Chapter 7

THEORY OF THE OPTICAL FARADAY EFFECT

M. W. Evans

Abstract

A circularly polarized pump laser applied to atomic or molecular matter rotates the plane of polarization of a linearly polarized probe laser whose propagation direction is parallel to that of the pump. We refer to this as the optical Faraday effect, and develop a theory based on the $\boldsymbol{B}^{(3)}$ field of the pump, a theory which shows that the molecular property tensors responsible for the optical Faraday effect (OFE) are the same as those which are used to describe the usual Faraday effect. This means that the effect of the $\boldsymbol{B}^{(3)}$ field of a pump laser is to shift an MCD spectrum upwards or downwards in frequency, depending on the sense of circular polarization of the pump laser. This effect occurs without distortion of the original MCD spectrum, and is precisely what has been observed experimentally recently by Sanford *et. al.* [1] in cadmium chromium selenium ferromagnetics. This is strong evidence for first order effects based on the $\boldsymbol{B}^{(3)}$ field of the pump laser. These first order OFE effects are distinguished from second order equivalents based on the conjugate product of the pump laser. The latter are mediated by different molecular property tensors, and would distort as well as shift the normal MCD profile of a sample treated with a circularly polarized pump laser. Such distortions were not observed by Sanford *et. al.* [1].

1. Introduction

Sanford *et. al.* [1] have recently observed an optical Faraday effect (OFE) due to optical pumping by a circularly polarized laser in $CdCr_2Se_4$. These authors noted a large shift in the Faraday rotation spectrum of this compound at 78 K, a shift caused by a circularly polarized pump laser. The shift was reversed by reversing the circular polarization of the pump, the displacement being to lower energy when right circularly polarized pump radiation was used. The intensity of the pump laser was only 0.7 watts cm^{-2}, and the sample thickness was 25 microns. It was observed that the complete Faraday rotation spectrum near 1000 nm was shifted without distortion to higher or lower frequencies by the pump laser, an effect which was interpreted in terms of a pump induced increment in the effective magnetic field. This effect was observed at remarkably low pumping levels and was reported to be ten times greater in circular than in linear polarization.

In this article we interpret these experimental results with the recently proposed magnetic field $\boldsymbol{B}^{(3)}$ of a circularly polarized laser, a field which is defined through the well known conjugate product [2–5]

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)}. \tag{1}$$

Here $\boldsymbol{B}^{(1)}$ is the complex conjugate of $\boldsymbol{B}^{(2)}$, the magnetic component of the usual transverse plane wave in free space. The symbol $B^{(0)}$ denotes the scalar amplitude of this magnetic component. From Eq. (1) it is seen that $\boldsymbol{B}^{(3)}$ is parallel to the propagation axis of the laser, and is frequency independent, as is the pure imaginary conjugate product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$. The field $\boldsymbol{B}^{(3)}$ reverses sign [6–9] with circular polarization and disappears in linear polarization. In S.I. units its free space magnitude is about $10^{-7} I_0^{1/2}$ tesla, where I_0 is the beam intensity in watt per square meter. For a beam of 10,000 watt per square meter (1.0 watt per square centimeter) its value is therefore about 10^{-5} tesla. This is about the same as that expected from the pump intensity (0.7 watt per square centimeter) reported by Sanford *et. al.* [1]. Since $\boldsymbol{B}^{(3)}$ is a magnetic field, it is expected to produce an optical Faraday effect, which is the subject of this article. Therefore we expect a Faraday spectrum to be shifted without distortion by $\boldsymbol{B}^{(3)}$ to higher or lower frequencies, as observed by Sanford *et. al.* [1].

In Sec. 2, it is shown that the introduction of the new longitudinal field $\boldsymbol{B}^{(3)}$ is consistent with conservation of electromagnetic energy. In Sec. 3, a simple example of the operation of $\boldsymbol{B}^{(3)}$ is given through the inverse Faraday effect for one electron. In Sec. 4, these considerations are extended to the inverse and optical Faraday effects in atoms and molecules, and it is shown that $\boldsymbol{B}^{(3)}$ is expected to produce both effects from a circularly polarized pump laser. The results of the $\boldsymbol{B}^{(3)}$ theory are compared with the experimental data of Sanford *et. al.* described already.

2. Conservation of Energy in the Presence of $\boldsymbol{B}^{(3)}$

That there is a connection between longitudinal and transverse solutions of Maxwell's equations in vacuo emerges from a consideration of the antisymmetric part of light intensity, i.e. the antisymmetric part of the tensor (in S.I. units),

$$I_{0ij} \equiv c\epsilon_0 E_i E_j, \tag{2}$$

where ϵ_0 is free space permittivity and c the velocity of light in vacuo. In vector notation, the antisymmetric part of Eq. (2) is

$$\boldsymbol{I}_0^{(A)} \equiv c\epsilon_0 \boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = c^3\epsilon_0 \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{3}$$

where $\boldsymbol{E}^{(2)}$ is the complex conjugate of $\boldsymbol{E}^{(1)}$, the transverse electric part of the plane wave in vacuo. If ϕ is the phase of the plane wave then

$$\boldsymbol{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \qquad \boldsymbol{E}^{(2)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-i\phi}, \tag{4}$$

where $E^{(0)}$ is the scalar amplitude in volts per meter and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit polar vectors in X and Y if the beam propagates in Z. The corresponding magnetic field solutions are

$$\boldsymbol{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \qquad \boldsymbol{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i\phi}, \tag{5}$$

where this time $\boldsymbol{i}$ and $\boldsymbol{j}$ signify unit axial vectors in X and Y. From Eqs. (3) and (4) it is seen that the electric field conjugate product [6–9] is

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = iE^{(0)2}\boldsymbol{k} = ic^2B^{(0)}\boldsymbol{B}^{(3)}, \tag{6}$$

which is independent of the phase of the plane wave. Woźniak *et. al*. [10] have recently expressed the inverse Faraday effect directly in terms of this conjugate product multiplied by hyperpolarizability components. The unit vector on the right hand side of Eq. (6) is axial by definition, because it is formed from a vector cross product of two polar vectors. The conjugate product is therefore $\hat{P}$ positive and it has been demonstrated [11] that it is $\hat{T}$ negative. Here $\hat{P}$ is the parity inversion operator and $\hat{T}$ the motion reversal operator. It is therefore proportional to a phase independent *magnetic* field, which is longitudinal and denoted by $\boldsymbol{B}^{(3)}$. Using the free space S.I. relation $E^{(0)} = cB^{(0)}$ it is easy to show that

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}. \tag{7}$$

A simple extension of this derivation shows that,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} = iB^{(0)}\boldsymbol{B}^{(3)}, \tag{8a}$$

$$\boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} = iB^{(0)}\boldsymbol{B}^{(1)*} = iB^{(0)}\boldsymbol{B}^{(2)}, \tag{8b}$$

$$\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} = iB^{(0)}\boldsymbol{B}^{(2)*} = iB^{(0)}\boldsymbol{B}^{(1)}, \tag{8c}$$

which can also be shown to be part of the Lie algebra [12] of the Poincaré or Lorentz groups of electromagnetism.

Transverse components of the electromagnetic plane wave in vacuo imply the presence of a novel longitudinal component $\boldsymbol{B}^{(3)}$ *which is phase independent*.

This is an obvious departure from the conventional notion [13, 14] that transverse and longitudinal solutions of Maxwell's equations in free space are unrelated. Many aspects of the properties of $\boldsymbol{B}^{(3)}$ are discussed elsewhere [6–9], but it is useful to note here that if $\boldsymbol{B}^{(3)}$ is zero, the symmetric, cyclic, Lie algebra (8) collapses. Also, from Eqs. (8b) and (8c) we see that if $\boldsymbol{B}^{(3)}$ is assumed zero, as is conventional, then $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ vanish, a paradoxical result. The Lie algebra (8) conserves $\hat{P}$, $\hat{T}$, and $\hat{C}$, which is charge conjugation symmetry, and therefore obeys the $\hat{C}\hat{P}\hat{T}$ theorem [15]. There is a close connection between the algebra (8) and fundamental three and four dimensional geometry (Euclidean and Minkowski). This is illustrated with reference to the unit vectors in the following circular basis, first used by Tanaś and Kielich [16],

$$\boldsymbol{e}^{(1)} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}); \quad \boldsymbol{e}^{(2)} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j}); \quad \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)}. \tag{9}$$

In this basis, it is easily seen that

$$\left.\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*} = i\boldsymbol{e}^{(3)}, \\ \boldsymbol{e}^{(2)} \times \boldsymbol{e}^{(3)} &= i\boldsymbol{e}^{(1)*} = i\boldsymbol{e}^{(2)}, \\ \boldsymbol{e}^{(3)} \times \boldsymbol{e}^{(1)} &= i\boldsymbol{e}^{(2)*} = i\boldsymbol{e}^{(1)}, \end{aligned}\right\} \tag{10}$$

which is exactly the same structure as (8). Tautologically, if $\boldsymbol{e}^{(3)}$ is asserted to be zero, this geometry becomes meaningless in three dimensions. From the relations [11],

$$\left.\begin{aligned} \boldsymbol{B}^{(1)} &= i\boldsymbol{e}^{(1)}B^{(0)}e^{i\phi}, \\ \boldsymbol{B}^{(2)} &= -i\boldsymbol{e}^{(2)}B^{(0)}e^{-i\phi}, \\ \boldsymbol{B}^{(3)} &= B^{(0)}\boldsymbol{e}^{(3)}, \end{aligned}\right\} \tag{11}$$

it becomes clear that the assertion $\boldsymbol{B}^{(3)} = 0$ implies $\boldsymbol{e}^{(3)} = 0$, and so the former is geometrically incorrect in Euclidean space. The same is also true in Minkowski space [15], in which

$$\left.\begin{aligned}\hat{B}^{(1)} &= -B^{(0)}\hat{J}^{(1)}e^{i\phi},\\ \hat{B}^{(2)} &= -B^{(0)}\hat{J}^{(2)}e^{-i\phi},\\ \hat{B}^{(3)} &= iB^{(0)}\hat{J}^{(3)},\end{aligned}\right\} \tag{12}$$

where the $\hat{J}$'s are rotation operators of the Poincaré and Lorentz groups [15]. It is well known that these rotation operators are each non-zero 4 x 4 complex matrices [15], and we again see that the conventional assertion $\hat{B}^{(3)} = 0$ is incorrect. From a consideration of the $\hat{J}$ matrices, the quantum nature of the $\hat{B}$ fields of electromagnetism is easily deduced,

$$\left.\begin{aligned}\hat{B}_q^{(1)} &= -B^{(0)}\frac{\hat{J}^{(1)}}{\hbar}e^{i\phi},\\ \hat{B}_q^{(2)} &= -B^{(0)}\frac{\hat{J}^{(2)}}{\hbar}e^{-i\phi},\\ \hat{B}_q^{(3)} &= iB^{(0)}\frac{\hat{J}^{(3)}}{\hbar},\end{aligned}\right\} \tag{13}$$

where $\hbar$ is the unit of angular momentum in quantum field theory, i.e. the angular momentum of one photon. From Eq. (13) it is seen that the photon has an elementary longitudinal magnetic operator component $\hat{B}_q^{(3)}$ [6–9]. This result was first derived in Ref. (6) using an independent method. Since $\hat{J}^{(3)}$ is non-zero from fundamental first principles [15] it is incorrect to assert that $\hat{B}_q^{(3)}$, and its expectation value $\boldsymbol{B}^{(3)}$, is zero.

Since $\boldsymbol{B}^{(3)}$ is a magnetic field we expect optical magnetization effects, the subject of this paper. These have actually been reported experimentally [1, 17–23] on several occasions. In this section, it is demonstrated using classical electrodynamics that $\boldsymbol{B}^{(3)}$ does not violate the fundamental Poynting theorem on the conservation of energy.

In S.I. units the classical Poynting theorem is the continuity equation [24],

$$\nabla\cdot\boldsymbol{N} = -\frac{\partial U}{\partial t} - c^2\epsilon_0\boldsymbol{E}\cdot\boldsymbol{J}, \tag{14}$$

where the vector field $\boldsymbol{N}$, the Poynting vector, is the electromagnetic power per unit area, or flux of electromagnetic energy. The scalar field U is the electromagnetic power per unit volume, or energy density of the electromagnetic field. $\boldsymbol{J}$ is the material current density as the result of charge density ρ moving with velocity $\boldsymbol{v}$. The product $c^2\epsilon_0\boldsymbol{E}\cdot\boldsymbol{J}$ is the rate per unit volume at which the electromagnetic field does work on a charge distribution [25], i.e. the rate at which the field loses energy to the charges. More generally [25], $c^2\epsilon_0\boldsymbol{E}\cdot\boldsymbol{J}$ is the density of power lost from the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ as a result of radiation-matter interaction. The power density $c^2\epsilon_0\boldsymbol{E}\cdot\boldsymbol{J}$ is identified classically [25] with the density of loss of field energy to particle motion. The notion of field energy takes meaning [25] only when there is coupling between field and matter. Formally, however, the Poynting theorem is written in free space (S.I.) as

$$\nabla\cdot\boldsymbol{N} = -\frac{\partial U}{\partial t}. \tag{15}$$

We wish to see how $\boldsymbol{B}^{(3)}$ affects the validity of Eq. (14), the fundamental law of conservation of energy in the field-matter interaction. We first answer this question classically, then extend the arguments to the quantum theory.

From the Maxwell equations,

$$\boldsymbol{J} = \frac{\partial\boldsymbol{P}}{\partial t} + \nabla\times\boldsymbol{M} + \ldots, \tag{16}$$

in S.I. units, where we have omitted the quadrupole term [24] for simplicity. Here $\boldsymbol{P}$ and $\boldsymbol{M}$ are material polarization and magnetization. In S.I. units,

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}, \tag{17a}$$

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}, \tag{17b}$$

where $\boldsymbol{H}$ is the magnetic field strength (amp m^{-1}); μ_0 the free space magnetic permeability; and $\boldsymbol{D}$ the electric displacement (C m^{-2}). With these definitions, the material energy density per unit volume, in S.I., is [24],

$$U_m = \mu_0\boldsymbol{H}\cdot\boldsymbol{M} + \boldsymbol{E}\cdot\boldsymbol{P} + \text{quadrupole term}, \tag{18}$$

the work done by the external classical electromagnetic field on the material is dU_m/dt; and the work done by the material on the electromagnetic field is $-dU_m/dt$. The total energy [24] stored in the electromagnetic field, in S.I. units, is [24],

$$U_S = \mu_0\boldsymbol{H}\cdot\boldsymbol{H} + \epsilon_0\boldsymbol{E}\cdot\boldsymbol{E} = \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{B} + \epsilon_0\boldsymbol{E}\cdot\boldsymbol{E} - 2\boldsymbol{M}\cdot\boldsymbol{B} + \mu_0\boldsymbol{M}\cdot\boldsymbol{M}, \tag{19}$$

In the absence of magnetization ($M = 0$), U_S depends only on $\boldsymbol{B}$, the magnetic flux density of the field; and on $\boldsymbol{E}$, its electric field strength (volt m^{-1}). The energy U_S in free space is therefore,

$$U_S\,(\textit{free space}) = \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}+\epsilon_0\boldsymbol{E}\cdot\boldsymbol{E}. \tag{20}$$

It is seen from Eq. (19) that the energy stored in the electromagnetic field in the presence of matter depends also on $\boldsymbol{M}$, but not on $\boldsymbol{P}$. *Magnetization contributes to U_S in the presence of matter*,

$$\Delta U_S = (U_S - U_S\,(\textit{free space})) = -2\boldsymbol{M}\cdot\boldsymbol{B}+\mu_0\boldsymbol{M}\cdot\boldsymbol{M}. \tag{21}$$

We now consider the role of the field $\boldsymbol{B}^{(3)}$ in Eqs. (19) to (21). In order to proceed it is first necessary to consider whether $\boldsymbol{B}^{(3)}$ is accompanied by a longitudinal electric component, $\boldsymbol{E}^{(3)}$, which we shall assert to be complex in general (i.e. to have a real and imaginary part in general, as for the transverse components). From Eqs. (20) and (15), both U_S and $\boldsymbol{N}$ in free space would then depend on both $\boldsymbol{B}^{(3)}$ *and* $\boldsymbol{E}^{(3)}$. At first sight, therefore, it appears that there would be a longitudinal contribution to U_S in free space,

$$U^{(3)} \equiv U_S\ (\text{free space, longitudinal}) \overset{?}{=} \frac{1}{\mu_0}\boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)}+\epsilon_0\boldsymbol{E}^{(3)}\cdot\boldsymbol{E}^{(3)}. \tag{22}$$

However, since $\boldsymbol{B}^{(3)}$ is parallel to $\boldsymbol{E}^{(3)}$ the longitudinal contribution to the Poynting vector in free space is zero,

$$\boldsymbol{N}^{(3)}\ (\text{free space, longitudinal}) = \frac{1}{\mu_0}\boldsymbol{E}^{(3)}\times\boldsymbol{B}^{(3)} = \boldsymbol{0}. \tag{23}$$

The longitudinal electromagnetic power per unit area, or flux of longitudinal electromagnetic energy, is zero. From Eq. (15),

$$\frac{\partial U^{(3)}}{\partial t} = 0, \tag{24}$$

and therefore $\boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}\cdot\boldsymbol{E}^{(3)}$ must be independent of time. A possible solution of Eq. (24) is $U^{(3)}=0$, in which case there is no free space longitudinal electromagnetic energy density, or free space electromagnetic power per unit volume. In this case, the longitudinal energy stored in the electromagnetic field in the presence of matter *becomes purely magnetic in nature*. From Eq. (21),

$$U_S^{(3)} = -2\boldsymbol{M}^{(3)}\cdot\boldsymbol{B}^{(3)}+\mu_0\boldsymbol{M}^{(3)}\cdot\boldsymbol{M}^{(3)}, \tag{25}$$

where $\boldsymbol{M}^{(3)}$ is the magnetization due to $\boldsymbol{B}^{(3)}$.

