble, we proceed in this section to emphasize the physical reality of the four-vectors E_μ and B_μ by defining them in terms of the well known [15] boost and rotation generators, respectively, of Minkowski space-time.

The boost and rotation generators, denoted respectively $\hat{K}_i$ and $\hat{J}_i$ are unitless, complex, four by four matrices which define the structure of the Lorentz group through the following commutator algebra [15] in space-time,

$$\begin{aligned} [\hat{J}_X, \hat{J}_Y] &= i\hat{J}_Z, \quad \text{and cyclic permutations} \\ [\hat{K}_X, \hat{K}_Y] &= -i\hat{J}_Z, \quad \text{"} \quad \text{"} \quad \text{"} \\ [\hat{K}_X, \hat{J}_Y] &= i\hat{K}_Z, \quad \text{"} \quad \text{"} \quad \text{"} \\ [\hat{K}_X, \hat{J}_X] &= 0, \quad \text{etc.} \end{aligned} \tag{144}$$

an algebra which can be summarized by [15],

$$\hat{J}_{\mu\nu}(\mu, \nu = 0, \ldots, 3) \begin{bmatrix} \hat{J}_{ij} = -\hat{J}_{ji} = \epsilon_{ijk}\hat{J}_k \\ \hat{J}_{io} = -\hat{J}_{oi} = -\hat{K}_i \end{bmatrix} \quad (i, j, k = 1, 2, 3) \tag{145}$$

This summary can in turn be displayed as a traceless, antisymmetric, four by four matrix of matrices,

$$\hat{J}_{\mu\nu} \equiv -\begin{bmatrix} 0 & -\hat{J}^{(3)} & \hat{J}^{(2)} & -\hat{K}^{(1)} \\ \hat{J}^{(3)} & 0 & -\hat{J}^{(1)} & -\hat{K}^{(2)} \\ -\hat{J}^{(2)} & \hat{J}^{(1)} & 0 & -\hat{K}^{(3)} \\ \hat{K}^{(1)} & \hat{K}^{(2)} & \hat{K}^{(3)} & 0 \end{bmatrix}. \tag{146}$$

The structure of the matrix in Eq. (146) is identical with that of the electromagnetic field tensor $F_{\mu\nu}$ in Eq. (123), suggesting that there is a proportionality between the two matrices. Equation (123) however describes a matrix of vectors and Eq. (146) a matrix of four by four matrices in space-time, and if there is a proportionality between the two it means that a complete description of electric and magnetic fields requires space-time. This is precisely the requirement fulfilled by the four-vectors E_μ and B_μ, which are, by definition, physically meaningful in space-time. That there does, indeed, exist a proportionality is already clear through our development in Sec. 4, leading to Eqs. (112) in Euclidean space. These equations show that when the magnetic field vectors appearing in $F_{\mu\nu}$ are re-expressed as rank two anti-symmetric polar tensors, they become directly proportional to the rotation generators of O(3). In Eq. (112), these were expressed in a circular basis appropriate to a consideration of the classical Maxwellian field in vacuo. The transition from Euclidean space to Minkowski space-time occurs for the rotation generators through the four by four complex matrices defined in Eq. (113). These are the rotation generators in the Lorentz group [15], and occur in Eq. (146) as off-diagonal elements, $\hat{J}^{(i)}$, of the matrix of matrices, $\hat{J}_{\mu\nu}$. We have therefore established an ineluctable chain of logic which relates the vectors $\boldsymbol{B}^{(i)}$, appearing in Eq. (123), to the four by four matrices occurring in Eq. (146).

This reasoning leads to the result that the three by three magnetic field matrices appearing in Eq. (112) occur in Minkowski space-time as four by four matrices. For example, the four by four matrix defining $\hat{B}^{(3)}$ is,

$$\hat{B}^{(3)} = iB^{(0)}\hat{J}^{(3)} = iB^{(0)}\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (147)$$

and this is *specifically a property of space-time in the same way that* $\hat{J}^{(3)}$ *in Eq. (113) is a property of space-time.* Thus, all three components $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ of the original vector $\boldsymbol{B}$ in Euclidean space can be expressed as components in space-time, and it follows that the complete representation of the magnetic part of the electromagnetic plane wave in vacuo must be a four-vector B_μ, thus reinforcing our earlier reasoning. This conclusion re-asserts that in special relativity, it is not possible to isolate Euclidean space from time. The space-time rotation generators $\hat{J}_i$ can also be expressed [15] (in a Cartesian basis) as differential operators,

$$\hat{J}_X = -i\left(Y\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial Y}\right), \quad \hat{J}_Y = -i\left(Z\frac{\partial}{\partial X} - X\frac{\partial}{\partial Z}\right), \quad \hat{J}_Z = -i\left(X\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial X}\right), \qquad (148)$$

which, within a factor $\hbar$, are the angular momentum operators of quantum mechanics. Quantized magnetic fields are therefore proportional to angular momentum operators, which generate rotations, thus revealing the fundamental nature of magnetism.

We must now establish a similar link between the electric part of the electromagnetic plane wave in vacuo and the boost generator $\hat{K}_i$ of the Lorentz group, in order to reinforce our earlier argument in favor of E_μ, the electric field four-vector. From Eq. (118) of Sec. 4 it is clear that cross products of electric fields can be expressed in terms of commutators of magnetic fields, the latter having been re-expressed mathematically as matrices. The commutators on the right hand side of Eq. (118) can be written in space-time through rotation generators expressed either as four by four matrices or differential operators. It follows therefore that the vector products of *electric* fields on the left hand side of Eq. (118) can also be expressed in space-time. Using the first two commutators of Eq. (144), we obtain the result,

$$[\hat{K}_X, \hat{K}_Y] = -[\hat{J}_X, \hat{J}_Y] = -i\hat{J}_Z, \text{ and cyclic permutations.} \qquad (149)$$

In a Cartesian basis for space-like components, therefore, the commutator of rotation generators in space-time can be expressed directly in terms of the commutator of boost generators.

We conclude that the vector cross product in Euclidean space of electric fields, the left hand side of Eq. (118), can be expressed in space-time as a commutator of boost generators.

Care must be taken to use consistently either a circular basis, as in Eq. (118), or a Cartesian basis, as in Eq. (144), otherwise, we have shown that the electric field of an electromagnetic plane wave in vacuo can be expressed as a boost generator of the Lorentz group in space-time, thus establishing the nature of the electric field as a fully covariant four-vector E_μ. The boost generator can be expressed in a Cartesian basis as a differential operator [15],

$$\hat{K}_X = i\left(t\frac{\partial}{\partial X} + X\frac{\partial}{\partial t}\right), \quad \hat{K}_Y = i\left(t\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial t}\right), \quad \hat{K}_Z = i\left(t\frac{\partial}{\partial Z} + Z\frac{\partial}{\partial t}\right), \qquad (150)$$

definitions which intrinsically involve space and time, and whose origin [15] is the Lorentz transformation. Boost generators can also be described [15] as dimensionless, complex, four by four matrices, which, in a Cartesian space-like basis, are,

$$\hat{K}_X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{K}_Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{K}_Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \qquad (151)$$

These matrices arise [15] from considerations of a boost Lorentz transformation connecting two inertial frames moving with a relative speed v.

The electric fields defined in Eq. (86) are polar vectors in Euclidean space. Using the above results, they can now be expressed in space-time in terms of boost generators, recognizing that

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \rightarrow [\hat{K}_X, \hat{K}_Y] = -i\hat{J}_Z, \qquad (152)$$

i.e. that the cross product of two Cartesian, polar, unit vectors, $\boldsymbol{i}$ and $\boldsymbol{j}$, in Euclidean space, to give the axial unit vector $\boldsymbol{k}$ in this space, is equivalent to the commutation of the boost generators $\hat{K}_X$ and $\hat{K}_Y$ in space-time to give the rotation generator $-i\hat{J}_Z$. In the circular basis the unit polar vectors of Eq. (152) become the boost generators,

$$\hat{e}^{(1)} \equiv i\hat{K}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -i & -1 & 0 & 0 \end{bmatrix}, \qquad \hat{e}^{(2)} = i\hat{K}^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 \end{bmatrix}, \tag{153}$$

finally allowing the electric fields to be expressed in space-time as boost generators in the circular basis multiplied by appropriate phases and scalar amplitudes,

$$\hat{E}^{(1)} = E_0\hat{e}^{(1)}e^{i\phi}, \qquad \hat{E}^{(2)} = E_0\hat{e}^{(2)}e^{-i\phi}, \tag{154}$$

$$[\hat{E}^{(1)}, \hat{E}^{(2)}] = E_0^2[\hat{e}^{(1)}, \hat{e}^{(2)}] = E_0 cB^{(0)}\hat{J}^{(3)} = -iE_0 c\hat{B}^{(3)}. \tag{155}$$

We conclude this part of Sec. 5 by recognizing that the electric and magnetic components of the electromagnetic plane wave in vacuo are proportional respectively to boost and rotation generators in Minkowski space-time. The magnetic fields therefore form a Lie algebra of the Lorentz group.

5.2. The Poincaré Group, Electric and Magnetic Fields as Pauli Lubansky Vectors

We have seen that the four components of E_μ and B_μ are present in the well known electromagnetic field tensor $F_{\mu\nu}$, through which electromagnetism is described in gauge theory (Sec. 3), the novel aspect of our treatment in this context being the demonstration that the time-like component $B^{(0)}$ is not zero. This development suggests that since $F_{\mu\nu}$ is the four-curl of A_μ, and since the individual components of E_μ and B_μ are related to individual components of A_μ, there might be a relation between each of E_μ and B_μ and $F_{\mu\nu}$. This section develops this method within the context of the inhomogenous Lorentz group, or Poincaré group [15] of special relativity. The Poincaré group is formed from the Lorentz group by adjoining the generator of translations in space-time, a generator introduced by Wigner [67] in 1939,

$$P_\mu = i\frac{\partial}{\partial x_\mu}. \tag{156}$$

Systems invariant under the Poincaré group [15] are characterized by mass and spin, which define the two Casimir invariants of the group. For finite mass, spin is characterized by a *rotation* group symmetry SU(2) in three, physically meaningful, space-like dimensions. For zero mass, this is no longer possible, leading to obscurity in the interpretation of the hypothetical "massless" photon. Before embarking on the main theme of this sub-section, it is instructive to explain the origin of this obscurity, because it illustrates the fundamental difficulties associated with the concept of zero photon mass.