We see that even if the free space longitudinal electromagnetic energy density is zero, the field $\boldsymbol{B}^{(3)}$ can still cause magnetization $\boldsymbol{M}^{(3)}$ in material matter; a magnetization which is experimentally observable.

This is an important result, because it is well known [26] that the Planck radiation law is derived (for example Bose [27]) *on the grounds that there are only two photon polarizations*, both being transverse, i.e. left and right circular. This law is obeyed precisely by experimental data, and on these experimental grounds, it is evident that $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ *in combination* do not contribute to the Planck law. In other words, on experimental grounds, $U^{(3)}$ is zero. Using Eq. (22) this result is satisfied if it is assumed that $\boldsymbol{E}^{(3)}$ is: a) phase independent; b) pure imaginary,

$$U^{(3)} = \frac{1}{\mu_0}\boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)}+\epsilon_0 i\boldsymbol{E}^{(3)}\cdot i\boldsymbol{E}^{(3)} = 0, \tag{26}$$

with $i\boldsymbol{E}^{(3)} = iE^{(0)}\boldsymbol{k};\ E^{(0)} = cB^{(0)}$).

What theoretical grounds are there for asserting that the real $\boldsymbol{B}^{(3)}$ is accompanied by an imaginary $i\boldsymbol{E}^{(3)}$?

One answer is to use a form of the Lorentz Lemma [28] as follows. It is assumed that,

$$\boldsymbol{E}_a = \boldsymbol{E}+\boldsymbol{E}^{(3)}, \qquad \boldsymbol{B}_a = \boldsymbol{B}+\boldsymbol{B}^{(3)}, \tag{27}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the transverse electromagnetic waves. The phase independent $B^{(3)}$ is real and we assume only that $\boldsymbol{E}^{(3)}$ is in general complex. Using the vector identities,

$$\nabla\cdot(\boldsymbol{E}^a\times\boldsymbol{B}^a) = \nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})+\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}^{(3)})+\nabla\cdot(\boldsymbol{E}^{(3)}\times\boldsymbol{B})+\nabla\cdot(\boldsymbol{E}^{(3)}\times\boldsymbol{B}^{(3)}), \tag{28}$$

and

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}^{(3)}) = \boldsymbol{B}^{(3)}\cdot(\nabla\times\boldsymbol{E})-\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B}^{(3)}), \tag{29}$$

and the Maxwell equation,

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{30}$$

with

$$\nabla\times\boldsymbol{B}^{(3)} = \boldsymbol{0}, \tag{31}$$

leads to the following form of the Lorentz Lemma [28],

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}^{(3)}) = \nabla\cdot(\boldsymbol{E}^{(3)}\times\boldsymbol{B}) . \tag{32}$$

This can be re-expressed, using the Maxwell equations, as

$$\boldsymbol{B}^{(3)}\cdot\frac{\partial\boldsymbol{B}}{\partial t}+\frac{1}{c^2}\boldsymbol{E}^{(3)}\cdot\frac{\partial\boldsymbol{E}}{\partial t} = 0, \tag{33}$$

from which it can be seen that the contributions of the products $\boldsymbol{B}^{(3)}\cdot\boldsymbol{B}$ and $\boldsymbol{E}^{(3)}\cdot\boldsymbol{E}$ to electromagnetic energy density in free space is zero. This is of course consistent with the fact that $\boldsymbol{B}^{(3)}$ is orthogonal to $\boldsymbol{B}$, and $\boldsymbol{E}^{(3)}$ is orthogonal to $\boldsymbol{E}$. Furthermore, integrating Eq. (15) over all space, and using the Divergence theorem,

$$\int(\boldsymbol{E}\times\boldsymbol{B}^{(3)} - \boldsymbol{E}^{(3)}\times\boldsymbol{B})\cdot d\boldsymbol{a} = 0, \tag{34}$$

which implies that the integrand is zero. Taking the $\boldsymbol{B}^{(1)}$ and $\boldsymbol{E}^{(1)}$ components, for example,

$$\boldsymbol{E}^{(1)}\times\boldsymbol{B}^{(3)} = \boldsymbol{E}^{(3)}\times\boldsymbol{B}^{(1)}, \tag{35}$$

and for the $\boldsymbol{B}^{(2)}$ and $\boldsymbol{E}^{(2)}$ components,

$$\boldsymbol{E}^{(2)}\times\boldsymbol{B}^{(3)} = \boldsymbol{E}^{(3)}\times\boldsymbol{B}^{(2)} . \tag{36}$$

Using the transverse solutions in circular polarization, Eqs. (4), it follows that if $\boldsymbol{B}^{(3)}$ is real, then it is accompanied by an imaginary $i\boldsymbol{E}^{(3)}$. *This is consistent with experimental data on the Planck radiation law.*

In arriving at this result we have assumed only that the longitudinal $\boldsymbol{B}^{(3)}$ is accompanied by an assumed complex longitudinal electric field, and have shown that this must be the imaginary and phase independent $i\boldsymbol{E}^{(3)}$. This is consistent with the fact that phase independent magnetic effects due to circularly polarized light have been reported, and have been predicted [24] in terms of the conjugate product. There appear to be no such electric counterparts. Since $i\boldsymbol{E}^{(3)}$ is imaginary, it has no real part in Maxwell's theory of light and therefore has no physical effects. All this is further consistent with the fact that the energy stored in the electromagnetic field in Eq. (25) is magnetic in nature, not electric.

The main result of this section is that the classical Poynting theorem in free space, Eq. (15), is unchanged by the existence of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$. These fields do not upset the Planck radiation law and do not contradict classical conservation of energy in field-matter interaction, *provided* that there is a

magnetization $\boldsymbol{M}^{(3)}$. From the Lie algebra (8) the source of $\boldsymbol{B}^{(3)}$ is the same as the source of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ and it is incorrect to assert that $\boldsymbol{B}^{(3)}$ vanishes because it has no source. Clearly if $\boldsymbol{B}^{(3)}$ vanished then from Eqs. (8) so would $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$, leaving no electromagnetism. This result depends logically on there always being a non-zero $B^{(0)}$, but if $B^{(0)}$ were zero, then there would be no transverse or longitudinal magnetic components, i.e. no electromagnetism.

We now turn our attention to the interpretation of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ in the "old quantum theory" [29].

Since $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are independent of frequency they do not contribute to the energy density of radiation per unit volume in the old quantum theory, i.e. they do not contribute to the Planck distribution, the Wien law, or Steffan Boltzmann law, because the frequency (ν) associated with $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ is zero in these laws. This corresponds to the fact that the Maxwellian energy density in free space is zero for $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ combined, as we have argued, despite the fact that $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are non-zero from the Lie algebra (8) and the Lorentz lemma (32). In the old quantum theory this is consistent with the fact that the Planck law can be derived [30] from Bose statistics with the energy held constant as the only constraint. In the old quantum theory the energy is directly proportional to frequency, and if the latter is zero, then so is the energy. Bose's original paper, reviewed by Pais [29] introduces the thermal equilibrium law for massless particles with two states of polarization. This paper is the first mention of a particle with two states of polarization, and his derivation [27] of the Planck law replaces the counting of electromagnetic wave frequencies by the counting of cells in one particle phase space. The one particle phase space element dxdp is integrated over the volume V and over all momenta between p and p + dp [29]. *A further factor 2 is now supplied to count polarizations* [29]. He then applies the photon momentum formula $p = h\nu/c$ to obtain the multiplier in the Planck formula. This is the source of the present day notions that the photon has two helicities and is massless. As explained by Pais [29], Bose introduced the factor 2 for polarization because it *seemed* to be a requirement for his method to produce the Planck law. A factor different from 2 would not give the known Planck constant h. At that time, (1924), the notion of a particle with two states of polarization was unprecedented, and remains obscure to this day, because [29] there is no rest frame definition of spin for the massless photon, and gauge invariance renders ambiguous [29] the distinction between orbital and intrinsic spin. These well known difficulties are discussed by Ryder [15], for example, and include the loss of manifest covariance in electromagnetic theory [13] based on the four-potential A_μ.

By introducing, as in this paper, the Maxwellian $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ it becomes immediately clear that the photon as defined by Bose [27, 29] can have *three* polarizations, *without changing the Planck law*, because there is no field frequency, energy or linear momentum associated with the longitudinal (third degree of) polarization. Similar conclusions hold for the Wien and Steffan Boltzmann laws, and for the Rayleigh-Einstein-Jeans law. Similarly equipartition is not disturbed by the existence of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$, and neither are the spectroscopic rules of photon absorption, emission, the Einstein A and B coefficients, and so forth. Nevertheless, as we have argued, $\boldsymbol{B}^{(3)}$ still acts as a magnetic field capable of producing magnetization and the effect which is the main subject of this paper, the optical Faraday effect observed by Sanford *et. al.* [1]. The introduction of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ makes possible the further important development that the photon can be massive, a recurring theme [31, 32] since the days of Cavendish. A photon mass (however tiny) would immediately allow three degrees of photon polarization, and it has been proposed recently [33] that finite photon mass need not upset gauge invariance and the important contemporary work on unified field theory.

In the new quantum theory the situation is changed slightly because the Planck distribution is changed by the existence of rest energy [34], which emerges from Schrödinger's equation for the harmonic oscillator, a standard textbook problem within the tenets of the new quantum theory. This rest energy is carried over [35] into contemporary photonics, and is therefore associated with the traditional *transverse* photon polarizations, or left and right photon spin. In the contemporary theory [36] it has been shown [6] that the $\boldsymbol{B}^{(3)}$ field becomes an operator directly proportional to the photon angular momentum through the $\hat{C}$ negative amplitude $B^{(0)}$,

$$\hat{B}^{(3)} = B^{(0)}\frac{\hat{J}}{\hbar}, \qquad (37)$$

an equation which obviously conserves $\hat{C}$, $\hat{P}$, *and* $\hat{T}$ symmetries as required. The classical $\boldsymbol{B}^{(3)}$ is then simply the expectation value of the operator $\hat{B}^{(3)}$. In the new quantum theory therefore, energy conservation is again satisfied, and Eq. (37) has the advantage of allowing a clear and unambiguous physical interpretation of $\hat{B}^{(3)}$ as an operator with all the properties of photon angular momentum. Therefore $\hat{B}^{(3)}$ propagates with the photon in the axis of propagation, about which there is angular momentum $\hbar$ for one photon. For one photon, therefore, the expectation value for $\hat{J}/\hbar$ is the classical axial unit vector $\boldsymbol{k}$ in the propagation axis Z of the photon and we recover the classical equation,

$$\boldsymbol{B}^{(3)} = \langle\hat{B}^{(3)}\rangle = B^{(0)}\boldsymbol{k} = \left(\frac{\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}}{iB^{(0)}}\right), \qquad (38)$$

which is identical with Eq. (8a). This answers the fundamental questions of why can $\hat{B}^{(3)}$ propagate when it has no frequency dependence, and what is its source. The answers in both cases are the same as for the angular momentum $\hbar$ of one photon.

It is quite impossible to assert that $\hat{B}^{(3)}$ is zero if we accept the notion that light intensity can have an antisymmetric component. The latter has long been accepted as a physically meaningful concept, and is the traditional explanation for the inverse Faraday effect [37]. In the next section we

demonstrate from first principles the use of $\boldsymbol{B}^{(3)}$ in the one electron inverse Faraday effect, the simplest possible case. It is interesting to note that Selleri [38] has recently described a novel approach to the paradoxes of the new quantum theory by invoking the double slit experiment. In Selleri's view [38] radiation has a corpuscular and undulatory nature, the former being, loosely speaking, the photon, and the latter the field. The corpuscular component carries energy and linear momentum, for one photon $h\nu$ and $h\nu/c$ respectively; is particulate in nature and can pass through only one slit at a time. The undulatory component can pass through both slits simultaneously if both are open. This means [38] that there can exist "empty waves", which have no energy or linear momentum, and cannot be observed directly. Their physical reality may nonetheless be verified experimentally [38], but indirectly [39]. The $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ fields generate no Maxwellian electromagnetic energy density by dint of Eq. (26), and their wavelength is infinite (more accurately the radius of the universe), so that their wavenumber is zero. Nonetheless, $\boldsymbol{B}^{(3)}$, being a real magnetic field, can influence matter as in the optical Faraday effect [1]. If the photon, as proposed originally by Einstein [40], is to be regarded as a unit of radiation *energy* (and linear momentum), then the classical $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ add nothing to its energy. Therefore they add nothing to the Planck law, as we have argued. The ability of the Maxwellian $\boldsymbol{B}^{(3)}$ to magnetize has been interpreted in the new quantum theory [6] as the $\hat{B}^{(3)}$ operator, proportional to the photon's *angular momentum* $\hbar$. If the photon has finite rest mass its angular momenta are $+\hbar$, 0, and $-\hbar$. It is interesting to think of $\hat{B}^{(3)}$ in Selleri's theory [38] has having been formed from the *corpuscular* $\hbar$ by multiplication by the scalar $B^{(0)}$.

For these reasons we refer to $\hat{B}^{(3)}$ *as the elementary [6] longitudinal magnetic flux density of one photon.*

Since the classical $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are states of the electromagnetic field that correspond to zero frequency, $\nu = 0$, not a positive definite frequency ν, and zero wavevector, they cannot be absorbed by an atom with energy levels $E_m > E_n$. This is because

$$E_m - E_n = h\nu, \qquad (39)$$

in the quantum theory, and if $\nu = 0$, $E_m = E_n$. The energy of one photon is $h\nu$; its linear momentum is $h\nu/c$; however, its angular momentum is $\hbar$, and the latter is frequency independent. Its angular energy is however $\hbar\omega = h\nu$. The only quantity from these four that is non-zero for $\omega = 2\pi\nu = 0$ is the photon's angular momentum $\hbar$, to which $\hat{B}^{(3)}$ is directly proportional. (Because of $\hat{T}$ and $\hat{P}$ symmetries, $\boldsymbol{B}^{(3)}$ can be proportional to neither linear nor angular momentum.) It is our contention in this paper that although $\boldsymbol{B}^{(3)}$ generates no Maxwellian electromagnetic energy density in free space, because of Eq. (26), and although $\hat{B}^{(3)}$ has no quantum energy, it can cause observable physical effects. One of these is the optical Faraday effect [1], another is the inverse Faraday effect (IFE).

It is shown in the next section that if $\boldsymbol{B}^{(3)}$ were zero, the IFE for one electron would disappear, contradicting experimental experience [17–23].

3. The Inverse Faraday Effect for One Electron

It is well known that the inverse Faraday effect is magnetization by circularly polarized light [37] and was first observed experimentally by van der Ziel *et. al.* [17] in molecular liquids and doped glasses. The effect was first observed experimentally in plasma by Deschamps *et. al.* [18]. Its influence on ferromagnetism has been observed by Sanford *et. al.* [1], and on conductivity in phthalocyanines by Barrett *et. al.* [19]. It is well established theoretically, but always in terms of the conjugate product $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$. For the interaction of light with one electron it has been described by Talin *et. al.* [20], who considered the angular momentum of an electron driven in a circular orbit by the rotating electric field of the light beam. From relativistic electrodynamics this angular momentum is proportional to the square of the scalar amplitude $E^{(0)}$ of the electric field strength in volts m^{-1}. In this section it is shown that the basically important results of Talin *et. al.* [20] can be expressed in terms of $\boldsymbol{B}^{(3)}$, and that if $\boldsymbol{B}^{(3)}$ were zero, the inverse Faraday effect would vanish, contrary to experience [17–23].

From first principles, it is clear that the effect of the rotating electric field of a circularly polarized plane wave, driving an electron around in a circular orbit, is entirely equivalent to that of a magnetic field. The circularly polarized beam drives the electron around in an orbit with radius [20],

$$r = \frac{ecB^{(0)}}{m_0 \omega^2 \gamma}, \tag{40}$$

where e is the charge on the electron, c the speed of light in vacuo, $B^{(0)}$ the magnetic flux density amplitude of the light beam, m_0 the electron rest mass, ω the angular frequency of the light, and γ is a relativistic factor which is unity to an excellent approximation at, say, visible frequencies and intensities of about a watt per square centimeter. The transverse momentum of the electron is given from these first principles as

$$p_t = \frac{ecB^{(0)}}{\omega}, \tag{41}$$

so that its angular momentum is

$$J_z = rp_t = \frac{e^2c^2}{m_0\omega^3}B^{(0)2}, \tag{42}$$

which is proportional to $B^{(0)2}$. Finally, the induced magnetic dipole moment is the gyromagnetic ratio multiplied by the angular momentum,

$$m_z = -\frac{e}{2m_0}J_z = \frac{-e^3c^2}{2m_0^2\omega^3}B^{(0)}|\boldsymbol{B}^{(3)}|. \tag{43}$$

The magnetization is, evidently, proportional to the square of $\boldsymbol{B}^{(3)}$ through a one electron hyperpolarizability and in this mechanism, the traditional one [20], is proportional to the intensity of the light beam. Clearly, if $\boldsymbol{B}^{(3)}$ were zero, there would be no effect.

It is shown in this section that the action of a rotating electric field, just described, is entirely equivalent to that of a magnetic field, leading to a first order inverse Faraday effect mediated by $\boldsymbol{B}^{(3)}$, which exists in addition to the usual second order mechanism just described.

Consider the textbook problem of an electron in a static magnetic field $\boldsymbol{B_s}$. The radius of the electron's orbit is [41]

$$r_0 = \frac{V_{ot}}{\Omega}, \tag{44}$$

where V_{ot} is the initial transverse linear velocity and Ω is the frequency defined by the Lorentz equation,

$$\Omega = \frac{e}{m_0}|\boldsymbol{B_s}|. \tag{45}$$

This angular frequency is the ratio e/m_0 multiplied by the magnetic field $\boldsymbol{B_s}$. The transverse momentum of the electron is [41],

$$p_t = er_0|\boldsymbol{B_s}|, \tag{46}$$

and the angular momentum of the electron in $\boldsymbol{B_s}$ is

$$L_z = er_0^2|\boldsymbol{B_s}|. \tag{47}$$

The angular momentum L_z is therefore er_0^2 multiplied by the magnetic field $\boldsymbol{B_s}$. The magnetic dipole moment induced by $\boldsymbol{B_s}$ is therefore

$$m_Z = -\left(\frac{e^2 r_0^2}{2m_0}\right)|\boldsymbol{B}_s|, \tag{48}$$

where the quantity in brackets is the one electron susceptibility.

The result (43) for the one electron Faraday effect is the traditional theory [20] expressed in terms of $B^{(0)}|\boldsymbol{B}^{(3)}|$ (i.e. $B^{(3)2}$) through Eq. (1). It is second order in the magnetic flux density amplitude of the beam ($B^{(0)}$). However, because $\boldsymbol{B}^{(3)}$ is non-zero by Eq. (1), and has the units and symmetry of magnetic flux density there is an additional *first order inverse Faraday effect*, given by Eq. (48), with $\boldsymbol{B}_s$ identified as $\boldsymbol{B}^{(3)}$,

$$m_Z^{(1)} = -\left(\frac{e^2 r_0^2}{2m_0}\right)|\boldsymbol{B}^{(3)}|. \tag{49}$$

The magnetic dipole moment induced by the beam is therefore the sum of first and second order effects,

$$\boldsymbol{m} = -\left(\frac{e^2 r_0^2}{2m_0}\right)\boldsymbol{B}^{(3)} - \left(\frac{e^3 c^2}{2m_0^2\omega^3}\right)B^{(0)}\boldsymbol{B}^{(3)}. \tag{50}$$

For ω about 10^{15} rad sec^{-1}; and for a first order electron radius of about 10 Å (10^{-9} m) this is, roughly

$$|\boldsymbol{m}| \sim 10^{-26}|\boldsymbol{B}^{(3)}| - 10^{-25}B^{(3)2}. \tag{51}$$

For a beam intensity of about 10^{14} watt m^{-2}; $B^{(0)}$ is about one tesla, and the second order effect is ten times bigger than the first order one. It would appear in this case that there were only a second order effect present, but under different conditions, the first order effect should become visible, superimposed on the second order induced dipole moment. The latter is proportional to the square root of light intensity. Therefore the complete effect is a sum of (43) and (49), which is the conclusion of this section.

From Eq. (49) it is seen that the second order effect can be expressed through an "effective" magnetic field $\boldsymbol{B}_{eff}$ defined by

$$\boldsymbol{B}_{eff} \doteq \left(\frac{eB^{(0)}}{\omega m_0}\right)\boldsymbol{B}^{(3)}. \tag{52}$$

This effective magnetic field is what is usually referred to in the literature on the inverse Faraday effect, and was first mentioned in references (17) to (23). The *physical* magnetic field $\boldsymbol{B}^{(3)}$ is an elementary field carried by the photon angular momentum, as we have argued in Sec. 2. It is clear that the inverse Faraday effect for one electron depends on a non-zero $\boldsymbol{B}^{(3)}$, however, at first and second orders, and would obviously disappear if $\boldsymbol{B}^{(3)}$ were zero, contrary to observation [17–23]. In linear polarization, the *net* $\boldsymbol{B}^{(3)}$ is zero, because there is 50% of $+\boldsymbol{B}^{(3)}$ and 50% of $-\boldsymbol{B}^{(3)}$ present in the beam simultaneously. Thus, no inverse Faraday effect is observed in linear polarization, but interestingly, there is an inverse Cotton Mouton effect [42] proportional to $\boldsymbol{B}^{(3)}$ squared, which does not vanish in linear polarization. From Eq. (1) it is seen that $\boldsymbol{B}^{(3)}$ is the elementary unit of the conjugate product, and therefore of the antisymmetric part of light intensity in free space. As always, radiation becomes manifest only when it interacts with matter, and similarly for $\boldsymbol{B}^{(3)}$. From Eq. (52), if $\boldsymbol{B}^{(3)}$ did not exist, then $\boldsymbol{B}_{eff}$ would also vanish. In setting up the effect of Eq. (49) we have simply assumed that the physical magnetic field causes magnetization at first order through a one electron susceptibility, i.e. that $\boldsymbol{B}^{(3)}$ behaves classically as a magnetic field. This appears to be reasonable, but since $\boldsymbol{B}^{(3)}$ is a novel property of light, it is not a regular magnetostatic field, generated, for example, by a wound solenoid. Since all magnetic fields are governed classically by Maxwell's equations, it appears reasonable to assert that $\boldsymbol{B}^{(3)}$ has all the properties of magnetic flux density. It is important to investigate this further by experiments on magnetization by light, for example the IFE, OFE, light shifts and optical NMR and ESR.

4. The First Order Optical Faraday Effect Due to $\boldsymbol{B}^{(3)}$

In this section the field $\boldsymbol{B}^{(3)}$, looked upon as Maxwellian in nature, is used to propose a first order optical Faraday effect proportional to the square root of pump laser intensity. The method used is an adaptation of the semi-classical theory of Woźniak, Evans and Wagnière [10] for the inverse Faraday effect. The field $\boldsymbol{B}^{(3)}$ is, as we have argued, the expectation value of the operator $\hat{B}^{(3)} = B^{(0)}\hat{J}/\hbar$, which is for one photon the elementary *photomagneton*, and in the quantum theory, $\hat{B}^{(3)}$ must be used as an operator directly proportional to the angular momentum operator. Progress in this direction has recently been initiated in the optical Zeeman effects [43, 44]. Henceforth, the field $\boldsymbol{B}^{(3)}$ is referred to as the ghost field (Gespensterstrahlung) because in the quantum theory it is associated with no energy ($h\nu = 0$), and no linear momentum ($h\nu/c = 0$). However it is associated with the non-zero angular momentum $\hbar$, through which the photomagneton $\hat{B}^{(3)}$ is defined. The ghost field $\boldsymbol{B}^{(3)}$ is generated by the spin of the photon through $\boldsymbol{B}^{(3)} = B^{(0)}\langle\hat{J}/\hbar\rangle$, where $\langle \;\; \rangle$ denote expectation value. The *ghost* field is defined classically by Eq. (1). The transverse magnetic *wave* fields $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ on the other hand are represented in the quantum theory with finite photon energy $h\nu$, because they have finite frequency ν; and with finite photon linear momentum $h\nu/c$. Equation (1) as represented in the quantum theory is therefore a relation between spins.

The basic assumption of this section is that the ghost field $\boldsymbol{B}^{(3)}$ can act as a classical magnetic field, so that its overall effect on a Faraday rotation would be to shift it upwards or downwards in frequency without changing its bandshape. This is what is demonstrated [1] by Sanford *et. al.* in their Figure (2). It was shown experimentally that the displacement of the spectrum was to lower frequency when right circularly polarized light was used from the pump laser; and vice-versa. Circularly polarized pumping was shown to be an order of magnitude more effective than unpolarized pumping. These very large light induced shifts of the Faraday spectrum around 1000 nm were observed with a sample thickness of only 25 microns, and a beam intensity of only 0.7 watts per square centimeter. If the ghost field is a standard Maxwellian field the standard Faraday effect theory [45] can be adopted for the description of the optical Faraday effect. This implies that the angle of rotation in a linearly polarized probe laser is (in S.I. units) [45],

$$\Delta\Theta \sim \frac{1}{12}\omega\mu_0 c l N|\boldsymbol{B}^{(3)}|\epsilon_{\alpha\beta\gamma}\left({}^{e}\gamma^{em}_{\alpha\beta\gamma}(f) + \frac{m_{n\alpha}}{kT}\alpha''_{\beta\gamma}(f) \right), \tag{53}$$

where ω is the angular frequency of the probe laser, μ_0 the free space permeability; c the speed of light; l the sample thickness; N the number of atoms or molecules per cubic meter in the sample, $|\boldsymbol{B}^{(3)}| \sim 10^{-7} I_0^{\frac{1}{2}}$, where I_0 is the intensity of the pump laser in watts per square meter; $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol; kT the thermal energy per molecule; and ${}^{e}\gamma^{em}_{\alpha\beta\gamma}(f)$, $m_{n\alpha}$, and $\alpha''_{\beta\gamma}$ the standard Faraday effect molecular property tensors [45]. Respectively, these are the hyperpolarizability, magnetic dipole moment, and antisymmetric polarizability. The structure of the hyperpolarizability is given by Woźniak *et. al.* [10] in semi-classical perturbation theory, and is closely related to the hyperpolarizability of the inverse Faraday effect [45]. The structure of the antisymmetric polarizability and the magnetic dipole moment is given in standard perturbation theory [45].

It is clear that Eq. (53) for $\Delta\Theta$ from the pump laser's ghost field $\boldsymbol{B}^{(3)}$ will produce a shift of the original Faraday spectrum, i.e. $\Delta\Theta(\omega)$, *symmetrically* upwards or downwards in frequency with right or left circular polarization. This is because the ghost field will be added to or subtracted from the permanent magnetic field $\boldsymbol{B}$ of the Faraday effect spectrometer, giving a net field $\boldsymbol{B} \pm \boldsymbol{B}^{(3)}$. *This is what is reported by Sanford et. al. [1] in their Figure 2.* No intensity dependence of this effect is reported by Sanford *et. al.* [1], however, and their interpretation is given in terms of $\pm\frac{3}{2}J'f$, where J' is the first order intra-Cr exchange and f the fraction of chromium sites with excited electrons. They estimate $J' = 0.42\,eV$ and $f = 1.8 \times 10^{-3}$, i.e. the fractional population of pump excited carriers per chromium atom site. Therefore $\pm\frac{3}{2}J'f$ is a spin-spin interaction between the chromium magnetic spin, $\frac{3}{2}$, and a photo-excited spin $J'f$. In the notation of Eq. (53), this corresponds to the term in $m_{n\alpha}\alpha''_{\beta\gamma}$; a tempera-

ture dependent term. As the temperature is changed from 78 K, however, where the Faraday rotation peaks are close to the fixed pump frequency, Sanford *et. al.* [1] report a gradual loss of symmetry in shifts for the right and left circularly polarized pump. This is caused [1] by the temperature dependence of the energy splitting between sub-bands, a mechanism which is allowed for in Eq. (53) through the fact that the optical resonance structure of the molecular property tensors depends on the energy splitting between sub-levels and on temperature. The resonance structure of the inverse Faraday effect spectrum obtained theoretically by Woźniak *et. al.* [10] shows these effects. Under conditions defined by Woźniak *et. al.* [10] the hyperpolarizability of the inverse and standard Faraday effects has the same structure in semi-classical perturbation theory. Sanford *et. al.* [1] refer to their interpretation of these interesting data as a first order magnetic coupling; and Eq. (53) is an equation to first order in the ghost field $\boldsymbol{B}^{(3)}$ of the pump laser.

It is important in further work to study these effects as a function of pump laser intensity, in order to detect effects in $\boldsymbol{B}^{(3)}$ proportional to the square root of pump laser intensity. If detected, and separated from the accompanying second order effects proportional directly to pump laser intensity (see below), experimental evidence would have become available for the ability of the ghost field $\boldsymbol{B}^{(3)}$ to act at first order as a magnetic field. The unequivocal data obtained by van der Ziel *et. al.* [17] for the inverse Faraday effect is already interpretable in terms of $\boldsymbol{B}^{(3)}$ at *second* order [17]. Clearly, if the ghost field were zero, $\boldsymbol{B}^{(3)} = \boldsymbol{0}$, then $B^{(3)2} = 0$, and there would be no inverse Faraday effect, contrary to experience [17–23]. Therefore the ghost field $\boldsymbol{B}^{(3)}$ is not zero and effects due to it at second order have been detected unequivocally [7, 17]. Effects due to it at *first* order have not yet been detected unequivocally, although there are signs [8] that data recently obtained by Frey *et. al.* [46] on the optical Faraday effect in magnetic semiconductors show the required square root intensity dependence. More work urgently needs to be done to clarify the fundamental properties of the ghost field and photomagneton of light.

From Eq. (53) and the results of Sanford *et. al.* [1] a pump of 0.7 watts per square centimeter intensity produces at a fixed frequency an ordinate shift of about 10° (roughly 0.2 rad) in the Faraday spectrum at 1000 nm (roughly $6\pi \times 10^{14}$ rad s^{-1}); depending on whether the beam is left or right circularly polarized at 78 K, for a sample thickness of 25 microns. Assuming that there are of the order 10^{26} atoms per cubic meter of sample, we obtain rough estimates of the orders of magnitude of $m_{n\alpha}\alpha''_{\beta\gamma}$ and ${}^{e}\gamma^{em}_{\alpha\beta\gamma}$ in Eq. (53) as follows:

$$|\,{}^{e}\gamma^{em}_{\alpha\beta\gamma}(f)\,| \sim 10^{-34}\ Am^4\ V^{-2}; \qquad |m_{n\alpha}\alpha''_{\beta\gamma}| \sim 10^{-53}\ C^2\ m^2\ T^{-1}. \tag{54}$$

In a diamagnetic liquid such as carbon disulphide, Woźniak *et. al.* [47] have calculated a value for the hyperpolarizability of the standard Faraday effect: $\sim 10^{-45}$ A m^4 V^{-2}. In a paramagnetic with a permanent magnetic dipole moment, the term $|m_{n\alpha}\alpha''_{\beta\gamma}|$ is very roughly of the order 10^{-69} C^2 m^2 T^{-1}. Consistently, therefore, the atomic property tensors in the ferromagnetic sample used by Sanford *et. al.* [1] are much greater in magnitude than in a diamagnetic or paramagnetic. This is consistent with the fact that some ferromagnetics, such as magnetic semiconductors of the type used by Frey *et. al.* [46] show giant Zeeman and Faraday effects orders of magnitude greater than those encountered in diamagnetics such as water. Sanford *et. al.* [1] have elegantly amplified the effect further by tuning the pump laser to peaks in the original Faraday effect spectrum of their sample. This is precisely what was proposed for the inverse Faraday effect by Woźniak *et. al.* [10], who have isolated the resonance spectrum for an idealized three level atom.