In a Cartesian basis the generator of space-time translation is

$$x_\mu \rightarrow x_\mu + a_\mu, \tag{157}$$

and therefore has the same dimensions as eA_μ (Sec. 3). In the presence of electromagnetic radiation, therefore, P_μ is proportional to A_μ. The first Casimir invariant of the Poincaré group, the mass invariant, is defined as

$$C_1 = P_\mu P_\mu, \tag{158}$$

and the second Casimir invariant, the spin invariant, as

$$C_2 = W_\mu W_\mu. \tag{159}$$

Here W_μ is the Pauli Lubansky pseudo-vector [15], defined through

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma, \tag{160}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the fully antisymmetric unit tensor in four dimensions (vide infra). Quantities invariant under the most general type of Lorentz transformation are therefore characterized by mass and spin, and by their respective invariants. No other kinematic quantity is needed other than mass and spin.

For a massive particle in its rest frame, for example the massive photon, the energy-momentum four-vector is defined by

$$p_\mu = k_\mu \equiv (0, 0, 0, im_0c), \qquad p_\mu p_\mu = -m_0^2c^2, \tag{161}$$

so that the first Casimir invariant by definition proportional to mass. The Pauli Lubansky pseudo-vector W_μ in the rest frame is orthogonal to p_μ and is therefore space-like. The subgroup of the Poincaré group that leaves p_μ invariant, the little group, is a *rotation group* [15] in *three* dimensional space, and is identified [15] as SU(2). Therefore, Lorentz transformation of any particle with rest mass requires a representation of SU(2) in all three space-like dimensions. Since A_μ is proportional to the rotation generator through e, the same conclusion must hold for A_μ for a massive photon, i.e. for A_μ with a non-zero time-like component. Representations of the Lorentz group are given by representations of SU(2), the rotation group, for a time-like four-vector such as that representing the massive photon in its rest frame. *This is the fundamental explanation for spin in special relativity.* Hence, the massive photon acquires by Lorentz transformation the fundamental kinematic property known as spin, and spin is an inevitable consequence of the most general type of Lorentz transformation. We note that spin defined in this way, through representations of SU(2), is inevitably linked with mass.

If it is now *asserted* that mass is zero in the rest frame, the fundamental kinematic structure just described is destroyed completely [15]. Representations of the SU(2) rotation group in three space-like dimensions can no longer be used. The Lie algebra of SU(2) is contradicted, and the little group is changed to the physically obscure E(2), which is the group of rotations and translations *in a plane*, i.e. in only two out of the three space-like dimensions. This leads to the habitual conclusion, based on zero photon rest mass, that the photon has only two helicities, and only two out of the three possible spin eigenvalues for a boson, i.e. 1 and -1. If the photon is considered to have finite rest mass, however small in numerical magnitude, the eigenvalues of spin become 1, 0, and -1, and the longitudinal component is restored.

In the theory of special relativity, therefore, zero photon mass removes one space dimension, resulting in physical obscurity. Evidently, experimental data are acquired in three space-like dimensions, not two, and zero photon mass is therefore fundamentally incorrect. A similar conclusion is arrived at for the neutrino, and it is no longer asserted in the literature that the neutrino is massless. In special relativity, a massless neutrino results in the loss of a space dimension, i.e. is a concept which is geometrically incorrect.

Returning to the theme of this sub-section, it follows from this line of argument that fundamental special relativity requires electrodynamics, both classical and quantized, to be modified to incorporate longitudinal as well as transverse polarizations in the electromagnetic plane wave in vacuo. We have seen in Sec. 4 that the Maxwell equations, which are consistent with special relativity, and originally led to the concepts underpinning special relativity, guide us towards the acceptance of physically meaningful longitudinal polarization. This is algebraically evident through novel [24–30] cyclical relations such as (7). In one sense, therefore, the Maxwell equations point towards the existence of finite photon mass, although such a concept is not explicit in the equations themselves. Finite m_0 is found in the Proca equation, of which the d'Alembert equation is a limiting form, defined by $m_0 \to 0$. The novel Lie algebra developed in Sec. 4, and exemplified in Eq. (7), shows that in this limit, there remain well defined, physically meaningful, longitudinal fields which are inextricably related to the transverse fields normally used in electrodynamics. Accepting this, it becomes immediately necessary to explain why Planck's radiation law is unaffected by longitudinal polarizations, and this leads, as we have seen, to the establishment of E_μ and B_μ as physically meaningful four-vectors of the theory of special relativity.

It is well known that under Lorentz transformation, electric and magnetic fields regarded as three-vectors in space, behave in such a way that the electric field acquires a magnetic component and vice versa. This result is based essentially on the structure of the four-curl of A_μ, the tensor $F_{\mu\nu}$. For example, the simple (boost) Lorentz transformation of an electric field from frame (X, Y, Z, ict) to frame (X′, Y′, Z′, ict′) results in

$$E_X' = \frac{E_X - vB_Y}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}, \qquad E_Y' = \frac{E_Y + vB_X}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}, \qquad E_Z' = E_Z, \tag{162}$$

where v is the velocity in the Z axis of the primed frame, with respect to the first frame. We obtain the well known result of classical electrodynamics proposed by Einstein in 1905 in his first paper on special relativity. We have asserted, however, that there exist E_μ and B_μ, and therefore the Lorentz transformation in vacuo of these novel four-vectors must not contradict the well verified equations (162). That there is no contradiction is demonstrated through the intermediacy of the relations,

$$cB_Y = E_X, \qquad cB_X = -E_Y, \tag{163}$$

which come from the Maxwell equations in vacuo. From (163) in (162), we obtain,

$$E_X' = \xi E_X, \qquad E_Y' = \xi E_Y, \qquad E_Z' = E_Z, \qquad \xi = \frac{\left(1 - \frac{v}{c}\right)}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}, \tag{164}$$

with a similar result for the magnetic fields. Equation (164) is the result of a Lorentz boost transformation applied to the space-like components of E_μ in a Cartesian basis in vacuo. Our theory goes further than this, however, because it considers the amplitude $E^{(0)}$ to be the time-like component of E_μ. The boost Lorentz transformations of E_μ and B_μ are therefore expressible as,

$$E_\mu' = c_{\mu\nu}E_\mu, \tag{165}$$

and

$$B_\mu' = d_{\mu\nu}B_\mu, \tag{166}$$

as for any legitimate four-vector. Therefore the Lorentz transformation of the electric and magnetic parts of the electromagnetic plane wave in vacuo can be expressed consistently either as equation of the type (162), where the Cartesian components in the primed frame of the electric field become mixtures of electric and magnetic fields in the original (observer) frame; or in the form (164), which is entirely equivalent but expressed in terms of electric components only. The link between these two representations, Eq. (163), is a direct consequence of the

Maxwell equations in vacuo expressed in the frame of the experimental observer (the unprimed frame of reference). In this development we have restricted consideration, as elsewhere in this article, to the vacuum state, when material is present, however, there is still no contradiction between special relativity and the representation of electric and magnetic components of light as four-vectors. Care must be taken, however, to use Maxwellian (or other) linking equations such as (163) consistently: a) in the same frame of reference for magnetic and electric components; b) in the same basis, e.g. Cartesian or circular.

A working relation between E_μ, B_μ and A_μ must be established now in terms of the four-curl of A_μ.

There must be a relation between E_μ, B_μ and $F_{\mu\nu}$. It is also apparent from the foregoing that P_μ is dimensionally the same as eA_μ; and that the $\mathcal{J}_{\mu\nu}$ matrix has the same antisymmetric structure as $F_{\mu\nu}$, the four-curl of A_μ. From these observations, there is evidently a Pauli Lubansky vector which can be defined by analogy with that of W_μ in Eq. (160), but one which is made up of a product of $F_{\mu\nu}$ with a translation generator in space-time,

$$W_{F\mu} \propto \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\nu\rho}P_\sigma . \tag{167}$$

The concept of space-time translation is missing from the Maxwell equations, which are relations between space-like electric and magnetic fields conventionally identified with the off-diagonal elements of the four-curl of A_μ. The latter has four components in space-time, however. The four-curl of A_μ, the matrix $F_{\mu\nu}$, has components which can be expressed as boost and rotation generators of the Lorentz group, but there is no direct reference within $F_{\mu\nu}$ to space-time translation. The Maxwell equations (139) therefore do not explicitly refer to the space-time translation generator P_μ. However, we have seen that eA_μ contributes to the translation generator in the presence of electromagnetism, so that the d'Alembert equation effectively considers a quantity, A_μ, with the same dimensions as P_μ/e, and allows for the fact that the origin of a frame of reference in space-time may translate. The generator P_μ, as we have seen, was introduced by Wigner in 1939 [67] and could not therefore have been considered by Einstein in his 1905 demonstration of the covariance of Maxwell's equations. This means that the description of electric and magnetic components of the electromagnetic plane wave in vacuo in terms of the space-like $\boldsymbol{E}$ and $\boldsymbol{B}$ vectors is not fully consistent with the structure of the Poincaré group, also known as the inhomogeneous Lorentz group.

A fully consistent description requires that the four-vectors E_μ and B_μ be expressed explicitly in terms of the translation generator P_μ, which adjoins the Lorentz group to form the Poincaré group [15]. The following appears to be a satisfactory method of achieving this aim. Following a recent paper by the present author [68], the notation is slightly different from the foregoing, and the differences are highlighted.