There accompanies the first order effect described in Eq. (53) a second order effect, proportional to the square of the ghost field, the modulus of the conjugate product in equation (1). This can be written [45] as

$$\Delta\boldsymbol{\Theta} \sim \frac{1}{2}\omega\mu_0 c^3 I N B^{(3)2}\left(\langle\boldsymbol{\alpha}^{(\pi)}_{1XYZ}(f)\rangle + \frac{1}{kT}\langle\boldsymbol{\alpha}''_{1Zn}\boldsymbol{\alpha}''_{1XY}(f)\rangle\right), \tag{55}$$

where the angular brackets denote averaging over appropriate molecular property tensors, defined in Ref. [45]. This effect depends on the conjugate product, a vector quantity, and as such is also expected to change sign between right and left pump laser polarizations. In general it accompanies the first order effect, Eq. (53), but is directly proportional to the pump intensity itself, and not to the square root of the pump intensity, as in the first order effect. The different contributions can therefore be separated in principle through an analysis of the intensity dependence of the optical Faraday effect.

It must be emphasized that the effects at first *and* second order in $\boldsymbol{B}^{(3)}$ obviously depend on the fact that $\boldsymbol{B}^{(3)}$ is not zero, and both effects show the existence of the fundamental ghost field and photomagneton of light.

Conclusions

There exists a Lie algebra between the transverse (wave) fields, $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ and longitudinal (ghost) field $\boldsymbol{B}^{(3)}$ of electromagnetic radiation. The classical ghost field can provide an explanation for the optical and inverse Faraday effects, and in the quantum field theory is directly proportional to the angular momentum of the photon beam. For one photon the quantum equivalent of $\boldsymbol{B}^{(3)}$ is the elementary photomagneton $\hat{B}^{(3)}$. The existence of the latter allows the photon three degrees of polarization without affecting the fundamental laws of radiation, based on the Planck law. These three degrees of polarization allow for the fact that the photon may be a massive boson and that the electromagnetic four potential A_μ is physically meaningful and manifestly covariant.

Acknowledgements

The Leverhulme Trust, Cornell Theory Center and Penn State University are thanked for support during the formative stages of this work.

References

[1] N. Sanford, R. W. Davies, A. Lempicki, W. J. Miniscalco and S. J. Nettel, *Phys. Rev. Lett.* **50**, 1803 (1983).
[2] G. Wagnière, *Phys. Rev. A* **40**, 2437 (1989).
[3] F. J. Ward, *Rev. Mod. Phys.* **37**, 1 (1965).
[4] M. W. Evans, *Phys. Rev. A* **41**, 4601 (1990).
[5] S. Woźniak, G. Wagnière and R. Zawodny. *Phys. Lett. A* **154**, 259 (1991).
[6] M. W. Evans, *Physica B* **182**, 227 (1992).
[7] M. W. Evans, *Physica B* **183**, 103 (1993).
[8] M. W. Evans, and S. Kielich, Eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, Eds., (Wiley Interscience, New York, 1993).
[9] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1993).
[10] S. Woźniak, M. W. Evans and G. Wagnière, *Mol. Phys.* **75**, 81, 99 (1992).
[11] M. W. Evans, in I. Prigogine and S. A. Rice eds., *Advances in Chemical Physics*, Vol. 81 (Wiley Interscience, New York, 1992).
[12] E. P. Wigner, *Ann. Math.* **40**, 149 (1939).
[13] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[14] R. M. Whitner, *Electromagnetics* (Prentice Hall, Englewood Cliffs, 1962).
[15] L. H. Ryder, *Elementary Particles and Symmetries* (Gordon and Breach, London, 1986); L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).
[16] see vol. 85(1) of Ref. 8.
[17] J. P. van der Ziel, P. S. Pershan and L. D. Malmstrom, *Phys. Rev. Lett.* **15**, 190 (1965); *Phys. Rev.* **143**, 574 (1966).
[18] J. Deschamps, M. Fitaire and M. Lagoutte, *Phys. Rev. Lett.* **25**, 1330 (1970); *Rev. Appl. Phys.* **7**, 155 (1972).
[19] T. W. Barrett, H. Wohltjen and A. Snow, *Nature* **301**, 694 (1983).
[20] B. Talin, V. P. Kaftandjian and L. Klein, *Phys. Rev. A* **11**, 648 (1975).
[21] P. F. Liao and G. C. Bjorklund, *Phys. Rev. Lett.* **36**, 584 (1976).
[22] M. Ayadi and J. Ferre, *Phys. Rev. Lett.* **50**, 274 (1983).
[23] reviewed by R. Zawodny, in Ref. 8, Vol. 85(1).
[24] P. S. Pershan, *Phys. Rev.* **130**, 919 (1963).
[25] B. W. Shore, *The Theory of Coherent Atomic Excitation*, Vol. 1 (Wiley, New York, 1990), Chapter 9.
[26] P. W. Atkins, *Molecular Quantum Mechanics*, 2nd edn. (Clarendon, Oxford, 1983).
[27] S. N. Bose, *Z. Phys.* **26**, 178 (1924).

[28] D. Corson and P. Lorrain, *Introduction to Electromagnetic Waves and Fields* (Freeman, San Francisco, 1962).
[29] A. Pais, *The Science and the Life of Albert Einstein* (Oxford University Press, New York, 1982).
[30] A. Einstein, *Verh. Schw. Naturf. Ges.* **105**, 85 (1924), reviewed in Ref. 29, chapter 23.
[31] L. de Broglie, *La Mécanique Ondulatoire du Photon* (Gauthier-Villars, Paris, 1936, 1940, 1957).
[32] J. P. Vigier, "Present Experimental Status of the Einstein / de Broglie Theory of Light." *Proc. I.S.Q.M.* 1992 Tokyo Workshop on Quantum Mechanics.
[33] M. W. Evans and J. P. Vigier, *Opt. Commun. in press.*
[34] Ref. 25, Vols. 1 and 2.
[35] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics* (Wiley, New York, 1989).
[36] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, 3rd edn. (Wiley Interscience, New York, 1980).
[37] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984).
[38] F. Selleri, Ed. A. van der Merwe, *Quantum Paradoxes and Physical Reality* (Kluwer, Dordrecht, 1990).
[39] discussed in Ref. 32.
[40] A. Einstein, *Ann. Phys.* **17**, 132 (1905), discussed in Ref. 29.
[41] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).
[42] B. A. Zon, V. Ya. Kuperschmidt, G. V. Pakhomov and T. T. Urazbaev, *JETP Lett.* **45**, 273 (1987).
[43] M. W. Evans, *Physica B* **182**, 237 (1992).
[44] M. W. Evans, in Ref. 9.
[45] L. D. Barron, *Molecular Light Scattering and Optical Activity* (Cambridge University Press, Cambridge, 1982).
[46] J. Frey, R. Frey, C. Flytzannis and R. Triboulet, *Opt. Commun.* **84**, 76 (1991).
[47] S. Woźniak, B. Linder and R. Zawodny, *J. Phys. Paris* **44**, 403 (1983).

Chapter 8

CLASSICAL RELATIVISTIC THEORY OF THE LONGITUDINAL GHOST FIELDS OF ELECTROMAGNETISM

M. W. Evans

Abstract

The classical relativistic theory is developed of electric and magnetic fields in terms of boost and rotation generators, respectively, of the Lorentz group of space-time. This development shows that Minkowski geometry requires that there be *three* states of polarization of radiation in free space. The magnetic components in a circular basis are right and left circular and longitudinal. The longitudinal component is real and physical, and proportional to one of the three, non-zero, rotation generators of the Lorentz group. The longitudinal electric component is pure imaginary, and proportional to one of the three boost generators. These theoretical arguments conform with experimental data from the Planck radiation law and from magnetic effects of light such as the inverse Faraday effect.

1. Introduction

It is well known that the Maxwell equations in free space are relativistically invariant, as was first shown by Lorentz in 1904. Shortly afterwards, Poincaré showed that all the equations of electrodynamics are similarly invariant. These results were proven independently by Einstein in 1905, and shown in the theory of special relativity to be generally valid. Einstein based his theory on two principles, the first asserts that the laws of physics take the same form in all Lorentz frames; the second asserts that the constant c is the same in all Lorentz frames. If the photon is regarded as being without mass, c is the speed of light in vacuo; otherwise, if the photon has mass, its speed varies from frame to frame, giving rise to "tired light" [1,2]. Unless otherwise specified, we shall restrict our attention in this paper to the massless photon. However, our results can and will be generalized in further work to the case of finite photon mass, in which the d'Alembert equation is replaced by the Proca equation, or variants thereof such as the de Broglie equation or Duffin-Kemmer--Petiau equations [3–6].

As is well known, the postulated constancy of c allows a connection to be made between Minkowski space-time coordinates in different frames, customarily labelled K and K′, the latter moving at v with respect to the former along the Z axis. The pseudo-Euclidean frame of reference is (X, Y, Z, ict) for K and (X′, Y′, Z′, ict′) for K′, and the assumed linear transformation is based on the relation between frames,

$$X'^2 + Y'^2 + Z'^2 - c^2t'^2 = \lambda^2(X^2 + Y^2 + Z^2 - c^2t^2), \tag{1}$$

where λ is a function of v such that $\lambda(0) = 1$. It can be shown [7] that λ is unity for all v. The Lorentz transform is then defined as,

$$X' = X, \quad Y' = Y, \quad Z' = \gamma(Z - vt), \quad t' = \gamma\left(t - \beta\frac{Z}{c}\right), \tag{2}$$

where $\beta = v/c$ and $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. This can be written in the four by four matrix form

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ ict \end{bmatrix}, \tag{3}$$

which in tensor notation becomes

$$x'_\mu = a_{\mu\nu}x_\nu . \tag{4}$$

The four by four matrix in Eq. (3) is the Lorentz transformation matrix and defines the boost generators to be used in this paper, generators which participate in the Lie algebra of the Lorentz group. There are three boost generators, one for each space axis, and each is a pure real four by four matrix by definition [8, 9]. Additionally, there are three rotation generators in the Lorentz group [8, 9], being four by four pure imaginary matrices by definition.

The boost generators, $\hat{K}$, and rotation generators, $\hat{J}$, are based directly on Minkowski geometry, i.e., the pseudo-Euclidean geometry of space-time. In Sec. 2 of this paper it is shown that axial unit vectors in the three dimensions of Euclidean space can be expressed as antisymmetric 3 x 3 unit tensors which are related directly to the rotation generators of three dimensional space, and by simple extrapolation, to those of the four dimensional Lorentz group, generators which are four by four matrices in Minkowski space-time. It follows that magnetic fields, which can be expressed in terms of axial vectors in three dimensional (3-D) space, become rotation generators in the 4-D Lorentz group. By using a suitable circular basis, the $\hat{J}$ matrices are used to define magnetic fields from Maxwell's equations in vacuo. This method leads directly and geometrically to the conclusion that there is a real longitudinal magnetic field from these equations in vacuo. This solution is also related geometrically to the two transverse ones (right and left circular polarization) using the Lie algebra of the rotation generators $\hat{J}$.

In Sec. 3, this method is extended to electric fields, which in 3-D space can be expressed in terms of polar unit vectors. These have no 3 x 3 matrix equivalents in 3-D, but it is well known that the vector cross product of two polar vectors in 3-D is an axial vector. In 4-D the equivalent is that the commutation of two boost generators is a rotation generator, showing that a polar vector in 4-D is a boost generator of the Lorentz group. With this result, electric fields in space-time also become boost generators, which are expressed in a circular basis.

Section 4 is a summary of the complete Lie algebra of electric and magnetic fields in vacuo, expressed respectively as boost and rotation generators of the Lorentz group of special relativity, using a suitable circular basis to relate these transverse and longitudinal fields to solutions of Maxwell's equations in vacuo. In this way, it is demonstrated geometrically that there exists a real, physically meaningful, longitudinal magnetic field, which in space-time is directly proportional to the rotation generator $\hat{J}^{(3)}$ in a circular basis. The generator $\hat{J}^{(3)}$ is a non-zero 4 x 4 matrix as a direct result of Minkowski geometry itself, i.e., as a result of the nature of space-time. Similarly, there exists a pure imaginary electric field $i\hat{E}^{(3)}$ which is directly proportional to the non-zero boost generator $\hat{K}^{(3)}$ of the Lorentz group. The electric and magnetic field solutions of Maxwell's equations therefore form the complete Lie algebra of the Lorentz group and part of that of the Poincaré group, longitudinal and transverse components being related by this Lie algebra.

Finally a discussion is given of the experimental support for this conclusion, and of experimental consequences in electrodynamics.

2. Magnetic Fields as Rotation Generators

It can be demonstrated in an elementary way [10–12] that there exists a cyclically symmetric relation between the transverse and longitudinal magnetic field solutions of Maxwell's equations in vacuo,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \tag{5a}$$

$$\boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} = iB^{(0)}\boldsymbol{B}^{(1)*}, \tag{5b}$$

$$\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} = iB^{(0)}\boldsymbol{B}^{(2)*}, \tag{5c}$$

where $B^{(0)}$ is the scalar magnetic flux density magnitude in tesla, and where the fields are defined in vacuo as the following solutions of Maxwell's equations,

$$\boldsymbol{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \tag{6a}$$

$$\boldsymbol{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}}(-i\boldsymbol{i}+\boldsymbol{j})e^{-i\phi}, \tag{6b}$$

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}. \tag{6c}$$

Here the oscillating transverse component (1) is the complex conjugate of component (2), and is expressed in terms of the phase $\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}$ of a travelling plane wave in vacuo whose angular frequency is ω at an instant t and whose wave vector is $\boldsymbol{\kappa}$ at a point $\boldsymbol{r}$ in space. Note that component (3) is phase free, so that $\nabla\cdot\boldsymbol{B}^{(3)} = 0$.

The magnetic flux densities $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are expressed in a natural, circular basis defined [10–12] by unit vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, and $\boldsymbol{e}^{(3)}$

$$\boldsymbol{e}^{(1)} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}), \tag{7a}$$

$$\boldsymbol{e}^{(2)} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j}), \tag{7b}$$

$$\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)*} = i\boldsymbol{k}, \tag{7c}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are Cartesian unit vectors in X, Y, and Z respectively, Z being the propagation axis of the plane wave. In Eqs. (5a) to (5c), * denotes "complex conjugate." It is obvious that in this circular basis, which naturally defines right and left circular polarization, $\boldsymbol{B}^{(2)}$ must be the complex conjugate of $\boldsymbol{B}^{(1)}$, so that $\boldsymbol{B}^{(3)}$ must be phase free. The perfectly cyclical symmetry of Eq. (5) is therefore a direct geometrical consequence of the isotropy of 3-D space. If $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, then $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ also disappear, showing immediately that the standard approach [13–16], in which $\boldsymbol{B}^{(3)}$ is unrelated to $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$, is *geometrically* unsound in 3-D. The standard approach violates the isotropy of free space and unnaturally reduces it to a plane. The ramifications of this conclusion have been discussed elsewhere [10–12]. The assertion that $\boldsymbol{B}^{(3)}$ is zero is usually [13–16] based on the Maxwell equation $\nabla\cdot\boldsymbol{B} = 0$. A longitudinal field in vacuo that depends on the phase of a plane wave [13–16] cannot satisfy this equation, so it is conventionally assumed that $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, or is otherwise "irrelevant" or "unrelated" to the transverse components. The existence of the simple cyclic algebra (5) appears never to have been realized prior to Refs. [10–12]. However, from Eq. (5a), the standard $\boldsymbol{B}^{(3)} = \boldsymbol{0}$ means that the conjugate product [17–19] $\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ is also zero, a result which means, for example, that the inverse Faraday effect [20–26] (phase free, or "static" magnetization by light) disappears. The inverse Faraday effect is, however, well demonstrated experimentally [20–26] and is theoretically [17] *directly proportional* to

$\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$, which is in turn algebraically the same as $iB^{(0)}\boldsymbol{B}^{(3)}$. This line of reasoning, based on experimental data, exposes a basic paradox in standard electrodynamics, a paradox which is embodied in special relativity [27], for example, in the well known fact that the Wigner little group [28, 29] in the conventional approach (with $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$) is the unphysical E(2), an Euclidean planar group, and not the natural group of rotations in three dimensions. This in turn leads to the well known loss of manifest covariance in the vector potential A_μ, something which is fundamentally at odds with the Bohm-Aharonov effect [29, 30] which shows A_μ to be *physically* meaningful. It is illogical in special relativity to assert that a physically meaningful four-vector is not manifestly covariant. In other words, all four components of A_μ must be physically meaningful as for any other four-vector. The usual assertion $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$ leads to absurdity therefore in 3-D and 4-D, an absurdity which is habitually accepted in the standard approach to electrodynamics and the U(1) sector of field theory.

With this preamble, the aim of this and the next two sections is to devise the relativistically self-consistent Lie algebra of the electric and magnetic solutions of Maxwell's equations in vacuo without making the usual assertion $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$. In this section, it is shown that $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ are each directly proportional to a standard rotation generator of the Lorentz group. These rotation generators are *all* non-zero by Minkowski geometry.

The starting point of the proof of this result is to express the Cartesian *axial* unit vectors as antisymmetric matrices using the fact that an axial rank one vector is also a polar antisymmetric rank two tensor [31]. The three rank two tensors are

$$\hat{i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{k} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8}$$

which in the circular basis (7) become

$$\hat{e}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{bmatrix}, \quad \hat{e}^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{bmatrix}, \quad \hat{e}^{(3)} = \hat{k}. \tag{9}$$

The latter form a classical commutator algebra which is cyclically symmetric in Euclidean space,

$$
\left.
\begin{aligned}
[\hat{e}^{(1)}, \hat{e}^{(2)}] &= -i\hat{e}^{(3)*} = -i\hat{e}^{(3)}, \\
[\hat{e}^{(2)}, \hat{e}^{(3)}] &= -i\hat{e}^{(1)*} = -i\hat{e}^{(2)}, \\
[\hat{e}^{(3)}, \hat{e}^{(1)}] &= -i\hat{e}^{(2)*} = -i\hat{e}^{(1)}.
\end{aligned}
\right\} \tag{10}
$$

In these equations, all three commutators are formed from geometrically meaningful component matrices which are all non-zero. Obviously, if one of these is arbitrarily set to zero, the other two vanish, and the geometrical structure is destroyed. This is the geometrical basis for the existence of a real three dimensional matrix representation of the magnetic part of free space electromagnetism. Specifically,

$$
\hat{B}^{(1)} = iB^{(0)}\hat{e}^{(1)}e^{i\phi}, \qquad \hat{B}^{(2)} = -iB^{(0)}\hat{e}^{(2)}e^{-i\phi}, \qquad \hat{B}^{(3)} = B^{(0)}\hat{e}^{(3)}, \tag{11}
$$

from which emerges the classical commutative algebra equivalent to the vectorial algebra (5),

$$
\left.
\begin{aligned}
[\hat{B}^{(1)}, \hat{B}^{(2)}] &= -iB^{(0)}\hat{B}^{(3)*} = -iB^{(0)}\hat{B}^{(3)}, \\
[\hat{B}^{(2)}, \hat{B}^{(3)}] &= -iB^{(0)}\hat{B}^{(1)*} = -iB^{(0)}\hat{B}^{(2)}, \\
[\hat{B}^{(3)}, \hat{B}^{(1)}] &= -iB^{(0)}\hat{B}^{(2)*} = -iB^{(0)}\hat{B}^{(1)},
\end{aligned}
\right\} \tag{12}
$$

an algebra which is again cyclically symmetric in Euclidean space. This algebra can now be expressed in terms of the rotation generators, $\hat{J}$, of the O(3) group [29] of three dimensional space, but using the circular basis (7) instead of the usual [29] Cartesian one,

$$
\hat{J}^{(1)} = \frac{\hat{e}^{(1)}}{i}, \qquad \hat{J}^{(2)} = \frac{-\hat{e}^{(2)}}{i}, \qquad \hat{J}^{(3)} = \frac{\hat{e}^{(3)}}{i}. \tag{13}
$$

Note that the generator $\hat{J}^{(3)}$ is pure imaginary. The magnetic field matrices and rotation generators are then linked by

$$
\hat{B}^{(1)} = -B^{(0)}\hat{J}^{(1)}e^{i\phi}, \qquad \hat{B}^{(2)} = -B^{(0)}\hat{J}^{(2)}e^{-i\phi}, \qquad \hat{B}^{(3)} = iB^{(0)}\hat{J}^{(3)}, \tag{14}
$$

a representation which accounts naturally for the phase, and which is of key importance in recognizing that the commutative algebra of the magnetic part of free space electromagnetism *is part of the Lie algebra [28, 29] of the Lorentz group of Minkowski space-time*. A Lie algebra is that of generators of a group, and therefore, magnetic components of free space electromagnetism are directly proportional to the *rotation* generators of the Lorentz group. Therefore,

commutative relations between magnetic fields are relations between spins and a magnetic field is a property of space-time itself. Therefore there are three components of a magnetic field in space because there are three rotation generators. Finally, magnetic field components are interrelated by commutators, in the same way as rotation generators.

The generalization of rotation generators from O(3) to the Lorentz group occurs as follows [29]:

$$
\hat{J}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad
\hat{J}^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad
\hat{J}^{(3)} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}
$$

It follows that magnetic fields in the four coordinates of space-time are also 4 x 4 matrices. There are three rotation generators in space-time, which obey the classical commutative algebra,

$$
\left.
\begin{aligned}
[\hat{J}^{(1)}, \hat{J}^{(2)}] &= -\hat{J}^{(3)*} = \hat{J}^{(3)}, \\
[\hat{J}^{(2)}, \hat{J}^{(3)}] &= -\hat{J}^{(1)*} = -\hat{J}^{(2)}, \\
[\hat{J}^{(3)}, \hat{J}^{(1)}] &= -\hat{J}^{(2)*} = -\hat{J}^{(1)},
\end{aligned}
\right\} \tag{16}
$$

which becomes the more familiar [29]

$$
[\hat{J}_X, \hat{J}_Y] = i\hat{J}_Z, \qquad [\hat{J}_Y, \hat{J}_Z] = i\hat{J}_X, \qquad [\hat{J}_Z, \hat{J}_X] = i\hat{J}_Y, \tag{17}
$$

in a Cartesian basis.

By geometry, therefore, it becomes absurd to assert $\hat{B}^{(3)} =? \hat{0}$, because by Eq. (14) this means $\hat{J}^{(3)} =? \hat{0}$, in direct contradiction with Eq. (15). This vividly and conclusively demonstrates that the standard approach [13–16] unnaturally reduces isotropic space to a plane.

The classical commutative algebra of rotation generators is, within a factor $\hbar$, *the commutator algebra of angular momentum operators in quantum mechanics*. Realizing this immediately leads to the quantization of the magnetic fields of the plane wave in vacuo, giving the result,

$$
\hat{B}^{(1)} = -B^{(0)}\frac{\hat{J}^{(1)}}{\hbar}e^{i\phi}, \qquad \hat{B}^{(2)} = -B^{(0)}\frac{\hat{J}^{(2)}}{\hbar}e^{-i\phi}, \qquad \hat{B}^{(3)} = iB^{(0)}\frac{\hat{J}^{(3)}}{\hbar}. \tag{18}
$$

where $\hat{B}^{(i)}$ are now operators in quantum field theory. In particular, the longitudinal operator $\hat{B}^{(3)}$ is the elementary quantum of magnetic flux density in the propagation axis. We refer to this hereinafter as the *photomagneton*. We refer to the expectation value of $\hat{B}^{(3)}$ as *the ghost field*, because $\hat{B}^{(3)}$ has no Planck energy [10-12], being phase free. In consequence, $\hat{B}^{(3)}$ is not absorbed or emitted by an atom or molecule, and can be detected only by its magnetization of matter in such phenomena as the inverse Faraday effect [20–26]. The ghost field is therefore far more difficult to detect experimentally than the everyday, oscillating $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$. This is probably why $\hat{B}^{(3)}$ has not been considered in electrodynamics. However, it is clear from Eq. (5a), for example, that if $\boldsymbol{B}^{(3)}$ were zero, then the experimentally observed [20–26] inverse Faraday effect would not exist, because the conjugate product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ would vanish. Finally, the source of $\boldsymbol{B}^{(3)}$ is the same as the source of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$, and $\hat{B}^{(3)}$ is directly proportional to the angular momentum of the photon through Eq. (18c). The eigenvalues of $\hat{B}^{(3)}$ are therefore $B^{(0)}$ and $-B^{(0)}$. These are, of course, defined in the *longitudinal* axis (3), i.e., Z of the Cartesian basis. The assertion $\hat{B}^{(3)} =? \hat{0}$ in the quantum field theory is therefore absurd, because it means that the eigenvalues of photon spin are zero, whereas they are well known to be $\pm\hbar$, an irremovable property of the photon [32].

3. Electric Fields as Boost Generators

An electric field is a polar vector in three, Euclidean dimensions, and unlike an axial vector, cannot be put into a 3 x 3 matrix form such as embodied in Eq. (8). The cross product of two polar vectors is, however, an axial vector in Euclidean space. For example, the product:

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \tag{19}$$

produces the Cartesian, axial, unit vector $\boldsymbol{k}$, which in the circular basis is $\boldsymbol{e}^{(3)}$. In Minkowski space-time the axial vector $\boldsymbol{k}$ is known from the arguments in Sec. 2 to be a 4 x 4 matrix, related directly to the rotation generator $\hat{J}^{(3)}$, of the Lorentz group. It follows that a rotation generator in space-time is the result of a classical commutation of two matrices which play the role of polar vectors. From the well established Lie algebra [29] of the generators of the Lorentz group, these matrices are *boost generators*, 4 x 4 real matrices. The equivalent of Eq. (19) in Minkowski space-time is therefore

$$\left[\hat{K}_X, \hat{K}_Y\right] = -i\hat{J}_Z, \tag{20}$$

and cyclic permutations. In the circular basis (1), (2), (3) (rather than the Cartesian basis X, Y, Z) this commutator algebra becomes the cyclically symmetric,

$$\left.\begin{aligned} \left[\hat{K}^{(1)}, \hat{K}^{(2)}\right] &= -i\hat{e}^{(3)*} = -\hat{J}^{(3)*}, \\ \left[\hat{K}^{(2)}, \hat{K}^{(3)}\right] &= -i\hat{e}^{(1)*} = -\hat{J}^{(1)*}, \\ \left[\hat{K}^{(3)}, \hat{K}^{(1)}\right] &= -i\hat{e}^{(2)*} = +\hat{J}^{(2)*}. \end{aligned}\right\} \tag{21}$$

Therefore, although polar vectors cannot be put in a matrix form in Euclidean space, they correspond to boost generators, 4 x 4 matrices, in Minkowski space-time.

This essentially geometrical result leads directly to the conclusion that *electric fields in space-time are proportional to boost generators* because electric fields in Euclidean space are proportional to polar unit vectors. Therefore the fundamental geometry of Minkowski space-time demands that magnetic fields are composed of rotation generators (imaginary 4 x 4 matrices) and electric fields are composed of boost generators (real 4 x 4 matrices). Furthermore, boost and rotation generators are linked by the Lie algebra of the Lorentz group, which is written out in full in the following section. It follows that electric and magnetic fields in space-time also form a Lie algebra of the Lorentz group, in any suitable basis, e.g. Cartesian or circular for the space part of space-time. The circular basis is suited naturally for solutions of Maxwell's equations, because of the fact that there is a right and left circular polarization, mutually orthogonal to the longitudinal polarization of the propagation axis.

In Euclidean space, electric field solutions to Maxwell's equations are conventionally regarded as the transverse, oscillatory counterparts of Eqs. (6a) and (6b),

$$\boldsymbol{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \tag{22a}$$

$$\boldsymbol{E}^{(2)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-i\phi}, \tag{22b}$$

which can be written directly in terms of the unit vectors of the circular basis,

$$\boldsymbol{E}^{(1)} = E^{(0)}\boldsymbol{e}^{(1)}e^{i\phi}, \qquad \boldsymbol{E}^{(2)} = E^{(0)}\boldsymbol{e}^{(2)}e^{-i\phi}. \tag{23}$$

In Minkowski space-time, the equivalents are therefore

$$\hat{E}^{(1)} = E^{(0)}\hat{K}^{(1)}e^{i\phi}, \qquad \hat{E}^{(2)} = E^{(0)}\hat{K}^{(2)}e^{-i\phi}. \tag{24}$$

(The phase ϕ is a Lorentz invariant [13], and remains the same in space-time and Euclidean space.) The boost generators appearing in Eq. (24) are written in a circular basis,

$$\hat{K}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \end{bmatrix}, \qquad \hat{K}^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 \end{bmatrix}, \tag{25}$$

and correspond to the complex, polar, unit vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ in Euclidean space.

By reference to the Lie commutator algebra (21) it is clear that the commutation of $\hat{K}^{(1)}$ and $\hat{K}^{(2)}$ is $\hat{J}^{(3)} = -\hat{J}^{(3)*}$, a *rotation* generator, directly proportional to a *magnetic* field. The equivalent result in Euclidean space is

$$\hat{e}^{(1)} \times \hat{e}^{(2)} = i\hat{e}^{(3)} \tag{26}$$

It is not possible to form an electric field from the cross product of $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ and this conforms with fundamental symmetry [10–12]. The question therefore arises as to what is the electric field proportional to the third boost generator $\hat{K}^{(3)}$ of Minkowski space-time, or equivalently, the *polar* vector $\mathbf{k}$ of Euclidean space. That there must be a longitudinal electric component is clear from both 3-D and 4-D geometry, and from the Maxwell equation in vacuo $\nabla \cdot \mathbf{E} = 0$, this component must be phase free. By writing out the longitudinal rotation and boost generators,

$$\hat{J}^{(3)} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \hat{K}^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{27}$$

it is seen that the former is pure imaginary and the latter is pure real. It follows that there are two possibilities:

1. that the longitudinal $\hat{B}^{(3)}$ is pure real and the longitudinal $i\hat{E}^{(3)}$ is pure imaginary;
2. vice versa.

Choice (1) follows, however, from a consideration of the nature of the unit vectors $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$ of the circular basis (7), in which the *axial* $\mathbf{e}^{(3)}$ *is pure real, and equal to the real, Cartesian, axial* $\mathbf{k}$. From Eq. (6c), multiplying this real, axial, unit vector by the amplitude $B^{(0)}$ (a real scalar) gives a *real* $\mathbf{B}^{(3)}$ in Euclidean space, and a real $\hat{B}^{(3)}$ in space-time. This *real* $\hat{B}^{(3)}$ is therefore defined as

$$\hat{B}^{(3)} \equiv iB^{(0)}\hat{J}^{(3)} \tag{28}$$

in terms of the *imaginary* rotation generator $\hat{J}^{(3)}$. It follows that the *imaginary* $i\hat{E}^{(3)}$ must be defined as

$$i\hat{E}^{(3)} \equiv iE^{(0)}\hat{K}^{(3)} \tag{29}$$

in terms of the *real* boost generator $\hat{K}^{(3)}$.

In the next section, we write out the complete Lie algebra of electric and magnetic fields in the Lorentz group.

4. The Lie Algebra of Electric and Magnetic Fields in the Lorentz Group

The complete Lie algebra of the boost and rotation generators of the Lorentz group can be written in a Cartesian or circular basis as follows. In a Cartesian basis [29],

$$\hat{K}_X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{K}_Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{K}_Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\hat{J}_X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{J}_Y = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{J}_Z = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{30}$$

so that

$$\begin{aligned} &[\hat{J}_X, \hat{J}_Y] = i\hat{J}_Z, && \text{and cyclic permutations,} \\ &[\hat{K}_X, \hat{K}_Y] = -i\hat{J}_Z, && \text{" " "}, \\ &[\hat{K}_X, \hat{J}_Y] = i\hat{K}_Z, && \text{" " "}, \\ &[\hat{K}_X, \hat{J}_X] = 0, && \text{etc.} \end{aligned} \tag{31}$$

This can be summarized concisely [29] as,

$$\hat{\mathcal{J}}_{\mu\nu}(\mu,\nu=0,\ldots,3)\begin{cases}\hat{\mathcal{J}}_{ij} = -\hat{\mathcal{J}}_{ji} = \epsilon_{ijk}\hat{\mathcal{J}}_k \\ \hat{\mathcal{J}}_{io} = -\hat{\mathcal{J}}_{oi} = -\hat{K}_i\end{cases} \quad (i,j,k=1,2,3) \tag{32}$$

and displayed as the 4 x 4 antisymmetric matrix of matrices,

$$\hat{\mathcal{J}}_{\mu\nu} = \begin{bmatrix} 0 & -\hat{\mathcal{J}}_3 & \hat{\mathcal{J}}_2 & -\hat{K}_1 \\ \hat{\mathcal{J}}_3 & 0 & -\hat{\mathcal{J}}_1 & -\hat{K}_2 \\ -\hat{\mathcal{J}}_2 & \hat{\mathcal{J}}_1 & 0 & -\hat{K}_3 \\ \hat{K}_1 & \hat{K}_2 & \hat{K}_3 & 0 \end{bmatrix} \tag{33}$$

The antisymmetric structure of $\hat{\mathcal{J}}_{\mu\nu}$ is reminiscent of the well known electromagnetic tensor $\hat{F}_{\mu\nu}$ of special relativity (the four-curl of A_μ); which in S.I. units is,

$$\hat{F}_{\mu\nu} = \epsilon_0 \begin{bmatrix} 0 & -cB_3 & cB_2 & -iE_1 \\ cB_3 & 0 & -cB_1 & -iE_2 \\ -cB_2 & cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix} \tag{34}$$

where ϵ_0 is the permittivity in vacuo. By comparing Eq. (34), which is a matrix of Euclidean space field vector components, with Eq. (33), a matrix of Minkowski boost and rotation generators, it becomes intuitively clear that there is a relation between electric and magnetic fields and boost and rotation generators of the type developed in Secs. 2 and 3. We come to see that the Maxwell equations, two of which [29] can be written as,

$$\frac{\partial \hat{F}_{\mu\nu}}{\partial x_\mu} = 0 \tag{35}$$

where $x_\mu \equiv (X, Y, Z, ict)$, *are relations between boost and rotation generators* which can be written formally as

$$\frac{\partial \hat{\mathcal{J}}_{\mu\nu}}{\partial x_\mu} = 0 \tag{36}$$

In comparing $\hat{F}_{\mu\nu}$ of standard electrodynamics [29] with $\hat{\mathcal{J}}_{\mu\nu}$ of standard special relativity, we see that:

1. $\hat{F}_{\mu\nu}$ already accounts for the fact that in general, *there is a real longitudinal cB_3 and an imaginary longitudinal iE_3.* These are obviously not set to zero in Eq. (34). The $\hat{F}_{\mu\nu}$ matrix, devised by Einstein in 1905 [33, 34], contains components all of which are valid solutions to Maxwell's equations, including the longitudinal solutions cB_3 and iE_3. These components are not zero in vacuo, because they are tied to the transverse components through fundamental space-time geometry.