We first define the *unit translation generator*,

$$\delta_\mu \equiv (0,\ 0,\ 1,\ -i), \tag{168}$$

which has the property

$$\delta_\mu \delta_\mu = 0 . \tag{169}$$

This generator can be considered as a delta function in the light-like condition corresponding to a massless particle moving at c, considered as the speed of light, in the Z space-like dimension. This concept is consistent with the classical Maxwellian limit $m_0 \to 0$, but is of course inconsistent with rigorously non-zero photon mass. In the presence of electromagnetism, the unit generator δ_μ becomes associated with a unit four-vector corresponding to the quantity eA_μ. Using (167), we are led to the following definitions,

$$\epsilon_0 E_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{F}_{\nu\rho}\delta_\sigma , \tag{170}$$

$$\epsilon_0 c B_\mu \equiv -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\delta_\sigma , \tag{171}$$

where the electric and magnetic four-vectors E_μ and B_μ are defined by

$$E_\mu \equiv (E^{(1)},\ E^{(2)},\ E^{(3)},\ -iE^{(0)}), \qquad B_\mu \equiv (B^{(1)},\ B^{(2)},\ B^{(3)},\ -iB^{(0)}), \tag{172}$$

In these equations, recall that if $\epsilon_{0123} = 1$, its other non-zero elements are +1 or -1, according as to whether ϵ_{0123} can be generated by an even or odd number of subscript pair permutations. Thus, for example,

$$\epsilon_{3120} = -1, \quad \epsilon_{3210} = 1, \quad \epsilon_{2310} = -1, \tag{173}$$

and so on. The elements of $F_{\mu\nu}$ in the definitions (170) and (171) are labelled explicitly as

$$F_{\nu\rho}(\nu,\ \rho = 0,\ 1,\ 2,\ 3) = \begin{bmatrix} 11 & 12 & 13 & 10 \\ 21 & 22 & 23 & 20 \\ 31 & 32 & 33 & 30 \\ 01 & 02 & 03 & 00 \end{bmatrix} . \tag{174}$$

With these definitions, and using the linking equations (163), it can be verified with tensor algebra [68] that the relations (170) and (171) do indeed give electric and magnetic four-vectors; Eq. (172), with the required properties

$$\boldsymbol{E}^{(3)} = E^{(0)}\boldsymbol{k}, \qquad \boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}, \tag{175}$$

for the longitudinal components. The transverse components are the usual transverse solutions (85) and (86) of the Maxwell equations in vacuo. Albeit with a slight change of notation, it has been shown that E_μ and B_μ are indeed simply related to the well known four-curl, $F_{\mu\nu}$, of A_μ, as expected. Once more, this reinforces the interpretation of E_μ and B_μ as physically meaningful four-vectors in Minkowski space-time.

Equation (171) covariantly defines B_μ as a Pauli Lubansky pseudo-vector, and Eq. (170) defines E_μ as a Pauli Lubansky vector. Equation (171) is dual with Eq. (170), because under the well known dual transformation [15] of special relativity,

$$F_{\rho\sigma} \rightarrow \tilde{F}_{\mu\nu}, \qquad E_\mu \rightarrow -icB_\mu. \tag{176}$$

The $\hat{C}$, $\hat{P}$, and $\hat{T}$ symmetries of B_μ and E_μ as defined in Eqs. (170) and (171) are consistent with those of $F_{\nu\rho}$ and $\boldsymbol{\delta}_\sigma$, bearing in mind that the latter is a unit space-time translation generator. We denote B_μ a pseudo-vector because its space-like components form a space-like pseudo-vector, and similarly, the space-like components of E_μ form a vector. Both E_μ and B_μ are orthogonal to $\boldsymbol{\delta}_\mu$ in space-time,

$$B_\mu \boldsymbol{\delta}_\mu = 0, \qquad E_\mu \boldsymbol{\delta}_\mu = 0. \tag{177}$$

Since E_μ and B_μ are defined covariantly, the time-like components E_μ and B_μ are both explicitly and implicitly stated to be physically meaningful in space-time. The products $E_\mu E_\mu$ and $B_\mu B_\mu$ are spin Casimir invariants of the Poincaré group, while $\boldsymbol{\delta}_\mu \boldsymbol{\delta}_\mu$ is a mass Casimir invariant. Because $\boldsymbol{\delta}_\mu$ has been defined in the light-like condition, corresponding to the Maxwellian field, it follows that

$$E_\mu E_\mu = 0, \qquad B_\mu B_\mu = 0, \qquad \boldsymbol{\delta}_\mu \boldsymbol{\delta}_\mu = 0, \tag{178}$$

i.e. the quantized Maxwellian field produces massless photons with helicity +1 and -1. These statements are of course modified fundamentally in the Proca field. From Eqs. (177) and (178) E_μ and B_μ are both orthogonal and proportional to $\boldsymbol{\delta}_\mu$ in space-time. The proportionality constant in the massless limit is the helicity.

By way of illustration, it may be verified explicitly that the conditions (178a) and (178b) are satisfied by the circularly polarized transverse components of Eq. (86) in combination with the longitudinal components of Eqs. (175a) and

(175b). For example,

$$\begin{aligned} E_\mu E_\mu &= E^{(1)2} + E^{(2)2} + E^{(3)2} - E^{(0)2} \\ &= E^{(0)2}(\hat{e}^{(1)} \cdot \hat{e}^{(1)} e^{2i\phi} + \hat{e}^{(2)} \cdot \hat{e}^{(2)} e^{-2i\phi} + \hat{e}^{(3)} \cdot \hat{e}^{(3)} - 1) \\ &= \frac{E^{(0)2}}{2}((\boldsymbol{i} - i\boldsymbol{j}) \cdot (\boldsymbol{i} - i\boldsymbol{j}) e^{2i\phi} + (\boldsymbol{i} + i\boldsymbol{j}) \cdot (\boldsymbol{i} + i\boldsymbol{j}) e^{-2i\phi}) \\ &= 0. \end{aligned} \tag{179}$$

This is the result quoted at the beginning of this development as Eq. (142), a result which shows that E_μ and B_μ are in a light-like condition. Equation (179) must not be misconstrued to mean that the intensity of light is zero. In Euclidean space, the customary representation of Eq. (179) is $\boldsymbol{E} \cdot \boldsymbol{E} = 0$ in the circular basis whose Lie algebra is Eq. (104) and which is used to define $\boldsymbol{E}$ through Eq. (86a) and (86b). The observable, time averaged electromagnetic energy density defines the scalar intensity of light in watts m^{-2},

$$I_0 = \epsilon_0 c E_0^2 = \frac{1}{2}\epsilon_0 c E_\mu E_\mu^*. \tag{180}$$

Here E_μ^* is the complex conjugate of E_μ in vacuo. To be consistent,

$$E_\mu \equiv (\boldsymbol{E}^{(1)}, \boldsymbol{E}^{(2)}, \boldsymbol{E}^{(3)}, -iE^{(0)}), \qquad E_\mu^* \equiv (\boldsymbol{E}^{(1)*}, \boldsymbol{E}^{(2)*}, \boldsymbol{E}^{(3)*}, -iE^{(0)}), \tag{181}$$

and

$$E_\mu E_\mu^* = \frac{E^{(0)2}}{2}((\boldsymbol{i} - i\boldsymbol{j}) \cdot (\boldsymbol{i} + i\boldsymbol{j}) + (\boldsymbol{i} + i\boldsymbol{j}) \cdot (\boldsymbol{i} - i\boldsymbol{j})) = 2E^{(0)2}. \tag{182}$$

This result shows that $E_\mu E_\mu^*$ is covariantly described because it is a constant in Minkowski space-time. The beam intensity is a scalar quantity which does not change with frame of reference, i.e. is invariant to frame (or Lorentz) transformation. Note that although E_μ^* is defined in Eq. (181) as the complex conjugate of E_μ, the sign of the time-like component, $-iE^{(0)}$, does not change, because the operation $E_\mu \rightarrow E_\mu^*$ takes place in a fixed frame of reference (X, Y, Z, -ict) in Minkowski space-time. Finally, if $i\boldsymbol{E}^{(3)}$ is assumed non-zero, it is invariant under $E_\mu \rightarrow E_\mu^*$. Thus $i\boldsymbol{E}^{(3)}$ and $-iE^{(0)}$ do not contribute to I_0, and this is consistent with the Planck radiation law.

We have seen in Sec. 4 that a well defined Lie algebra leads to the inescapable conclusion that there exist physically meaningful longitudinal field solutions of Maxwell's equations in vacuo. In this section we have provided support for the existence of the novel four-vectors E_μ and B_μ, whose properties have been briefly outlined. This four-vector representation has the important properties of being consistent with the Planck radiation law, with the Lorentz transformation of special relativity, with the Maxwell equations, and with the four-curl of A_μ, the well known electromagnetic field tensor $F_{\mu\nu}$. Electric and magnetic field components of the electromagnetic plane wave in vacuo have been expressed in terms of boost and rotation generators, respectively, of space-time. A fully covariant description of E_μ and B_μ has been proposed within the Poincaré group.

6. Quantization: The Four-Vector E_μ as Conjugate Momentum π_μ

It is shown in this section that the novel four-vector E_μ can be identified with the well known conjugate momentum π_μ [15] of the electromagnetic field in vacuo, i.e.

$$\epsilon_0 E_\mu = i\pi_\mu = i\frac{\partial \mathcal{L}_E}{\partial\left(\frac{\partial A_\mu}{\partial x^{(0)}}\right)}, \qquad x^{(0)} = ict, \tag{183}$$

where

$$\mathcal{L}_E = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{\partial A^{(0)}}{\partial x^{(0)}}\frac{\partial A_\mu}{\partial x_\mu}, \qquad A^{(0)} = \frac{i\phi}{c}, \tag{184}$$

is a novel Lagrangian with an appropriate Feynman gauge fixing term [15]. This is the second term on the right hand side of Eq. (184) and has the required dimensions, symmetry and scalar character. This term is also covariant, and invariant to gauge transformation of the second kind (Sec. 3) because of the Lorentz condition,

$$\frac{\partial A_\mu}{\partial x_\mu} = 0. \tag{185}$$

It replaces the habitual [15] Feynman gauge fixing term, i.e. $-\frac{1}{2}(\partial A_\mu/\partial x_\mu)^2$, in the theory of electromagnetic field quantization in the Lorentz gauge, and has the key advantage of producing the self consistent result,

$$\pi^{(0)} = -\frac{\partial A^{(0)}}{\partial x^{(0)}} = \epsilon_0 E^{(0)}, \tag{186}$$

which is Eq. (129b). This is demonstrated later, and removes a shortcoming of the conventional theory, described for example by Ryder [15], and which results in

$$\pi^{(0)} = 0, \tag{187}$$

making it difficult to quantize the electromagnetic field. The reason for this difficulty is that $A^{(0)}$ commutes with its conjugate momentum component $\pi^{(0)}$ if the latter vanishes. Thus $A^{(0)}$ becomes a c-number [15] and loses meaning as an operator. This is equivalent to a loss of manifest covariance in π_μ, which is properly a physically meaningful four-vector, because it is the conjugate canonical momentum of A_μ. The novel Lagrangian (184) restores meaning to π_μ, which becomes

$$i\pi_\mu = (\pi_i, i\pi^{(0)}), \tag{188}$$

and is fully covariant and fully consistent with the theory of special relativity.

It is important to note that Eq. (187) introduces difficulty into the quantization of the classical Maxwellian field, even with the use of the Lorentz gauge. We abandon the Coulomb gauge because of the novel Lie algebra of Sec. 4.

It is demonstrated at this stage that Eq. (184) is consistent with the d'Alembert equation, a demonstration that uses the Euler-Lagrange equation of motion,

$$\frac{\partial \mathcal{L}_E}{\partial A_\mu} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial \mathcal{L}_E}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)}\right) = 0. \tag{189}$$

We have [15]

$$\frac{\partial \mathcal{L}_E}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)} = -\left(\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}\right) - g_{\mu\nu}\frac{\partial A^{(0)}}{\partial x^{(0)}}, \qquad \frac{\partial \mathcal{L}_E}{\partial A_\mu} = 0, \tag{190}$$

where $g_{\mu\nu}$ is the Minkowski metric tensor, which vanishes for $\mu \neq \nu$. For $\mu = \nu = 0$, then $g_{00} = 1$, and

$$\pi^{(0)} \equiv \frac{\partial \mathcal{L}_E}{\partial\left(\frac{\partial A^{(0)}}{\partial x^{(0)}}\right)} = -\frac{\partial A^{(0)}}{\partial x^{(0)}} = \epsilon_0 E^{(0)}, \tag{191}$$

which is Eq. (186). The space-like part of π_μ is given from Eq. (190) by setting $\nu = 0$, $\mu \neq 0$, so $g_{\mu 0} = 0$, and

$$-i\pi_i \equiv \frac{\partial \mathcal{L}_E}{\partial\left(\frac{\partial A_i}{\partial x^{(0)}}\right)} = \frac{\partial A^{(0)}}{\partial x_i} - \frac{\partial A_i}{\partial x^{(0)}} = -i\epsilon_0 E_i, \qquad \pi_i = \epsilon_0 E_i. \tag{192}$$

There is therefore a simple proportionality between the manifestly covariant four-vectors π_μ and E_μ, proving that the latter is interpretable as the conjugate canonical momentum of A_μ. This result has been arrived at through an appropriate choice of gauge fixing term in the Lagrangian of the Euler-Lagrange equation of motion. This choice effectively adds zero to the original Lagrangian because we are working within the Lorentz gauge, and using the Lorentz condition (185). This link between E_μ and A_μ is a direct result of the fundamental Euler-Lagrange equation (189) given this gauge fixing term, and is another way of showing that E_μ is a physically meaningful four-vector if A_μ is such a vector. This implies that the scalar potential ϕ, defined by

$$A^{(0)} \equiv i\frac{\phi}{c}, \tag{193}$$

cannot be set to zero, as is the customary procedure [15]. The usual Feynman gauge fixing term $-\frac{1}{2}(\partial A_\mu / \partial x_\mu)^2$ is chosen so that $\pi^{(0)} = 0$, but if $E^{(0)}$ is non-zero, then $\pi^{(0)}$ is non-zero. As we have seen, the use of the four-vectors E_μ and B_μ satisfies the Planck radiation law given the novel Lie algebra of Sec. 4.

The d'Alembert equation is recovered from Eqs. (189) and (190) by using

$$\frac{\partial}{\partial x_\nu}\left(\frac{\partial \mathcal{L}_E}{\partial\left(\frac{\partial A_\mu}{\partial x_\nu}\right)}\right) = -\frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}\right) - \frac{\partial}{\partial x_\nu}\left(g_{\mu\nu}\frac{\partial A^{(0)}}{\partial x^{(0)}}\right) = 0, \tag{194}$$

i.e.,

$$\frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\mu}{\partial x_\nu}\right) = \Box A_\mu = 0, \tag{195}$$

and so the Lagrangian (184) is consistent with the d'Alembert equation in vacuo. The existence of E_μ (and B_μ) therefore does not contradict this fundamental relation.

Quantization of the Maxwellian field in the Lorentz gauge becomes a self-consistent procedure with Eqs. (188), (191) and (192). The "position-momentum" equal time commutator is written as [15]

$$\left[\hat{A}_\mu(\boldsymbol{x}, t), \hat{\pi}_\nu(\boldsymbol{x}', t)\right] = i g_{\mu\nu}\delta^3(\boldsymbol{x} - \boldsymbol{x}'), \tag{196}$$

in the usual way, but now π_ν *is fully covariant*, having been identified through Eq. (183) with E_μ. The basic field commutator is therefore the fully covariant

$$\left[\hat{A}_\mu(\boldsymbol{x}, t), \hat{E}_\nu(\boldsymbol{x}', t)\right] = -\frac{g_{\mu\nu}}{\epsilon_0}\delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{197}$$

It is also clear that

$$\left[\hat{A}_\mu(\boldsymbol{x}, t), \hat{A}_\nu(\boldsymbol{x}', t)\right] = \left[\hat{E}_\mu(\boldsymbol{x}, t), \hat{E}_\nu(\boldsymbol{x}', t)\right] = 0. \tag{198}$$

This commutator must be carefully distinguished from a commutator such as Eq. (155), which commutes $\hat{E}^{(1)}$, defined as a boost generator, with $\hat{E}^{(2)}$, the complex conjugate of this boost generator. In Eq. (198), $\hat{E}_\mu$ and $\hat{E}_\nu$ are not complex conjugates.

There is a critically important difference, therefore, between the method proposed here and the traditional method [15] in which $\pi^{(0)}$ (and thus $E^{(0)}$) *vanish*. In the traditional method, the existence of E_μ is not recognized, but the space-like π_i is at the same time identified with the space-like E_i. The traditional method *recognizes*, therefore, that there must be a four-vector π_μ, and that the space-like part of π_μ is directly proportional to E_i, but perversely sets its time-like component, $\pi^{(0)}$, to zero, and in this way destroys manifest covariance. Clearly, the time-like part of π_μ, i.e. $\pi^{(0)}$, should properly have the same units as its space-like component, and therefore should be proportional to electric field strength amplitude in volt m^{-1}. We propose that π_μ must be proportional to E_μ, whose time-like part, $E^{(0)}$, is non-zero in general and proportional to $\pi^{(0)}$ (Eq. (191)). This leads to a basic commutator, Eq. (197), which is fully consistent with the d'Alembert equation, and with the Euler-Lagrange equation of motion of the Maxwellian field. The four-vector E_μ becomes the canonical momentum of A_μ, both being fully, i.e. manifestly, covariant in the theory of special relativity. The four-vector B_μ is related to E_μ by the dual transforms of special relativity,

$$E_\mu \to icB_\mu, \qquad cB_\mu \to -iE_\mu, \tag{199}$$

which leaves Maxwell's equations invariant in vacuo. (Note that $E_\mu \to icB_\mu$ does not mean "E_μ is equal to icB_μ", but that E_μ is replaced by icB_μ in the Maxwell equations, which are then unchanged.) This is discussed in more detail in Sec. 7. By defining E_μ as being proportional to the conjugate momentum π_μ, of A_μ, it is clear that E_μ must behave under Lorentz transformation in the same way as

π_μ, which is in turn defined through A_μ by the Euler-Lagrange equation of the Maxwellian field. The Lagrangian of this equation contains the four-tensor $F_{\mu\nu}$, thus establishing a link between E_μ, A_μ, and $F_{\mu\nu}$.

The use of E_μ has the major advantage of retaining the Lorentz condition as a meaningful operator identity, because $\partial A_\mu/\partial x_\mu$ is no longer equal to $\pi^{(0)}$ as in the traditional method [15], which depends on the traditional Feynman gauge fixing term $-\frac{1}{2}(\partial A_\mu/\partial x_\mu)^2$. *If E_μ is recognized as a four-vector, therefore, the Lorentz condition no longer conflicts with the basic commutation relations, (196) and (198), of the quantized Maxwellian field, and quantization becomes a self-consistent procedure.*

The well known Gupta Bleuler condition of relativistic quantum field theory then emerges directly from the expectation value of the quantized operator corresponding to the Lorentz condition evaluated between eigenstates $|\psi\rangle$ of the quantized field,

$$\left\langle\psi\left|\frac{\partial\hat{A}_\mu}{\partial x_\mu}\right|\psi\right\rangle = \left\langle\psi\left|\frac{\partial\hat{A}_\mu^{(+)}}{\partial x_\mu}+\frac{\partial\hat{A}_\mu^{(-)}}{\partial x_\mu}\right|\psi\right\rangle = \left\langle\psi\left|\frac{\partial\hat{A}_\mu^{(-)}}{\partial x_\mu}\right|\psi\right\rangle = \left\langle\psi\left|\frac{\partial\hat{A}_\mu^{(+)}}{\partial x_\mu}\right|\psi\right\rangle^* = 0. \tag{200}$$

This result implies the operator identity

$$\frac{\partial\hat{A}_\mu^{(+)}}{\partial x_\mu}|\psi\rangle = 0, \tag{201}$$

where $\hat{A}_\mu^{(+)}$ contains only annihilation operators [15]. The operator $\hat{A}_\mu^{(-)}$ contains only creation operators, and can never act on an eigenstate $|\psi\rangle$ to produce zero, i.e. $\hat{A}_\mu^{(-)}$ must create a quantum state different from zero by definition. The d'Alembert equation (195) can be solved and used in Eq. (201) as in the standard theory [15], leading to the condition,

$$\hat{a}^{(0)}|\psi\rangle = \hat{a}^{(3)}|\psi\rangle, \tag{202}$$

where $\hat{a}^{(3)}$ and $\hat{a}^{(0)}$ are longitudinal and time-like photon annihilation operators.

In our approach, this condition is derived from the Lorentz *operator* condition (200), which is now fully consistent with the fundamental commutators (196) and (198) of the quantized Maxwellian field. Our theory is manifestly covariant, and fully consistent with special relativity. This implies that $\hat{a}^{(0)}$ and $\hat{a}^{(3)}$ must be *physically meaningful photon operators*. It follows from Eq. (202) that

$$\langle\psi|\hat{a}^{(0)+}\hat{a}^{(0)}|\psi\rangle = \langle\psi|\hat{a}^{(3)+}\hat{a}^{(3)}|\psi\rangle. \tag{203}$$

Now, it may be shown [15] that the total energy of a collection of photons is given by the Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3 2k_0} k_0\left(\sum_{\lambda=1}^{3}(\hat{a}^{(\lambda)+}(k)\,\hat{a}^{(\lambda)}(k) - \hat{a}^{(0)+}(k)\,\hat{a}^{(0)}(k))\right), \tag{204}$$

so that the contributions of the longitudinal and time-like photons cancel, *leaving only those of the transverse states.*

This procedure is consistent with and equivalent to the definition of classical electromagnetic energy density in terms of the covariant products $E_\mu E_\mu$ and $B_\mu B_\mu$, in which the longitudinal and time-like components cancel, leaving contributions only from the transverse components. This is consistent, in turn, with the Planck radiation law, and with the fact that longitudinal photons have no effective Planck energy (see introduction). This is, furthermore, consistent with the Lie algebra of Eq. (7) and Sec. 4, which shows that the longitudinal fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are independent of frequency. It becomes ever clearer that $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are physically meaningful fields in vacuo.