2. From a comparison of elements of $\hat{F}_{\mu\nu}$ and $\hat{\mathcal{J}}_{\mu\nu}$ it becomes clear that if the electric field components can be converted to boost generators, and the magnetic fields to rotation generators, $\hat{F}_{\mu\nu}$ becomes directly proportional to $\hat{\mathcal{J}}_{\mu\nu}$.

In order to demonstrate this proportionality we need to transfer to a circular basis, because plane wave solutions of Maxwell's equations are written naturally in a circular basis, multiplying a phase factor. It is also necessary to convert the electric and magnetic field components in Eq. (34) from Euclidean space to Minkowski space-time. This is precisely what has been achieved in Secs. 1 and 2. Specifically, the equivalents of Eq. (30) of the Cartesian basis are given in the circular basis of Eqs. (7) by Eqs. (15), (25) and (27). In the circular basis, Eqs. (31) become:

$$\begin{aligned} &[\hat{\mathcal{J}}^{(1)}, \hat{\mathcal{J}}^{(2)}] = -\hat{\mathcal{J}}^{(3)*}, && \text{and cyclic permutations,} \\ &[\hat{K}^{(1)}, \hat{K}^{(2)}] = -i\hat{e}^{(3)*}, && \text{" " "}, \\ &[\hat{K}^{(1)}, \hat{e}^{(2)}] = -i\hat{K}^{(3)*}, && \text{" " "}, \\ &[\hat{K}^{(1)}, \hat{\mathcal{J}}^{(1)}] = 0, && \text{etc.,} \end{aligned} \tag{37}$$

with

$$\begin{aligned} i\hat{e}^{(1)*} &= \hat{\mathcal{J}}^{(1)*}, & -i\hat{e}^{(2)*} &= \hat{\mathcal{J}}^{(2)*}, & i\hat{e}^{(3)*} &= \hat{\mathcal{J}}^{(3)*}, \\ i\hat{e}^{(2)} &= \hat{\mathcal{J}}^{(2)}, & -i\hat{e}^{(1)} &= \hat{\mathcal{J}}^{(1)}, & i\hat{e}^{(3)} &= -\hat{\mathcal{J}}^{(3)}. \end{aligned} \tag{38}$$

In terms of electric and magnetic fields, these cyclically symmetric geometrical relations can also be written as a complete algebra which is by implication a Lie algebra of the Lorentz group,

$$\begin{aligned} &[\hat{B}^{(1)}, \hat{B}^{(2)}] = iB^{(0)}\hat{B}^{(3)*}, && \text{and cyclic permutations,} \\ &[\hat{E}^{(1)}, \hat{E}^{(2)}] = -iE^{(0)2}\hat{e}^{(3)*} = -ic^2B^{(0)}\hat{B}^{(3)*}, && \text{etc.,} \\ &[\hat{E}^{(1)}, \hat{B}^{(2)}] = iB^{(0)}(i\hat{E}^{(3)}), && \text{etc.,} \\ &[\hat{E}^{(1)}, \hat{B}^{(1)}] = 0, && \text{etc.} \end{aligned} \tag{39}$$

These relations demonstrate that the assertion $\hat{B}^{(3)} =? \hat{0}$ [35] is relativistically incorrect. Although $i\hat{E}^{(3)}$ is imaginary, it too is not zero.

Discussion

In order to obtain the rigorously correct field algebra in 4-D, Eqs. (38), in free space-time, it is essential to use the fundamental geometry, Eq. (37) written in terms of boost and rotation generators. The following relations between fields and generators must then be substituted into Eqs. (37) to obtain Eqs. (39),

$$\begin{aligned}
\hat{B}^{(1)} &= -B^{(0)}\hat{J}^{(1)}e^{i\phi} = iB^{(0)}\hat{e}^{(1)}e^{i\phi},\\
\hat{B}^{(2)} &= -B^{(0)}\hat{J}^{(2)}e^{-i\phi} = -iB^{(0)}\hat{e}^{(2)}e^{-i\phi},\\
\hat{B}^{(3)} &= iB^{(0)}\hat{J}^{(3)} = B^{(0)}\hat{e}^{(3)},\\
\hat{E}^{(1)} &= E^{(0)}\hat{K}^{(1)}e^{i\phi},\\
\hat{E}^{(2)} &= E^{(0)}\hat{K}^{(2)}e^{-i\phi},\\
i\hat{E}^{(3)} &= iE^{(0)}\hat{K}^{(3)}.
\end{aligned} \qquad (40)$$

These relations emphasize that $i\hat{E}^{(3)}$ is pure imaginary and $\hat{B}^{(3)}$ pure real. In Euclidean space, they reduce to $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$, imaginary polar and real axial vectors respectively.

It is critically important to note that the classical electromagnetic energy density in vacuo [36],

$$U^{(3)} = \epsilon_0 i\boldsymbol{E}^{(3)}\cdot i\boldsymbol{E}^{(3)} + \frac{1}{\mu_0}\boldsymbol{B}^{(3)}\cdot\boldsymbol{B}^{(3)} = 0 \qquad (41)$$

In other words, the correct *combination* of $i\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ adds nothing to the Rayleigh-Jeans law, a classical limit of the Planck radiation law. This is another vivid demonstration of how the existence of $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ has been by-passed in electromagnetic theory, so that it has become habitual to neglect them. Recent comments in the literature [35] continue to assert erroneously that $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, an assertion that makes nonsense out of fundamental geometry.

Equation (41) is one illustration of why $\boldsymbol{B}^{(3)}$ is a ghost field. Its influence cannot be detected through measurements of light intensity leading to the Rayleigh-Jeans law. Following the rule that real fields are physical, imaginary fields are unphysical, it is expected that $\boldsymbol{B}^{(3)}$ will magnetize material, but that $i\boldsymbol{E}^{(3)}$ will not produce electric polarization. This conforms with what is available experimentally to date on the magnetizing effects of light. The inverse Faraday effect [20–26] for instance can be shown [17] to be directly proportional to $iB^{(0)}\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ at second order in $B^{(0)}$, the magnitude of $\boldsymbol{B}^{(3)}$. This shows *experimentally* that $\boldsymbol{B}^{(3)}$ is not zero, and that effects due to $\boldsymbol{B}^{(3)}$ are expected a priori at first order [10–12]. If $\boldsymbol{B}^{(3)}$ were zero, the inverse Faraday effect would not exist experimentally, and fundamental geometry would be invalidated. It remains to be seen experimentally whether $\boldsymbol{B}^{(3)}$ can act at first order. There appears at present to be no reason why not, and such an effect, proportional to the *square root* of intensity, would appear in the inverse Faraday effect under suitable circumstances, namely in material with a net magnetic dipole moment [10].

The ghost field $i\boldsymbol{E}^{(3)}$ is imaginary and unphysical for this reason, and no electric polarization due to $i\boldsymbol{E}^{(3)}$ can occur. Indeed, to date, no large first order polarizing effects of this nature have been observed experimentally.

Both $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are phase free, and in the quantum theory have no Planck energy, being associated with no frequency. They are not therefore absorbed [10–12] by an atom or molecule, and cannot be detected by ordinary techniques of spectroscopy such as infra red or Raman. The field $\boldsymbol{B}^{(3)}$ can be detected only in a difficult experiment such as the inverse Faraday effect, or in related phenomena such as light shifts [37], the optical Faraday effect [38], or light induced shifts in NMR [39]. The inverse Faraday effect is due to $iB^{(0)}\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)}$ at second order as we have seen. If careful future measurements on the inverse Faraday effect show that $\boldsymbol{B}^{(3)}$ cannot act at first order, it does not follow that $\boldsymbol{B}^{(3)}$ is zero, because that would mean that $iB^{(0)}\boldsymbol{B}^{(3)}$ itself would vanish, contrary to experience. A postulated inability of $\boldsymbol{B}^{(3)}$ to act at first order might mean, for example, that it takes two modes, (1) and (2), to define it, needing light to act at first order in intensity (second order in the amplitude $B^{(0)}$). However, these matters must be settled by further careful experimentation.

Free space electromagnetism, by causality, originates in a source infinitely removed from the vacuum. The source is made up of charged particles and currents, for example moving electrons which radiate. The source gives the free space amplitude $E^{(0)} = cB^{(0)}$ in S.I. units. If the nature of the source is changed so that the sign of $B^{(0)}$ is reversed (e.g. by replacing the electrons by positrons) it is clear from Eqs. (39) that *all six* field components are changed in sign precisely. This process is not equivalent to a phase shift (a change in ϕ) because in a phase shift, the sign of $B^{(0)}$ is not changed. (In other words the charge conjugation operator $\hat{C}$, *by definition*, changes the sign of $B^{(0)}$ but not of ϕ, because the latter is a spatio-temporal quantity [40].) If it is argued [35] that the change $B^{(0)}$ to $-B^{(0)}$ results in $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, then it must also result in $\boldsymbol{B}^{(1)} =? \boldsymbol{0}$ and $\boldsymbol{B}^{(2)} =? \boldsymbol{0}$. This argument [35], which is equivalent to the erroneous assertion that $\boldsymbol{B}^{(3)}$ violates $\hat{C}$, therefore *leads to the disappearance of electromagnetism*. In the presence of electromagnetism in vacuo, this is obviously not true, and the argument fails. We can see this clearly from Eqs. (5), for example, which conserve $\hat{C}$. (The $\hat{C}$ symmetry on both sides of all three Eqs. (5) is positive.)

On the most fundamental level, electromagnetic theory requires a sign to be allocated to the charge on the electron, i.e., e is negative by convention. In the same way, the equations of electrodynamics implicitly attach a positive sign to $B^{(0)}$ (or to $E^{(0)}$). It is assumed that free space electromagnetism has been generated by an infinitely distant source in such a way that the sign of $B^{(0)}$ is positive. The equations work equally well with a negative $B^{(0)}$, but by convention, $B^{(0)}$ is positive. At the most basic level this means that evolution in the universe, and in our solar system, has been such that e is negative and $B^{(0)}$ is positive. It is possible to work through all the equations of electrodynamics with a positive e and a negative $B^{(0)}$, but this produces a universe composed of anti-matter, and a source of free space electromagnetism made of anti-matter.

It remains true, however, that if a source were available in the laboratory that produced a negative $B^{(0)}$, the sign of $\mathbf{B}^{(3)}$ would be opposite from that produced by a source giving a positive $B^{(0)}$ for the same sense of circular polarization. This clearly requires the use of two different sources, one made up of moving electrons, the other of moving positrons. Any *first order* effect of $\mathbf{B}^{(3)}$ in this case would be opposite in sign for the two different sources. Effects at second order in $B^{(0)}$ would not be changed in sign.

Acknowledgments

Support for this work is acknowledged at various stages from the Leverhulme Trust, Cornell Theory Center, the Materials Research Laboratory, Pennsylvania State University, and North Carolina Supercomputing Center.

References

[1] L. de Broglie, *La Mèchanique Ondulatoire du Photon* (Gauthier-Villars, Paris, 1936, 1940, 1957).
[2] J.-P. Vigier, "Present Experimental Status of the Einstein-de Broglie Theory of Light.", *Proceedings I.S.Q.M., 1992 Tokyo Workshop on Quantum Mechanics*.
[3] A. S. Goldhaber and A. A. Nieto, *Rev. Mod. Phys.* **43(3)**, 277 (1971).
[4] A. Proca, *Comptes Rendues* **190**, 1377; **191**, 26 (1930).
[5] A. Proca, *J. Phys. et Rad.*, *VII* **7**, 347; **8**, 23 (1936).
[6] R. J. Duffin, *Phys. Rev.* **54**, 1114 (1938).
[7] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[8] M. W. Evans, and S. Kielich, Eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, Eds., (Wiley Interscience, New York, 1993).

[9] M. W. Evans, "On Longitudinal Free Space-time Electric and Magnetic Fields in the Einstein-de Broglie Theory of Light", *Proceedings de Broglie Centennial Conference at Trani*, A. van der Merwe, Ed. (Plenum, New York, 1993/1994).
[10] M. W. Evans, *Physica B* **182**, 227, 237 (1992); **183**, 103 (1993).
[11] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1993).
[12] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon*, "Fundamentals of Physics", A. van der Merwe, Ed. (Kluwer, Dordrecht) in preparation.
[13] R. M. Whitner, *Electromagnetics* (Prentice Hall, Englewood Cliffs, New Jersey, 1962).
[14] A. F. Kip, *Fundamentals of Electricity and Magnetism* (McGraw-Hill, New York, 1962).
[15] M. Born and E. Wolf, *Principles of Optics*, 6th edn. (Pergamon, Oxford, 1975).
[16] B. W. Shore, *The Theory of Coherent Atomic Excitation*, Vols. 1 and 2 (Wiley, New York, 1990).
[17] S. Woźniak, M. W. Evans and G. Wagnière, *Mol. Phys.* **75**, 81, 99 (1992).
[18] G. Wagnière, *Phys. Rev. A* **40**, 2437 (1989).
[19] R. Zawodny, R. Tanaś and S. Kielich in Ref. 8, Vol. 85(1).
[20] J. P. van der Ziel, P. S. Pershan and L. D. Malmstrom, *Phys. Rev. Lett.* **15**, 190 (1965); also *Phys. Rev.* **143**, 574 (1966).
[21] J. Deschamps, M. Fitaire, and M. Lagoutte, *Phys. Rev. Lett.* **25**, 1339 (1970); also *Rev. Appl. Phys.* **7**, 155 (1972).
[22] N. Sanford, R. W. Davies, A. Lempicki, W. J. Miniscalco and S. J. Nettel, *Phys. Rev. Lett.* **50**, 1803 (1983).
[23] T. W. Barrett, H. Wohltjen and A. Snow, *Nature*, **301**, 694 (1983).
[24] P. F. Liao and G. C. Bjorklund, *Phys. Rev. Lett.* **36**, 584 (1976).
[25] B. Talin, V. P. Kaftandjan and L. Klein, *Phys. Rev. A* **11**, 648 (1975).
[26] J. F. Ward, *Rev. Mod. Phys.* **37**, 1 (1965).
[27] A. Pais, *The Science and Life of Albert Einstein* (Oxford University Press, New York, 1982).
[28] E. P. Wigner, *Ann. Math.* **40**, 149 (1939).
[29] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).
[30] Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
[31] L. D. Barron, *Molecular Light Scattering and Optical Activity* (Cambridge University Press, Cambridge, 1982).
[32] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983).
[33] A. Einstein, *Ann. Phys.* **17**, 891 (1905).
[34] A. Sommerfeld, Ed., *The Principle of Relativity*, p. 37, (Dover, New York), reviewed in Ref. 27, p. 162.
[35] L. D. Barron, *Physica B* **190**, 307 (1993); M. W. Evans, *Physica B* **190**, 310 (1993).
[36] P. Pershan, *Phys. Rev.* **130**, 919 (1963).
[37] W. Happer, *Rev. Mod. Phys.* **44**, 169 (1972).