The traditional approach, on the other hand, fails to recognize the existence of the novel Lie algebra of Sec. 4, and incorrectly concludes [15] that longitudinal photon states are not physically meaningful. The existence of $i\boldsymbol{E}^{(3)}$ (and of $\boldsymbol{B}^{(3)}$) is not recognized, and the time-like component of E_μ is set to zero by a gauge fixing term. The critical failure of the traditional approach is the failure to recognize the Lie algebraic relation between longitudinal and transverse Maxwellian fields.

These points are emphasized, finally, through the well known relations between the electric and magnetic components of the Maxwellian electromagnetic field and annihilation operators. In S.I. units [69],

$$\hat{E}^{(0)} = \xi_E\hat{a}^{(0)}, \quad \hat{E}^{(3)} = \xi_E\hat{a}^{(3)}, \quad \hat{B}^{(0)} = \xi_B\hat{a}^{(0)}, \quad \hat{B}^{(3)} = \xi_B\hat{a}^{(3)}, \tag{205}$$

where $\xi_E = \left(\frac{2\hbar\omega}{\epsilon_0 V}\right)^{\frac{1}{2}}$, $\xi_B = \left(\frac{2\mu_0\hbar\omega}{V}\right)^{\frac{1}{2}}$, and where V is the quantization volume [70]. Using these relations, Eq. (202) becomes

$$\hat{E}^{(0)}|\psi\rangle = \hat{E}^{(3)}|\psi\rangle, \quad \hat{B}^{(0)}|\psi\rangle = \hat{B}^{(3)}|\psi\rangle, \tag{206}$$

and Eq. (203) becomes

$$\langle\psi|\hat{E}^{(0)*}\hat{E}^{(0)}|\psi\rangle = \langle\psi|\hat{E}^{(3)*}\hat{E}^{(3)}|\psi\rangle, \quad \langle\psi|\hat{B}^{(0)*}\hat{B}^{(0)}|\psi\rangle = \langle\psi|\hat{B}^{(3)*}\hat{B}^{(3)}|\psi\rangle. \tag{207}$$

Equations (206) and (207) are the quantized counterparts of the classical Maxwellian equations,

$$E^{(0)}\boldsymbol{k} = \boldsymbol{E}^{(3)}, \qquad B^{(0)}\boldsymbol{k} = \boldsymbol{B}^{(3)}, \tag{208}$$

and

$$E^{(0)2} = E^{(3)2}, \qquad B^{(0)2} = B^{(3)2}, \tag{209}$$

which emerge from the Lie algebra of Eq. (7) and Sec. 4. These relations imply that $E^{(0)}$ is the time-like component of E_μ and $B^{(0)}$ is the time-like component of B_μ.

6.1 Lorentz Transformation of The Four-Vector $\boldsymbol{E}_\mu$

In the foregoing we have followed the traditional notation for π_μ, and therefore for E_μ. From Eq. (190), however, it is clear that $\pi_{\mu\nu}$ is in general a four-tensor, and can be simply related to the well-known electromagnetic four-tensor $F_{\mu\nu}$,

$$\pi_{\mu\nu} = -\left(F_{\mu\nu} + g_{\mu\nu}\frac{\partial A^{(0)}}{\partial x^{(0)}}\right). \tag{210}$$

The notation described, for example, by Ryder [15], is, however, in terms of a four-vector as we have seen. Self-consistency of notation and meaning must therefore be obtained as follows. Scalar, time-like, elements of the tensor $\pi_{\mu\nu}$ are obtained by setting $\mu = \nu = 0$; and vector (space-like) elements by setting $\mu = \mu$, $\nu = 0$. We therefore write,

$$\pi^{(0)} \equiv \pi_{00}, \qquad \pi_i \equiv \pi_{\mu 0}. \tag{211}$$

The time-like element is thereby linked to the trace of $\pi_{\mu\nu}$ and the space-like element to the off-diagonals. Since $\pi_{\mu\nu}$ is directly related to $F_{\mu\nu}$ (Eq. (210)), the same conclusion holds for $F_{\mu\nu}$. *This is consistent with our novel analysis of $F_{\mu\nu}$* in Sec. 5, where the diagonal elements were shown to be related to the time-like $E^{(0)}$ and $B^{(0)}$. Since $\pi^{(0)}$ is time-like, π_i is space-like, they are components of the four-*vector* π_μ, which is the usual π_μ of the traditional

development [15]. Therefore π_μ, and therefore E_μ, is a four-vector whose elements can be related to elements of the four-tensor $\pi_{\mu\nu}$. The latter is related to the well-known electromagnetic four-tensor $F_{\mu\nu}$ through Eq. (210).

The Lorentz transformation of $\pi_{\mu\nu}$ is the same as that of $F_{\mu\nu}$ plus the novel term $g_{\mu\nu}\frac{\partial A^{(0)}}{\partial x^{(0)}}$. It is well known that, in a Cartesian basis,

$$F'_{\mu\nu} = a_{\mu\lambda}a_{\nu\sigma}F_{\lambda\sigma}, \tag{212}$$

where

$$a_{\mu\nu} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix}, \qquad \beta = \frac{v}{c}, \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}, \tag{213}$$

showing that the transformation of electric components of $F_{\lambda\sigma}$ produces a mixture of electric and magnetic components as in Einstein's original paper of 1905, a result whose space-like part is summarized in Eq. (162). From Eq. (210) this is seen to be consistent with the existence of π_i, and therefore of E_i. To complete the analysis it is necessary to consider the Lorentz transformation of the novel term $g_{\mu\nu}\frac{\partial A^{(0)}}{\partial x^{(0)}}$, a product of the metric tensor $g_{\mu\nu}$ and the quantity $\frac{\partial A^{(0)}}{\partial x^{(0)}}$, defining the non-zero time-like component $\pi^{(0)}$ or $E^{(0)}$. By definition, Lorentz transformation is the rotation in four-space, therefore each quantity must be transformed consistently, or covariantly. The metric $g_{\mu\nu}$, being a tensor, transforms in the same way as $F_{\mu\nu}$.

The transformation of the term $\partial A^{(0)}/\partial x^{(0)}$ is accomplished by using the transformation of the Lorentz condition. It is well known that

$$\frac{\partial A_\mu}{\partial x_\mu} = \frac{\partial A'_\mu}{\partial x'_\mu}, \tag{214}$$

i.e.,

$$\nabla\cdot\boldsymbol{A} + \frac{\partial A^{(0)}}{\partial x^{(0)}} = \nabla'\cdot\boldsymbol{A}' + \frac{\partial A^{(0)\prime}}{\partial x^{(0)\prime}}. \tag{215}$$

From the definitions, Eq. (129) of $E^{(0)}$ and $B^{(0)}$, and using the fact that $i\boldsymbol{E}^{(3)}$

and $\boldsymbol{B}^{(3)}$ always travel at the speed of light in the Maxwellian field,

$$B_Z' = B_Z, \qquad E_Z' = E_Z. \tag{216}$$

It follows that

$$B^{(0)\prime} = B^{(0)}, \qquad E^{(0)\prime} = E^{(0)}, \tag{217}$$

i.e. that

$$\nabla\cdot\boldsymbol{A} = \nabla'\cdot\boldsymbol{A}', \tag{218a}$$

and

$$\frac{\partial A^{(0)}}{\partial x^{(0)}} = \frac{\partial A^{(0)\prime}}{\partial x^{(0)\prime}}. \tag{218b}$$

The standard development [15] of the four-vector π_μ [15] implicitly reduces the four-*tensor* $\pi_{\mu\nu}$ to a four-vector. In this respect the standard treatment [15] is again confusing and incomplete. The confusion is compounded by the habitual use of reduced, (non-S.I.) units, in which $c = \hbar = 1$, and the permittivity in vacuo, ϵ_0, does not appear.

The existence of E_μ (and by implication B_μ) has therefore been shown to be consistent with the Euler-Lagrange equation of motion, the d'Alembert equation, the Lorentz transformation, and the four-curl of A_μ, the four-tensor $F_{\mu\nu}$. Quantization of the Maxwellian field using E_μ becomes manifestly covariant and consistent, the Lorentz condition becomes a well-defined operator condition from which follows the Gupta-Bleuler condition. These procedures are consistent with the Lie algebra of Sec. 4, and with the Planck radiation law.

On the other hand, the habitual procedure [15] fails to recognize the Lie algebra of Sec. 4, incorrectly asserts that $E^{(0)}$ is non-physical, so that the time-like component of the four-vector E_μ vanishes. This destroys manifest covariance, means that the time-like component of A_μ cannot be defined as an operator, means that the Hilbert space of photon particle states has an indefinite metric, and leads to negative expectation values for the Hamiltonian [15]. The key failure of the habitual theory, and of conventional electrodynamics in general, is its failure to recognize the Lie algebra of Sec. 4.

6.7 Quantization Of The Proca Field

The correct method of quantization of the Proca field has been briefly summarized in the introduction, following an article by Vigier [32]. The Proca equation of 1930 is a wave equation in Minkowski space-time, and is therefore an eigenfunction equation of the Schrödinger type, in which A_μ is regarded as a wavefunction. In view of the above development for the Maxwellian field, and in view of the evidence reviewed in the introduction for finite photon mass, the concept of E_μ must be introduced in such a way that it is consistent with the Proca field. The Lagrangian,

$$\mathcal{L}_E = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}m_0^2 c^4 A_\mu A_\mu - \frac{\partial A^{(0)}}{\partial x^{(0)}}\frac{\partial A_\mu}{\partial x_\mu}, \tag{219}$$

produces the required result, the only difference being the presence of the photon mass term. The conventional treatment of the Proca field [15] in an electromagnetic context is clearly inconsistent, because it relies on a mass term in the Lagrangian, but at the same time asserts that the mass of the photon is identically zero. Nevertheless, the conventional treatment does emphasize that for massive particles, quantization of the Proca field is self-consistent. From the treatment by Vigier [32] this is obviously due to the fact that the Proca equation is a Schrödinger type equation. It should be noted, however, that the traditional view again asserts that the time-like part of π_μ is identically zero, even in the Proca field. This is fundamentally inconsistent, in the electromagnetic field, with the Lie algebra of Sec. 4, which assumes central importance in electrodynamics, classical and quantum.

7 Discussion: Survey of Experimental Evidence for $\boldsymbol{B}^{(3)}$

It has been shown theoretically that there exist longitudinal solutions of Maxwell's equations in vacuo, denoted $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$, which are phase independent, and which are related to the usual transverse solutions by closed, i.e. cyclical, algebraic relations developed in this article. The experimentally observed and confirmed inverse Faraday effect is evidence for $\boldsymbol{B}^{(3)}$, and other magneto-optic effects such as light shifts, optical NMR, and the optical Faraday effect can be explained in terms of $\boldsymbol{B}^{(3)}$. In general, the well known conjugate product of nonlinear optics is directly proportional to $\boldsymbol{B}^{(3)}$ through the scalar E_0^2, and therefore $\boldsymbol{B}^{(3)}$ can be expressed in terms of the well known third Stokes parameter, S_3, of circularly polarized electromagnetic radiation in vacuo,

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}, \qquad B^{(0)} = \left(\frac{\epsilon_0}{I_0 c}\right)^{\frac{1}{2}}|S_3|. \tag{220}$$

It follows that well known optical phenomena, usually described in terms of S_3, can also be described in terms of $\boldsymbol{B}^{(3)}$, proving once again the latter's physical significance. Thus, S_3 can be replaced wherever it occurs in optics, or, for example, Rayleigh refringent scattering theory, by its equivalent in vacuo,

$$S_3 = \pm c^2 B^{(0)} |\boldsymbol{B}^{(3)}| = \pm c^2 \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} . \tag{221}$$

Kielich [72] has shown that in material media, as opposed to free space, linear and nonlinear optical activity depend on S_3, and in the Rayleigh theory [73] of natural optical activity in chiral media, it is well known that whatever the nature of the several molecular property tensors participating in the polarization and magnetization of the material, the observable of circular dichroism has pseudoscalar symmetry and is proportional to the third Stokes parameter. For different enantiomers for a given sense of circular polarization, or for one enantiomer for different sense of transverse circular polarization,

$$\frac{I_R - I_L}{I_R + I_L} = \pm \frac{S_3}{S_0} , \tag{222}$$

where I_R and I_L are the intensities of right and left components transmitted by structurally chiral material, with,

$$I_0 = I_R + I_L , \tag{223}$$

for the transmitted total beam intensity. Therefore

$$\pm \frac{\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)}}{B^{(0)2}} = \frac{I_R - I_L}{I_R + I_L} , \tag{224}$$

which shows that circular dichroism can be described in terms of $\boldsymbol{B}^{(3)}$ at all electromagnetic frequencies, $(I_R - I_L)$ being proportional to $\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)}$.

For all practical purposes therefore, *the same conclusion holds in the Proca field*, because $\boldsymbol{B}^{(3)}$ in the Maxwellian and Proca fields are indistinguishable practically. In this sense, therefore, circular dichroism, and all phenomena dependent on S_3, are consistent with finite photon mass, and are manifestations of the photon's longitudinal magnetic field.

The observable $I_R - I_L$ is therefore a spectral consequence of the interaction of $\boldsymbol{B}^{(3)}$ or $i\boldsymbol{E}^{(3)}$ with structurally chiral material, being proportional to the real quantity $\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)}$ after it emerges from the chiral material through which the beam has passed, i.e. after interaction has occurred between the flux

quantum $\boldsymbol{B}^{(3)}$ and the appropriate molecular properties. For one photon, the observable $I_R - I_L$ provides an experimental measure of the transmitted elementary $\boldsymbol{B}^{(3)}$ at each frequency. Although $\boldsymbol{B}^{(3)}$ is itself independent of that frequency, the interacting molecular property tensor is not. Semi-classical perturbation theory gives, for linear optical activity:

$$\frac{S_3}{S_0} = \frac{\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)}}{B^{(0)2}} = \tanh\left(\omega \mu_0 c\, l N \xi''_{XYZ}(g)\right) , \tag{225}$$

where μ_0 is the permeability in vacuo, ω the angular frequency of the beam, l the sample path length, and ξ'' is a combination of molecular property tensors which may be electric or magnetic in nature. For nonlinear optical activity, Eq. (225), as first shown by Kielich [74], contains additional terms. Therefore, every time natural optical activity is observed with $I_R - I_L$, as in circular dichroism, the field $\boldsymbol{B}^{(3)}$ has been measured. In the Proca formalism, this is inevitably associated with finite photon mass, and the Maxwellian counterpart is a practically indistinguishable limiting form where photon mass goes to zero.

These conclusions follow directly from the relation between $\boldsymbol{B}^{(3)}$ and the conjugate product, (expressible, for example, as Eq. (7a)), and the latter's well known relation to the third Stokes parameter S_3. (In the quantized field the latter becomes the well known third Stokes operator.) Note that S_3 is intrinsically frequency independent by definition, but can still be used to describe frequency dependent phenomena such as circular dichroism, as in Eq. (222). A similar conclusion follows for $\boldsymbol{B}^{(3)}$, the reason being that the frequency dependence of the spectral phenomenon is to be found in the molecular property tensor, in which it appears through perturbation theory. This is a new way of interpreting the well known and well measured phenomenon of circular dichroism, and, more generally, any phenomenon that depends on S_3 and therefore on $\boldsymbol{B}^{(3)}$. Through algebraic relations such as Eq. (7a), this includes the traditional interpretation, describable in a Cartesian basis by

$$\frac{S_3}{S_0} = -\frac{i\left(E_X E_Y^* - E_Y E_X^*\right)}{E_X E_X^* + E_Y E_Y^*} , \tag{226}$$

but extends its meaning to longitudinal fields, which can be associated with finite photon mass. There are several basic optical phenomena which are described customarily in terms of the Stokes parameter S_3, for example the development of ellipticity in an initially circularly polarized light beam. In the electric Kerr effect [75], beam ellipticity (η) is expressed in terms of S_3, and is induced with an electric field in a probe laser. The description of this phenomenon is therefore,

$$\frac{S_3}{S_0} = \frac{\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)}}{B^{(0)2}} = \sin(2\eta), \tag{227}$$

where η is the ellipticity developed in the transmitted probe as a result of the application of an electric field to a sample. This effect is therefore experimental evidence for $\boldsymbol{B}^{(3)}$. Proceeding in this way, it becomes clear that there are many different aspects of traditional linear optics that can be reinterpreted in terms of $\boldsymbol{B}^{(3)}$. The scalar magnitudes of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are $B^{(0)}$ and $E^{(0)}$ respectively, associated in the four-vector representations B_μ and E_μ with the time-like components. The time-like polarization always appears as an admixture with the longitudinal polarization, and both are physically meaningful because they are observed in fundamental optical phenomena. Equation (225) for example shows that circular dichroism is related to the molecular property tensor sum represented by ξ'', which is made up of the Rosenfeld tensor and the electric quadrupole tensor. However, ξ'' is a material property, while both $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are properties of free space-time which interact with matter, in the same way as S_3. The latter can be defined without any reference to matter, and the definition of S_3 in terms of $\boldsymbol{B}^{(3)}$ is unaffected by any material property. In Eq. (225), we have used the result that $I_R - I_L$ is directly proportional to the Stokes parameter S_3 in free space-time, and have replaced the Stokes parameter by a term proportional to $\boldsymbol{B}^{(3)2}$.

Realizing the link between S_3 and $\boldsymbol{B}^{(3)}$ shows that there is in fact copious evidence for $\boldsymbol{B}^{(3)}$, and that all of this evidence is consistent with finite photon mass. Through a Lie algebra such as that of Eq. (7), the customary description is supplanted in physical optics by a more complete understanding, one which is fully consistent with special relativity and one which does not arbitrarily discard $\boldsymbol{B}^{(3)}$. This leads in turn to an appreciation of the role played by finite photon mass.

More generally, ordinary optical absorption, described customarily by the Beer Lambert law:

$$\alpha(\bar{\nu}) = \frac{1}{d}\log_e\frac{I_0}{I}, \tag{228}$$

can be interpreted anew in terms of $\boldsymbol{B}^{(3)}$. Here I_0 is the incident beam intensity, I the transmitted beam intensity, and d the sample length. $\alpha(\bar{\nu})$ is the power absorption coefficient in neper cm^{-1}, a quantity which can be expressed in terms of the longitudinal field $i\boldsymbol{E}^{(3)}$ because the zeroth Stokes parameter S_0 is proportional to beam intensity. Therefore simple optical absorption (at any frequency) is a process which can be interpreted in terms of $\boldsymbol{B}^{(3)}$, and is therefore consistent with finite photon mass. The traditional interpretation remains valid as far as it goes, but is supplemented by the new type of algebra exemplified by Eq. (7), an algebra from which these conclusions

follow directly. All four Stokes parameters (or operators of the quantized field) can be expressed in terms of $\boldsymbol{B}^{(3)}$ as well as in terms of the traditional transverse components. It gradually becomes clear therefeore that the subject of optics in general is enriched by the realization that $\boldsymbol{B}^{(3)}$ is non-zero in vacuo, and that there are many new interpretations possible.

In nonlinear optics [28] the light beam is used to induce phenomena in material media (e.g. molecular matter); phenomena which depend nonlinearly on the electric and magnetic components of the intense laser beam. A large number of such phenomena are known [28], both in the classical and quantum field formalisms of magneto and electro-optics. In principle, all can be reinterpreted in terms of longitudinal fields and in this sense all are consistent with finite photon mass. For all practical purposes the Proca and Maxwellian formalisms give the same results in nonlinear optics of this type. Taking into consideration [32] however, astrophysical and cosmological evidence for finite photon mass, all laboratory phenomena must be interpreted consistently; i.e. if evidence is found for finite m_0 from cosmology, then all laboratory phenomena must also be described in terms of finite m_0. The novel longitudinal fields of this article provide a convenient means of doing so through the Maxwellian limit of the Proca field $\boldsymbol{B}^{(3)}$.