[38] J. Frey, R. Frey, C. Flytzannis and R. Triboulet, *Opt. Commun.*, **84**, 76 (1991).
[39] M. W. Evans, *J. Phys. Chem.* **95**, 2256 (1991); W. S. Warren, S. Mayr, D. Goswami and A. P. West, Jr., *Science* **255**, 1683 (1992).
[40] L. H. Ryder, *Elementary Particles and Symmetries* (Gordon and Breach, London, 1986).

Chapter 9

THE MAGNETIC FIELDS AND ROTATION GENERATORS OF FREE SPACE ELECTROMAGNETISM

M. W. Evans

Abstract

The relation is developed between rotation generators of the Lorentz group and the magnetic fields of free space electromagnetism. Using these classical relations, it is shown that in the quantum field theory there exists a longitudinal photomagneton, a quantized magnetic flux density operator which is directly proportional to the photon spin angular momentum. Commutation relations are given in the quantum field between the longitudinal photomagneton and the usual transverse magnetic components of quantized electromagnetism. The longitudinal component is phase free, the transverse components are phase dependent. All three components can magnetize material in general, but only the transverse components contribute to Planck's law. The photon therefore has three, not two, relativistically invariant degrees of polarization, an axial, longitudinal, polarization, and the usual right and left circular transverse polarizations. Since the longitudinal polarization is axial, it is a phase free magnetic field.

1. Introduction

In conventional, classical, electrodynamics it is customary to consider only the transverse, oscillating, phase dependent components of a travelling plane wave in free space. The transverse magnetic components can be written as two orthogonal electromagnetic modes or components in a complex, circular basis [1-4],

$$\boldsymbol{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i}+\boldsymbol{j})\,e^{i\phi}, \qquad \boldsymbol{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}}(-i\boldsymbol{i}+\boldsymbol{j})\,e^{-i\phi}, \tag{1}$$

where $\phi = \omega t-\boldsymbol{\kappa}\cdot\boldsymbol{r}$ is the phase. Component (1) is the complex conjugate of component (2). Here $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the X and Y axes of the laboratory frame (X, Y, Z), mutually orthogonal to the propagation axis, Z, of the plane wave. Here ω is the angular frequency in radians per second at an instant of time, t, and $\boldsymbol{\kappa}$ the wavevector in inverse meters at a position $\boldsymbol{r}$ in the laboratory frame. Modes (1) and (2) are both solutions of the free space Maxwell equations. Correspondingly, there are oscillating, transverse, electric fields, usually written as,

$$\boldsymbol{E}^{(1)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i}-i\boldsymbol{j})\,e^{i\phi}, \qquad \boldsymbol{E}^{(2)} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i}+i\boldsymbol{j})\,e^{-i\phi}, \tag{2}$$

in the same notation. In Eqs. (1) and (2), $B^{(0)}$ is the scalar magnetic flux density amplitude, and $E^{(0)}$ the scalar electric field strength amplitude.

These fields constitute the well known classical Maxwellian description of electrodynamics in free space, a description in which there are only two degrees of polarization, i.e., in a circular (or Cartesian) basis, one (longitudinal) degree of polarization is missing. This is easily seen by considering the following circular representation of three dimensional space,

$$\boldsymbol{e}^{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}), \quad \boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j}), \quad \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} = i\boldsymbol{e}^{(3)} = i\boldsymbol{k}, \tag{3}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are Cartesian unit vectors, defined by

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}. \tag{4}$$

Therefore if $\boldsymbol{i}$ and $\boldsymbol{j}$ are polar, $\boldsymbol{k}$ is axial; if $\boldsymbol{i}$ and $\boldsymbol{j}$ are axial, $\boldsymbol{k}$ is also axial. In the representation of transverse electric fields (Eqs. (2)), $\boldsymbol{i}$ and $\boldsymbol{j}$ are polar; for transverse magnetic fields (Eqs. (1)), $\boldsymbol{i}$ and $\boldsymbol{j}$ are axial. The unit vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$ and $\boldsymbol{e}^{(3)}$ in the circular basis form the cyclical algebra,

$$\left.\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*} = i\boldsymbol{e}^{(3)}, \\ \boldsymbol{e}^{(2)} \times \boldsymbol{e}^{(3)} &= i\boldsymbol{e}^{(1)*} = i\boldsymbol{e}^{(2)}, \\ \boldsymbol{e}^{(3)} \times \boldsymbol{e}^{(1)} &= i\boldsymbol{e}^{(2)*} = i\boldsymbol{e}^{(1)}, \end{aligned}\right\} \tag{5}$$

so that if any one is zero, the other two also vanish. The basic electrodynamical notion that there can be only two degrees of field polarization in three dimensional space is therefore geometrical nonsense. This simple illustration translates into well known [5] fundamental difficulties in the theory of electromagnetism. In the required language of special relativity, the electromagnetic four-potential A_μ loses manifest covariance, only two out of its four components are physically meaningful, and since A_μ is known to be physically meaningful through the Bohm-Aharonov effect [6], this is obviously not a satisfactory description.

In this paper these difficulties are resolved by the use of the longitudinal magnetic field $\boldsymbol{B}^{(3)}$, which is phase free and which contributes nothing to the Planck radiation law because its associated frequency is zero. For this reason it is referred to as the "ghost field" (Gespensterstrahlung) of electromagnetism. In the quantum field theory it is represented by the longitudinal "photomagneton," which is the operator $\hat{B}^{(3)}$, directly proportional to the well known spin angular momentum of the photon, whose eigenvalues are $\hbar$ and $-\hbar$ if the photon is considered to have no mass, and $\hbar$, 0, and $-\hbar$ otherwise. The $\hat{B}^{(3)}$ photomagneton is simply $B^{(0)}$ multiplied into the normalized photon angular

momentum operator, $\hat{J}/\hbar$. In Sec. 2, this result is derived from simple geometrical considerations. Evidence for the existence of $\hat{B}^{(3)}$ is available in the inverse Faraday effect [7–12] (frequency independent magnetization by light). If $\boldsymbol{B}^{(3)}$ were zero, then both $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ would vanish. Since $\boldsymbol{B}^{(3)}$, the expectation value of $\hat{B}^{(3)}$, is directly proportional through $B^{(0)}$ to unit angular momentum, represented by the axial unit vector $\boldsymbol{k}$, it is relativistically invariant in free space, a requirement of the Maxwell equations. In the quantum theory this is interpreted through the fact that photon spin angular momentum, $\hbar$, is also frequency independent, i.e., does not depend on ν, the frequency of the light. (In contrast, photon linear momentum, $h\nu/c$, is proportional to ν, and photon energy, as originally proposed [13], is the quantum of light energy $h\nu$.) It follows therefore that $\hat{B}^{(3)}$ is not absorbed at any frequency, because in the quantum field theory, absorption depends on the quantum of energy $h\nu$, and $\hat{B}^{(3)}$ has no energy. The photomagneton $\hat{B}^{(3)}$ is therefore far more difficult to detect experimentally than the usual transverse fields, and has no effect, as we have argued, on Planck's law of radiation. It can however, be detected because as a phase free magnetic field, it participates in the magnetization of matter by light — the well known inverse Faraday effect which has hitherto been interpreted in terms of nonlinear optics. This phenomenon occurs *without optical absorption* in general, meaning that it can occur without the absorption of a quantum of energy $h\nu$, in other words at frequencies where the sample is transparent to light [7–12].

In Sec. 2, the rigorous geometrical basis of $\hat{B}^{(3)}$ is developed using rotation operators in O(3) and in the Lorentz group of Minkowski space-time. It is shown there that the neglect of $\hat{B}^{(3)}$ implies that one rotation generator is erroneously asserted to be zero. In the quantum field theory this translates into the conclusion that one angular momentum is missing, this being the longitudinal angular momentum, and this is of course diametrically inconsistent with the basic assumption that the photon (considered as massless) have an ineluctable spin angular momentum, whose components in the *longitudinal* axis are $\hbar$ and $-\hbar$. Therefore, photon spin immediately implies the existence of $\hat{B}^{(3)}$.

In Sec. 3, the experimental basis for $\hat{B}^{(3)}$ and $\boldsymbol{B}^{(3)}$ is examined through the inverse Faraday effect. The main conclusion of this section is that if there were no $\hat{B}^{(3)}$, there would be no inverse Faraday effect, contrary to experimental data [7–12].

In Sec. 4, the nature of the longitudinal, concomitant electric field in free space is examined. It is clear from symmetry [14] and special relativity that there can be no *real* $\boldsymbol{E}^{(3)}$, because Fitzgerald-Lorentz contraction reduces any longitudinal *polar* axis to zero for any object travelling at the speed of light. For a *massless* photon, this axis remains zero in any frame of reference, because the Maxwell equations in free space do not vary under Lorentz transformation. For a *massive* photon [15], it becomes possible that there be, in the observer frame, an additional, phase dependent, longitudinal magnetic field $\boldsymbol{B}^{(3)}$ and electric field $\boldsymbol{E}^{(3)}$. It is shown that these questions can be resolved in an internally consistent manner by deducing that $\mathbf{B}^{(3)}$ is accompanied by a pure imaginary $i\mathbf{E}^{(3)}$. This implies that the *combined* contribution of $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ to the Poynting theorem in free space is zero, this being the classical statement equivalent to the fact that in the quantum field, $\hat{B}^{(3)}$, has no Planck energy because it has no frequency. (In Planck's postulate of 1900 [16] energy is directly proportional to frequency, so that $\hat{B}^{(3)}$ corresponds to an oscillator state of zero frequency.)

The use of $\mathbf{B}^{(3)}$ and $i\mathbf{E}^{(3)}$ makes the theory of electromagnetism in free space fully consistent and manifestly covariant.

2. Geometrical Basis for $\hat{B}^{(3)}$

The first indication of the existence of $\hat{B}^{(3)}$ in free space appeared [17] through its relation to a quantity known in non-linear optics as the conjugate product, the vector cross product $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$. This quantity is also referred to in non-linear optics as the antisymmetric, or vectorial, part of the light intensity tensor $\epsilon_0 c E_i E_j$ [18], and therefore has a well defined physical meaning. Any quantity to which $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is algebraically equal also has a well defined physical meaning by tautology. From Eq. (2), the conjugate product is the pure imaginary, *longitudinal* quantity

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = iE^{(0)2}\mathbf{k}, \tag{6}$$

where $\mathbf{k}$ is a unit *axial* vector. The product $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is therefore magnetic in nature, with positive parity inversion ($\hat{P}$) symmetry and negative motion reversal ($\hat{T}$) symmetry [19]. Using the fundamental free space result,

$$E^{(0)} = cB^{(0)}, \tag{7}$$

immediately gives the field $\mathbf{B}^{(3)}$,

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = c^2\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = ic^2B^{(0)}\mathbf{B}^{(3)}, \tag{8}$$

with

$$\mathbf{B}^{(3)} = B^{(0)}\mathbf{k}. \tag{9}$$

It is elementary to show, with these relations, that,

$$\left.\begin{aligned} \mathbf{B}^{(1)} \times \mathbf{B}^{(2)} &= iB^{(0)}\mathbf{B}^{(3)*} = iB^{(0)}\mathbf{B}^{(3)}, \\ \mathbf{B}^{(2)} \times \mathbf{B}^{(3)} &= iB^{(0)}\mathbf{B}^{(1)*} = iB^{(0)}\mathbf{B}^{(2)}, \\ \mathbf{B}^{(3)} \times \mathbf{B}^{(1)} &= iB^{(0)}\mathbf{B}^{(2)*} = iB^{(0)}\mathbf{B}^{(1)}, \end{aligned}\right\} \tag{10}$$

where the * denotes complex conjugation. *It is seen that there is a symmetric, cyclical algebra between the three magnetic field components in free space. This structure is that of the complex basis vectors* $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$, *and* $\mathbf{e}^{(3)}$, *Eqs. (5).*

This is precisely what is expected if there is a three dimensional, geometrical, relation between the transverse and longitudinal components of solutions of Maxwell's equations in free space. The longitudinal component, $\mathbf{B}^{(3)}$, must be phase free, because of the Maxwellian condition

$$\nabla \cdot \mathbf{B} = 0. \tag{11}$$

If it is accepted that $B^{(0)}$ is non-zero, (otherwise there is no electromagnetism), then, Eqs. (10) show that if $\mathbf{B}^{(3)}$ is zero, both $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ *vanish*. Conversely, if $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ are non-zero, then so is $\mathbf{B}^{(3)}$. This result once more emphasizes the fundamental inconsistency in the conventional approach [1–4], in which $\mathbf{B}^{(3)}$ bears no relation to the transverse $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$, and in most texts is not considered. Equation (10) shows that there is a well defined geometrical relation, which shows that $\mathbf{B}^{(3)}$ is physically meaningful. The converse of this result is that if it is asserted that $\mathbf{B}^{(3)}$ is zero, then the conjugate product vanishes, in contradiction with experimental data on the inverse Faraday effect [7–12]; and in contradiction to the fundamental theoretical structure of non-linear optics [17, 18].

The question now arises of the fundamental properties of $\mathbf{B}^{(3)}$ in the classical and quantum theories of fields. In this section it is shown using elementary tensorial methods that it is defined in the quantum field theory by the operator,

$$\hat{B}^{(3)} = B^{(0)}\frac{\hat{J}}{\hbar}, \tag{12}$$

where $\hat{J}$ is the photon angular momentum operator. For one (massless) photon the eigenvalues of $\hat{J}$ are $\pm\hbar$, meaning that the projections in the *longitudinal* (Z) axis are $+\hbar$ or $-\hbar$. In the quantum field theory therefore, $\hat{B}^{(3)}$ depends on the existence of $\hat{J}$, because $B^{(0)}/\hbar$ is a constant of the electromagnetism for a given intensity. Realizing this, any attempt to assert that $\hat{B}^{(3)}$ is zero becomes inconsistent with the fundamentals of quantum mechanics, because such an assertion would imply that the photon spin angular momentum is zero. It is well accepted that the photon spin is an intrinsic, irremovable property [20], and therefore so is $\hat{B}^{(3)}$.

The interpretation of $\hat{B}^{(3)}$, the longitudinal "photomagneton" of electromagnetism, is therefore simple in the quantum field theory - it is an operator generated directly from photon spin. The latter has eigenvalues $\pm\hbar$ *independent of the frequency (ν) and phase (ϕ) of the electromagnetic field*. Any attempt to understand the meaning of $\hat{B}^{(3)}$ must therefore be based on the meaning of $\hat{J}$. Similarly, the interaction of $\hat{B}^{(3)}$ with matter must be understood in the same way as that of $\hat{J}$ with matter. In particular, there must be conservation of angular momentum – the total angular momentum before and after the interaction must be the same. To understand this needs the theory of angular momentum coupling in quantum mechanics [20]. Therefore $\hat{B}^{(3)}$ is obviously not a static magnetic field in any conventional sense (e.g. a field generated from a bar magnet or a solenoid). It is a novel property of light.

It is also obvious that the source of $\hat{B}^{(3)}$ in free space is the source of $\hat{J}$ – usually thought of as a charge-current system at infinity, i.e., matter infinitely removed from free space. This is the same as the source of the usual transverse, oscillating, fields, which in the quantum field theory are thought of in terms of creation and annihilation operators. Therefore, *the existence of $\hat{B}^{(3)}$ does not require a separate source*. In the same way, $\hat{J}$, or photon spin, does not require a source in any way distinct or different from that of electromagnetism.

We arrive at the inescapable conclusion that if $\hat{J}$ be accepted, as usual, then so must $\hat{B}^{(3)}$.

The classical interpretation of $\boldsymbol{B}^{(3)}$, the expectation value of $\hat{B}^{(3)}$, depends on the cross product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ of negative and positive frequency transverse modes (1) and (2). This must also be reflected in the quantum field theory, so Eq. (1) is more fully written in the accepted notation [12, 18], as,

$$\hat{B}^{(3)}(0;-\omega,\omega) = B^{(0)}\frac{\hat{J}}{\hbar}(0;-\omega,\omega). \qquad (13)$$

This shows that $\hat{J}$ itself is generated from a phase free cross product of negative and positive frequency waves; i.e., from a particular combination of creation and annihilation operators [12]. This is the same combination as that which defines [17] the Stokes operator $\hat{S}_3$. The latter is well known to be an angular momentum operator, and the commutator relations between Stokes operators are the same as those of angular momentum operators in quantum mechanics. In this section, we shall develop commutator relations for $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$. These are also angular momentum commutator relations.