A comprehensive and rigorously systematic scheme for nonlinear optics is available [28] in the work of Kielich and co-workers. This is of course formulated in traditional terms, in which $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are not used specifically, but the whole of this *oeuvre* can be re-worked in principle in terms of the longitudinal fields, while at the same time retaining the intricate tensorial structure of the original work [28]. An example of how this may be done is the replacement of the nonlinear conjugate product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$ by $iE_0c\boldsymbol{B}^{(3)}$. This is more than a mere re-expression of the well known $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$, because $\boldsymbol{B}^{(3)}$, being a magnetic field, interacts in principle with a magnetic dipole moment (electronic or nuclear). This interaction occurs in addition to that of the product $iE_0c\boldsymbol{B}^{(3)}$, which of course is $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$, with antisymmetric electronic polarizability. This is one example of how the tensorial formalism of nonlinear optics [28] may be developed. In tensor notation the conjugate product is the antisymmetric part of $E_iE_j^*$, the light intensity tensor [28], an idea developed systematically by the Kielich School and many others. The antisymmetric part of the tensor is an axial vector.

We proceed to sketch a few suggestions for development based on classic papers by Kielich *et al*.

In an early work, Kielich [76] has considered frequency and spatially variable electric and magnetic polarization induced in nonlinear media by electromagnetic fields, using Born Infeld electrodynamics. The new term $\boldsymbol{B}^{(3)}$ should be systematically incorporated in this work, and novel nonlinear effects predicted. Nonlinear processes resulting from multipole interaction between molecules and electromagnetic fields [77] can also be considered in terms of $\boldsymbol{B}^{(3)}$ at first and higher orders, a structure that would be consistent with finite photon mass. Examples include scattering theory based on $\boldsymbol{B}^{(3)}$; the role of $\boldsymbol{B}^{(3)}$ in nonlinear optical processes where linear superposition is lost; investigations of the probability of an n photon process with magnetic transitions involving an incoming $\boldsymbol{B}^{(3)}$ flux quantum. In the theory [78] of nonlinear light scattering from colloidal media [79], $\boldsymbol{B}^{(3)}$ is expected to play a basic part in defining the polarization ratio, because $\boldsymbol{B}^{(3)}$ is proportional to $I_R - I_L$. In general in Rayleigh refringent scattering theory, the Stokes parameters can be described in terms of $\boldsymbol{B}^{(3)}$. The role of $\boldsymbol{B}^{(3)}$ in phenomena such as the Majorana effect and intensity dependent circular birefringence is also fundamental. Ellipse self rotation [80] by a circularly polarized laser is also fundamentally dependent on the longitudinal $\boldsymbol{B}^{(3)}$.

The simple measurement of beam intensity does not, however, appear to allow the detection of longitudinal photons, at least in a direct and simple way. Although finite photon mass implies the existence of a third degree of freedom for the photon, suggesting a 50% increase in the stored energy [6] of a system of photons, this cannot be observed experimentally. The Planck radiation law, for example, is derived customarily using only transverse polarizations, and is found to hold precisely in comparison with experimental data. Bass and Schrödinger [35] were among the first to explain how the Planck law can remain valid even for finite photon mass by showing that the approach to equilibrium of longitudinal photons in a cavity is very slow, and energy stored in transverse waves takes a time approximately similar to the age of the universe to be partitioned equally among three degrees of freedom. Farahi and Evans [27] have shown recently that longitudinal electric and magnetic solutions $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ in the Maxwellian limit do not contribute to the electromagnetic energy density. Their method was based on the simple assumption that the most general solution of Maxwell's equations are of the form,

$$\boldsymbol{E}^{G} = \boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{E}^{(3)}, \quad \boldsymbol{B}^{G} = \boldsymbol{B}(\boldsymbol{r}, t) + \boldsymbol{B}^{(3)}, \tag{229}$$

leading to the result that the continuity equation for electromagnetic radiation in vacuo is unchanged if (Appendix C)

$$\boldsymbol{B}^{(3)} \times \boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}^{(3)} \times \boldsymbol{B}(\boldsymbol{r}, t). \tag{230}$$

It was shown that [27] this result is compatible with the dual transformations (199), a demonstration that is valid for electromagnetic waves in free space, rather than for the cavity fields considered by Goldhaber and Nieto [6]. The conclusion by Farahi and Evans [27] is consistent with the representation of electromagnetic energy density in terms of products such as $E_\mu E_\mu$ and $B_\mu B_\mu$ in free space, products in which longitudinal and time-like components cancel; a result which is also obtained from the d'Alembert equation and given as Eq. (204) of this article.

Finally, in its quantized version, $\hat{B}^{(3)}$ can be expressed as [27]

$$\hat{B}^{(3)} = \frac{B^{(0)}}{2}\left(\hat{a}_X \hat{a}_Y^{+} - \hat{a}_Y \hat{a}_X^{+}\right)\boldsymbol{k}, \tag{231}$$

showing that $\hat{B}^{(3)}$ and the conjugate product operator $\hat{E}^{(1)} \times \hat{E}^{(2)}$ are rigorously proportional in quantum field theory. Both are described by the operator $\left(\hat{a}_X \hat{a}_Y^{+} - \hat{a}_Y \hat{a}_X^{+}\right)$ whose expectation value between quantum states of the electromagnetic field is always a constant, 2. The Stokes operator $\hat{S}_3$ in this notation is [27]

$$\hat{S}_3 = -\frac{E_0^2}{2}\left(\hat{a}_X \hat{a}_Y^{+} - \hat{a}_Y \hat{a}_X^{+}\right), \tag{232}$$

i.e. $\hat{B}^{(3)}$ is defined in terms of $\hat{a}_X \hat{a}_Y^{+} - \hat{a}_Y \hat{a}_X^{+}$, which operates on any number state $|n\rangle$ to give the constant expectation value of 2. The latter is independnet of the number state $|n\rangle$ of the photons, and generalizes the third Stokes parameter S_3 of the classical field. This is another way of seeing that $\hat{B}^{(3)}$ does not contribute to electromagnetic energy density, which is described by the zero'th order Stokes parameter, S_0. The expectation value of the energy of n photons is

$$\langle n|\hat{H}|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega, \tag{233}$$

and depends on n. Since the expectation value of $\hat{B}^{(3)}$ does not depend on n, it cannot contribute to electromagnetic energy.

Nevertheless, $\hat{B}^{(3)}$ acts as a field throughout electromagnetism, and is expected to provide useful new techniques such as optical NMR and optical magnetic resonance imaging, in which the image is enhanced by $\hat{B}^{(3)}$. These techniques are currently under development.

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Appendix A. Special Relativity and Consistency of Notation

Unfortunately, the literature in special relativity uses different types of notation, exemplified by that of Ryder [15], and that of Jackson [71]. For readers unfamiliar with this use of notation, this appendix briefly sketches the foundations of special relativity, and explains the notation used throughout this article. To avoid ambiguity and possible confusion, a comparison is provided of different notations whenever necessary.

A clear account of the foundations of special relativity is given by Jackson [71], chapter 11, and this is summarized here.

Maxwell's equations in vacuo are invariant (Appendix B) to the Lorentz transformation of special relativity. This was shown by Lorentz in 1904. Shortly afterwards, Poincaré showed that all the equations of electrodynamics are similarly invariant. These results were proven independently by Einstein in 1905, and shown in the theory of special relativity to be generally applicable in physics. Einstein based his theory on two general principles, the first asserts that the laws of physics take the same form in all Lorentz frames; the second asserts that the constant c is the same in all Lorentz frames. The latter is the speed of light in the Maxwellian theory of electrodynamics, but not in the Proca theory (see introduction to the text). The constant c is independent of the motion of the source. The postulated constancy of c allows a connection [71] to be made between Minkowski space-time coordinates in different frames, customarily labelled K and K', the latter moving at v with respect to the former along the Z axis.

In special relativity it is convenient to use pseudo-Euclidean space-time rather than Euclidean space. The pseudo-Euclidean frame of reference is written as (X, Y, Z, ict) in the notation of this article, following the original

proposal by Minkowski, circa 1906. In the notation of Ryder [15] however, the i is suppressed, and the time-like component written first. The Ryder notation is common in contemporary field theory. In our notation

$$K = (X, Y, Z, ict), \qquad K' = (X', Y', Z', ict'). \tag{A1}$$

Note that the only quantity that does not change from K to K' is ic, time changes in the Lorentz transformation as well as space, and space and time are no longer distinct.

It is assumed [71] that the transformation $K \to K'$,

$$X'^2 + Y'^2 + Z'^2 - c^2t'^2 = \lambda^2(X^2 + Y^2 + Z^2 - c^2t^2), \tag{A2}$$

where λ is a function of v such that $\lambda(0) = 1$. It can be shown [71] that λ is unity for all v. If $v = 0$, K' does not move with respect to K. In general, the Lorentz transform takes the form,

$$X' = X, \qquad Y' = Y, \qquad Z' = \gamma(Z - vt), \qquad t' = \gamma\left(t - \beta\frac{Z}{c}\right), \tag{A3}$$

where $\beta = \frac{v}{c}$, $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. This can be written in matrix form,

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ ict \end{bmatrix}, \tag{A4}$$

which in tensor notation becomes

$$x'_\mu = a_{\mu\nu} x_\mu . \tag{A5}$$

The four by four matrix in Eq. (A4) is known as the Lorentz transformation matrix and defines the boost generators of the text. Note that our boost generators are in Jackson notation, and are real. In Ryder type notation they are complex and look quite different. However, in both notations they obey the same commutation relations of the Lorentz and Poincaré groups. The Lorentz transformation matrix is also different in Ryder notation. The matrix (A4) defines the *boost generator* of the text as follows. We recognize that if

$$\gamma \equiv \cosh\phi, \quad \gamma\beta \equiv \sinh\phi, \quad \frac{v}{c} \equiv \tanh\phi, \quad \gamma^2(1-\beta^2) = 1, \tag{A6}$$

then

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\phi & i\sinh\phi \\ 0 & 0 & -i\sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ ict \end{bmatrix}, \tag{A7}$$

The boost generator in Z is defined as

$$\hat{K}_Z \equiv \frac{1}{i}\frac{\partial a_{\mu\nu}}{\partial\phi}\bigg|_{\phi=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{A8}$$

which is a *real*, unitless, antisymmetric, four by four matrix in the Jackson notation of special relativity. In the Ryder notation, $\hat{K}_Z$ is imaginary. *Throughout the text of this article we have used the Jackson notation, converted into S.I. units.*

For a frame K' moving with respect to K in the Y axis, we obtain

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & i\gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & -i\gamma\beta & 0 & \gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ ict \end{bmatrix}, \tag{A9}$$

and

$$\hat{K}_Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{A10}$$

Thirdly, for a frame K' moving with respect to K in the X axis,

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ ict' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ ict \end{bmatrix}, \tag{A11}$$

and

$$\hat{K}_X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{A12}$$

We see that the boost generators $\hat{K}_X$, $\hat{K}_Y$ and $\hat{K}_Z$ of the Lorentz group are antisymmetric 4 x 4 matrices. It can be checked that in the Jackson as well as the Ryder notation, they form the commutation relations of the Lorentz and Poincaré groups, for example,

$$[\hat{K}_X, \hat{K}_Y] = -i\hat{J}_Z \text{ \& cyclics,} \tag{A13}$$

as in the text.

Covariance of the Laws of Physics.

The first principle of Einstein asserts that the laws of physics have the same form in different Lorentz frames; meaning that the equations describing the physical laws must be *covariant* in form [71]. "Covariant" means that the equations can be written so that *both sides have the same, well-defined, transformation properties* under Lorentz transformation. Thus, physically meaningful equations must be relations between four-vectors, four-tensors, Lorentz scalars [71], and derivatives thereof. A relation valid in one frame must also be valid in the same form in another. We illustrate the Einstein principle of covariance (the first principle of special relativity) by reference to our novel Lagrangian (184) of the text, testing its covariance in the process. The Lagrangian is an energy, and therefore a scalar quantity which is the same in different Lorentz frames, i.e. is Lorentz invariant. Combining this with the principle of covariance means that the transformed Lagrangian must take the form:

$$\mathfrak{L}'_K = \mathfrak{L}_K = -\frac{1}{4}F_{\mu\nu}'F_{\mu\nu}' - \left(\frac{\partial A^{(0)'}}{\partial x^{(0)'}}\right)\left(\frac{\partial A'_\mu}{\partial x'_\mu}\right), \tag{A14}$$

in frame K'. It follows that

$$F_{\mu\nu}'F_{\mu\nu}' = F_{\mu\nu}F_{\mu\nu}, \qquad \frac{\partial A^{(0)'}}{\partial x^{(0)'}} = \frac{\partial A^{(0)}}{\partial x^{(0)}}, \qquad \frac{\partial A'_\mu}{\partial x'_\mu} = \frac{\partial A_\mu}{\partial x_\mu}, \tag{A15}$$

and this is indeed the case, as shown in more detail in Appendix B. Because of the Lorentz condition:

$$\frac{\partial A_\mu}{\partial x_\mu} = \frac{\partial A_\mu'}{\partial x_\mu'} = 0, \tag{A16}$$

the Lagrangians in both frames are also invariant to gauge transformation of the second kind (Sec. 3 of the text). The Lagrangian is therefore covariant and invariant in the required way.

Appendix B. The Formal Lorentz Transformation Properties of $\boldsymbol{E}_\mu$ and $\boldsymbol{B}_\mu$

The four-vectors E_μ and B_μ are light-like, as discussed in the text. The four- vector $x_\mu = (X, Y, Z, ict)$ is light-like if,

$$X^2 + Y^2 + Z^2 = c^2t^2, \tag{B1}$$

i.e., if,

$$x_\mu x_\mu = 0. \tag{B2}$$

As in the text

$$E_\mu E_\mu = 0, \qquad B_\mu B_\mu = 0, \tag{B3}$$

which is of the same form as Eq. (B2). The formal Lorentz transform of a light-like vector from frame K to K' takes place through the linear relation [71],

$$(X'^2 + Y'^2 + Z'^2 + c^2t'^2) = \lambda^2(X^2 + Y^2 + Z^2 + c^2t^2) = 0. \tag{B4}$$

For all λ,

$$X'^2 + Y'^2 + Z'^2 + c^2t'^2 = 0. \tag{B5}$$

Therefore the formal Lorentz transformation of x_μ gives another light-like four-vector x'_μ such that

$$x_\mu x_\mu = x'_\mu x'_\mu = 0 . \tag{B6}$$

Therefore,

$$E_\mu E_\mu = E'_\mu E'_\mu = 0 , \qquad B_\mu B_\mu = B'_\mu B'_\mu = 0 . \tag{B7}$$

The novel fields $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ of Sec. 4 of the text *vanish if there is no electromagnetic plane wave travelling at c in vacuo in the Maxwellian formalism of electromagnetism.* Therefore they are not conventional static electric and magnetic fields. The constant c, identified with the speed of light in the Maxwell equations in vacuo, is the same in all Lorentz frames by Einstein's second principle of relativity. This means that a Lorentz transformation from one frame to another cannot change the fact that $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ propagate at c in vacuo. To emphasize this point, consider the Maxwell equations in vacuo in frame K,

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0 . \tag{B8}$$

This means that the four-divergence of $F_{\mu\nu}$ vanishes. This four-divergence is by definition a four-vector in K, and Maxwell's equations can be interpreted to mean that this four-vector vanishes in K. In frame K', following Jackson [71], Eq. (B8) becomes,

$$\frac{\partial F'_{\lambda\sigma}}{\partial x'_\sigma} = 0, \tag{B9}$$

and the transformed four-vector also vanishes in frame K'. This transformed four-vector can be given the symbol $\boldsymbol{\alpha}'_\lambda$ in frame K'. By definition,

$$\alpha'_\mu \equiv a_{\mu\nu}\alpha_\nu , \tag{B10}$$

under the Lorentz transformation $a_{\mu\nu}$. Therefore,

$$a_{\lambda\mu}\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0, \tag{B11}$$

in frame K. We have simply back-transformed Eq. (B9) (frame K') to Eq. (B11) (frame K). Equation (B11) must be the same as Eq. (B8), and so,

$$a_{\lambda\mu}\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial F'_{\lambda\sigma}}{\partial x'_\sigma} = 0, \tag{B12}$$

showing that Maxwell's equations in frame K are the same as in frame K'. It is impossible to tell the difference between the original and transformed Maxwell equations *in vacuo*. Note that in matter, this is not the case, because [71]

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} \neq 0 . \tag{B13}$$

In vacuo, the transformation of a zero (i.e., $\frac{\partial F_{\mu\nu}}{\partial x_\nu}$) gives a zero.

If the Maxwell equations are indistinguishable in all Lorentz frames, then a four-vector defined *in vacuo*, such as E_μ, whose three space-like components are all solutions of Maxwell's equations in vacuo, will also appear the same in all Lorentz frames. Therefore,

$$E_\mu = E'_\mu , \qquad B_\mu = B'_\mu . \tag{B14}$$

From Eq. (B12) we have the result,

$$a_{\lambda\mu}\alpha_\mu = \alpha_\mu , \qquad \alpha_\mu = \frac{\partial F'_{\mu\nu}}{\partial x_\nu} , \tag{B15}$$

and this is possible if and only if

$$a_{\lambda\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{B16}$$

and in turn, this is compatible with the definition (Eq. (A14)) of $a_{\lambda\mu}$ as the Lorentz transformation matrix if and only if $v = 0$.

Physically, this means that if a quantity is moving at c, it is not possible, by Einstein's second principle, for c to be exceeded, and a formal Lorentz transformation is possible if and only if $v = 0$. There is no frame travelling faster than the speed of light, identified in the Maxwellian formalism with c.

In summary, it is possible to write, *formally*,

$$E'_\mu = a_{\mu\nu} E_\mu, \qquad B'_\mu = a_{\mu\nu} B_\mu, \tag{B17}$$

but v is always zero in $a_{\mu\nu}$ by Einstein's second principle of relativity. It is essential therefore to realize that $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are never static fields in the conventional meaning of electrostatics, well described by Jackson [71]. The field $\boldsymbol{B}^{(3)}$ for example is formed from the vector cross product (Eq. (7)) and Sec. 4) of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$, both of which are plane waves in vacuo, propagating at c by definition. Therefore, $\boldsymbol{B}^{(3)}$ must propagate at c, although its specific phase dependence has been removed by the cross product. As explained in the text, $\boldsymbol{B}^{(3)}$ is in general a quantum mechanical operator, and obviously differs from the static magnetic field of magnetostatics. The latter is not a quantized quantity, and must always be generated by a moving charge according to the Biot-Savart law [71]. The static electric field of electrostatics must be generated by a static point charge according to the Coulomb law [71].

It is well known that the Lorentz transformation of, for example, a static electric field in electrostatics produces a mixture of electric and magnetic fields. In S.I. units,

$$\begin{aligned}
E'_X &= \gamma(E_X - vB_Y), & E_X &= \gamma(E'_X + vB'_Y), \\
E'_Y &= \gamma(E_Y + vB_X), & E_Y &= \gamma(E'_Y - vB'_X), \\
E'_Z &= E_Z, & E_Z &= E'_Z, \\
B'_X &= \gamma\left(B_X + \frac{v}{c^2}E_Y\right), & B_X &= \gamma\left(B'_X - \frac{v}{c^2}E'_Y\right), \\
B'_Y &= \gamma\left(B_Y - \frac{v}{c^2}E_X\right), & B_Y &= \gamma\left(B'_Y + \frac{v}{c^2}E'_X\right), \\
B'_Z &= B_Z, & B_Z &= B'_Z.
\end{aligned} \tag{B18}$$

These well known space-like relations are compatible with Eq. (B18) *if and only if* $v = 0$. This means that E'_μ and B_μ can be defined if and only if they describe a plane wave travelling at c in vacuo in the Maxwellian formalism. The latter is for all practical (i.e. laboratory) purposes identical with the Proca formalism of electrodynamics in vacuo.

We claim no more and no less than this in the text of this article.

Appendix C. Conservation of Classical Electromagnetic Energy Density

If Eq. (230) is obeyed in classical electrodynamics, the longitudinal fields $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ cannot contribute to electromagnetic energy density in Maxwell's description [27]. This is observed experimentally, for example the Planck radiation law is built up from transverse components only. Equation (230) is obeyed in general by longitudinal fields with real and imaginary parts, for example it is obeyed if

$$\boldsymbol{B}^{(3)} = B_0(1 + i)\boldsymbol{k}, \qquad \boldsymbol{E}^{(3)} = E_0(i - 1)\boldsymbol{k}. \tag{C1}$$

If $\boldsymbol{B}^{(3)}$ is real (i.e. if its imaginary part is zero), then Eq. (230) is satisfied if and only if $i\boldsymbol{E}^{(3)}$ is imaginary, i.e. if the real part of $i\boldsymbol{E}^{(3)}$ is zero. This is consistent with the text, which shows that $\boldsymbol{B}^{(3)}$ from the Proca (or d'Alembert) equation is real, and that the real part of $i\boldsymbol{E}^{(3)}$ is zero, a result which was obtained on the basis of a gauge condition (2) which was obtained in turn from the assumption of finite photon mass.