Any interaction of $\hat{B}^{(3)}$ with matter must therefore reflect its fundamental character, i.e., account for the fact that it is defined as

$$\hat{B}^{(3)} \equiv \hat{B}^{(3)}(0;-\omega,\omega). \qquad (14)$$

Similarly,

$$\hat{J} \equiv \hat{J}(0;-\omega,\omega), \qquad \hat{S}_3 \equiv \hat{S}_3(0;-\omega,\omega). \qquad (15)$$

Our earlier description [17] of $\hat{B}^{(3)}$ as "static" obviously refers to the fact that it has no *net* (i.e., explicit) functional dependence on phase, $\phi = \omega t - \boldsymbol{\kappa}\cdot\boldsymbol{r}$. In precisely the same way, $\hat{J}$ has none, i.e., its eigenvalues are $\pm\hbar$, which are frequency independent quantities. Similarly, the Stokes operator $\hat{S}_3$ and parameter S_3 have no net phase dependence. For a given beam intensity in circular polarization, S_3 is a constant of magnitude $\pm E^{(0)2}$; + for left, - for right circular polarization.

We shall return to the question of how $\hat{B}^{(3)}(0;-\omega,\omega)$ interacts with matter in Sec. 3, when dealing with the inverse Faraday effect and time dependent perturbation theory. In the remainder of this section, Eq. (12) is derived from first principles.

The first step is to put the cyclic relations between $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ into *classical* commutative form by using the result from elementary tensor analysis [12, 18] that an axial rank one vector is equivalent to a polar rank two antisymmetric tensor,

$$B_i = \frac{1}{2}\epsilon_{ijk}\hat{B}_{jk}, \qquad (16)$$

where ϵ_{ijk} is the rank three, totally antisymmetric, unit tensor (the Levi-Civita symbol). The rank two tensor representation of the magnetic field, $\hat{B}_{jk}$, is entirely equivalent to the usual rank one vector B_i, but has the key advantage of being accessible to commutator algebra. This allows a straightforward transition to the quantum field theory through a factor $\hbar$. Commutator algebra also provides a means of expressing $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ in terms of O(3) rotation generators [21]. In so doing these magnetic fields are related directly to quantum mechanical angular momentum operators, and have the same commutator properties. This was originally deduced [17] using creation and annihilation operators, an independent method.

The classical fields $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ in free space are all axial vectors by definition, and it follows that their unit vector components must also be axial. In matrix form, they are, using tensor analysis of the type illustrated in Eq. (16),

$$\hat{i} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \hat{j} \equiv \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{k} \equiv \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (17)$$

It follows that the matrix representation of the unit vectors in the circular basis is,

$$\hat{e}_1 \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{bmatrix}, \quad \hat{e}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{bmatrix}, \quad \hat{e}_3 \equiv \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

and that these form a commutator Lie algebra which is mathematically equivalent to the vectorial Lie algebra (5),

$$[\hat{e}_1, \hat{e}_2] = -i\hat{e}_3^* = -i\hat{e}_3, \quad [\hat{e}_2, \hat{e}_3] = -i\hat{e}_1^* = -i\hat{e}_2, \quad [\hat{e}_3, \hat{e}_1] = -i\hat{e}_2^* = -i\hat{e}_1. \tag{19}$$

If it is arbitrarily asserted that $\hat{e}_3$ is zero, then both $\hat{e}_1$ and $\hat{e}_2$ vanish, i.e., the assertion is fundamentally inconsistent with three dimensional geometry, expressed in a circular basis (5) rather than a Cartesian basis. Nevertheless, this meaningless assertion is the root of the conventional approach to electrodynamics, an approach which considers only transverse components of the plane wave in free space, and does not recognize that the transverse field components are linked to the longitudinal field component. This conclusion becomes clear if the geometrical basis (19) is used to describe the plane wave in vacuo,

$$\hat{B}^{(1)} = iB^{(0)}\hat{e}^{(1)}e^{i\phi}, \quad \hat{B}^{(2)} = -iB^{(0)}\hat{e}^{(2)}e^{-i\phi}, \quad \hat{B}^{(3)} = B^{(0)}\hat{e}^{(3)}, \tag{20}$$

from which emerges the classical commutator relations between the three magnetic field components,

$$\left.\begin{aligned} [\hat{B}^{(1)}, \hat{B}^{(2)}] &= -iB^{(0)}\hat{B}^{(3)*} = -iB^{(0)}\hat{B}^{(3)}, \\ [\hat{B}^{(2)}, \hat{B}^{(3)}] &= -iB^{(0)}\hat{B}^{(1)*} = -iB^{(0)}\hat{B}^{(2)}, \\ [\hat{B}^{(3)}, \hat{B}^{(1)}] &= -iB^{(0)}\hat{B}^{(2)*} = -iB^{(0)}\hat{B}^{(1)}. \end{aligned}\right\} \tag{21}$$

This algebra can now be expressed in terms of the well known [5, 21] rotation generators of O(3) in three dimensional space. These generators are complex matrices [5, 21],

$$\left.\begin{aligned} \hat{J}^{(1)} &= \frac{\hat{e}^{(1)}}{i} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{bmatrix}, \\ \hat{J}^{(2)} &= \frac{-\hat{e}^{(2)}}{i} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{bmatrix}, \\ \hat{J}^{(3)} &= \frac{\hat{e}^{(3)}}{i} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}\right\} \tag{22}$$

providing the key link between the magnetic field matrices and rotation generators,

$$\hat{B}^{(1)} = -B^{(0)}\hat{J}^{(1)}e^{i\phi}, \quad \hat{B}^{(2)} = -B^{(0)}\hat{J}^{(2)}e^{-i\phi}, \quad \hat{B}^{(3)} = iB^{(0)}\hat{J}^{(3)}. \tag{23}$$

This classical result shows that the commutator algebra of the magnetic fields (21) is part of the Lie algebra of the Lorentz group of Minkowski space-time [5, 21]. It shows that the longitudinal component $\hat{B}^{(3)}$ is non-zero, because $\hat{J}^{(3)}$ is non-zero. An assertion to the contrary means that if $\hat{B}^{(3)} =? \hat{0}$, then $\hat{J}^{(3)} =? \hat{0}$, which by reference to Eq. (22), is incorrect.

Furthermore, the generalization of the rotation generator from classical three-space (O(3) group) to classical space-time (Lorentz group) is well known [21] to be

$$\left.\begin{aligned} \hat{J}^{(1)} &= \hat{J}^{(2)*} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \hat{J}^{(2)} &= \hat{J}^{(1)*} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \hat{J}^{(3)} &= -\hat{J}^{(3)*} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}\right\} \tag{24}$$

It follows that $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ can also be generalized in this way, and are also well defined properties of space-time in vacuo. This result is in turn consistent with the fact that all *three* magnetic components are well-defined solutions of Maxwell's equations, which in free space are invariant under Lorentz transformation [1, 22]. In this sense, $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ are defined directly by the rotation generators of the Lorentz group, generators that form a Lie algebra in space-time. By including $\hat{B}^{(3)}$ and not arbitrarily discarding it, as is the usual practice [1–4] electrodynamics in space-time actually becomes more fully self-consistent. For example, the Wigner little group [21] becomes O(3) instead of E(2): in other words three dimensional, not two dimensional. It is well known in field theory [21] that the Euclidean E(2) is physically meaningless, implying that classical electrodynamics is deeply flawed if the longitudinal field is not linked to the transverse fields as in this paper.

The fact that the approach that leads to E(2) is incorrect is seen through the fact that it leads to $\hat{J}^{(3)} =? \hat{0}$. Uncritical acceptance of such an obviously incorrect result has become habitual because of the assumption that Maxwell's equations deal only with transverse field components in free space. Equation (10) shows that the longitudinal and transverse field components in free space are linked geometrically. This finding is tautological in nature, because Maxwell's equations are written in three, not two, space dimensions. The assertion $\hat{B}^{(3)} =? \hat{0}$ not only makes nonsense out of Euclidean (and Minkowski) geometry, but also leads to difficulties throughout the gauge theory of electromagnetism, difficulties which are actually well known [5]. Many of these difficulties disappear when it is realized that $\hat{B}^{(3)}$ is a "spin field," which is phase free, and which therefore obeys the Maxwellian constraint $\nabla\cdot\boldsymbol{B} = 0$.

It is well known [5] that the rotation generators of O(3) form a Lie algebra; part of the Lie algebra of the Lorentz group. In a circular basis (19), this becomes the following classical commutator algebra,

$$\left.\begin{aligned} [\hat{J}^{(1)}, \hat{J}^{(2)}] &= -\hat{J}^{(3)*} = \hat{J}^{(3)}, \\ [\hat{J}^{(2)}, \hat{J}^{(3)}] &= -\hat{J}^{(1)*} = -\hat{J}^{(2)}, \\ [\hat{J}^{(3)}, \hat{J}^{(1)}] &= -\hat{J}^{(2)*} = -\hat{J}^{(1)}, \end{aligned}\right\} \tag{25}$$

which becomes

$$[\hat{J}_X, \hat{J}_Y] = i\hat{J}_Z, \qquad [\hat{J}_Y, \hat{J}_Z] = i\hat{J}_X, \qquad [\hat{J}_Z, \hat{J}_X] = i\hat{J}_Y, \tag{26}$$

in the Cartesian basis, and which is, within a factor $\hbar$, *identical with the commutator algebra of angular momentum operators in quantum mechanics*. This result provides a simple route to quantization of the magnetic fields of the plane wave in free space, giving the result,

$$\hat{B}^{(1)} = -B^{(0)}\frac{\hat{J}^{(1)}}{\hbar}e^{i\phi}, \qquad \hat{B}^{(2)} = -B^{(0)}\frac{\hat{J}^{(2)}}{\hbar}e^{-i\phi}, \qquad \hat{B}^{(3)} = iB^{(0)}\frac{\hat{J}^{(3)}}{\hbar}, \tag{27}$$

where $\hat{B}^{(i)}$ are now field operators in quantum mechanics. In particular, the longitudinal operator $\hat{B}^{(3)}$ is the elementary quantum of longitudinal magnetic flux density, *the photomagneton* of electromagnetic radiation in free space. The photomagneton is the pilot wave of photon spin in the Einstein-de Broglie interpretation of the quantum theory of light. In the Copenhagen interpretation, the quantized field operators $\hat{B}^{(1)}$, $\hat{B}^{(2)}$ and $\hat{B}^{(3)}$ form angular momentum commutators in free space,

$$\left.\begin{aligned} [\hat{B}^{(1)}e^{-i\phi}, \; \hat{B}^{(2)}e^{i\phi}] &= -iB^{(0)}\hat{B}^{(3)}, \\ [\hat{B}^{(2)}e^{i\phi}, \; \hat{B}^{(3)}] &= -iB^{(0)}\hat{B}^{(2)}e^{i\phi}, \\ [\hat{B}^{(3)}, \; \hat{B}^{(1)}e^{-i\phi}] &= -iB^{(0)}\hat{B}^{(1)}e^{-i\phi}. \end{aligned}\right\} \tag{28}$$

The fields can now be thought of in terms of creation and annihilation operators [12, 17] as usual. In the Copenhagen interpretation, the three field components appearing in each commutator relation cannot be specified simultaneously [20]. If $\hat{B}^{(3)}$ is specified, then the transverse components remain unspecified, as in any angular momentum commutator relation in quantum mechanics. This is consistent with the fact that the (3) (i.e., Z) component of photon angular momentum is usually specified as eigenvalues, *longitudinal* projections $\hbar$ and $-\hbar$; and that for a massless photon travelling at the speed of light, the transverse angular momentum components are mathematically indeterminate. The longitudinal component of angular momentum in an object travelling at c is relativistically invariant. Therefore $\hat{B}^{(3)}$ in free space is also relativistically invariant. This must be so because it is a solution of Maxwell's equations, which are also relativistically invariant in free space. The specification of $\hat{B}^{(3)}$ in terms of photon spin is therefore fully consistent with relativistic quantum field theory.

Therefore, $\hat{B}^{(3)}$ is a constant of motion [12, 20]; while $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ are governed by photon statistics and are subject to purely quantum effects such as light squeezing [12]. The field $\hat{B}^{(3)}$, being defined by photon spin, is not subject to light squeezing effects. In other words photon spin itself is not affected by light squeezing, its eigenvalues remain a constant $\hbar$ and $-\hbar$. If the photon is massive [23], the eigenvalues become $\hbar$, 0 and $-\hbar$. The constancy of the field $\hat{B}^{(3)}$ is consistent with the fact that in quantum mechanics the general expression for the rate of change of an expectation value is [20],

$$\frac{d}{dt}\langle\hat{B}^{(3)}\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{B}^{(3)}]\rangle, \tag{29}$$

so that $\hat{B}^{(3)}$ commutes with the Hamiltonian $\hat{H}$. This is consistent with the fact that $\hat{B}^{(3)}$ has no Planck energy, and does not contribute to classical electromagnetic energy density [24]. The expectation value of $\hat{B}^{(3)}$, being a constant of motion, is independent of time, and its eigenvalues are specified as the constant $\hbar$ and $-\hbar$. Similarly, the Stokes operator $\hat{S}_3$ is a constant of motion, so $\hat{B}^{(3)}$ is proportional to $\hat{S}_3$ [17]. Therefore the photomagneton $\hat{B}^{(3)}$ conserves angular momentum in free space, and this is a consequence of the isotropy of the Hamiltonian in free space [20], and therefore a consequence of three dimensional symmetry [20].

This conclusion is simply a way of saying that the spin of the massless photon is $\pm\hbar$; and that the photomagneton $\hat{B}^{(3)}$ is a direct consequence of photon spin. The classical $\boldsymbol{B}^{(3)}$ is therefore a direct consequence of the fact that there exists right and left circular polarization in electromagnetic radiation. This is an expression of Eq. (10) in words.

The expectation values of $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ on the other hand are not constants of motion, and do not commute with the Hamiltonian. This is consistent with the fact that $\boldsymbol{B}^{(1)}$ or $\boldsymbol{B}^{(2)}$ contribute to classical electromagnetic energy density in free space. Similarly, the expectation values of $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ are not constants of motion, and remain unspecified in the Copenhagen interpretation if $\hat{J}^{(3)}$ is specified. Such a result is consistent with special relativity, which deduces that the transverse classical angular momenta of an object travelling at c are indeterminate in the observer frame and that the longitudinal component is relativistically invariant. In other words, in special relativity,

$$J_Z = J_Z', \qquad J_Y = \gamma J_Y', \qquad J_X = \gamma J_X', \tag{30}$$

where $\gamma = (1 - v_Z^2/c^2)^{-\frac{1}{2}}$. We see that if the relative velocity of two frames, v_Z, is c; then J_Y and J_X (in the static, observer frame) become infinite unless $J_X' = J_Y' = 0$. In this condition, J_X and J_Y are mathematically indeterminate but $J_Z = J_Z'$ is well defined. This is what is indicated by the Copenhagen interpretation of Eqs. (28), the field $\hat{B}^{(3)}$ has specified, relativistically invariant, eigenvalues which are projections in the longitudinal axis of the propagating plane wave in free space. The transverse fields $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ are not constants of motion and are not specified if $\hat{B}^{(3)}$ is specified. It is well known that $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ are subject to quantum effects such as light squeezing, which are a consequence of the Heisenberg uncertainty principle applied to photons.

If we compare directly the classical and quantum equations,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)}, \tag{31a}$$

$$[\hat{B}^{(1)}, \hat{B}^{(2)}] = -iB^{(0)}\hat{B}^{(3)}, \tag{31b}$$

it becomes immediately obvious that Eq. (31a) is *a relation between spins* in the Maxwellian interpretation. Each spin component (1), (2) and (3) is formed from a vector cross product of the other two; this being a requirement of Euclidean geometry. In order for this geometrical requirement to simultaneously satisfy Maxwell's equation $\nabla\cdot\boldsymbol{B} = 0$, the longitudinal component $\boldsymbol{B}^{(3)}$ *must be phase free*, otherwise its divergence is non-zero because the phase has a Z dependence. In order to satisfy this and the other three Maxwell equations, the transverse components $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ must be phase dependent. Equations (10) tie these considerations together in a circular basis, in the same way that rotation generators are tied together.

At the fundamental level, Eq. (13) shows that photon spin *itself* is nonlinear in nature, being an angular momentum. In the quantum field theory this has eigenvalues $\hbar$ and $-\hbar$. In Maxwellian electrodynamics the classical equivalent to photon spin is obtained from the *nonlinear* conjugate product, $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ (or $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}$), *which removes the phase* ϕ. In this way the classical conjugate product and quantized photon spin are linked together in a relation which clarifies the physical meaning of both.

In the Copenhagen interpretation, the Heisenberg uncertainty principle applied to Eq. (31b) shows that

$$\delta\hat{B}^{(1)}\delta\hat{B}^{(2)} \geq \frac{1}{2}|B^{(0)}\hat{B}^{(3)}|, \tag{32}$$

where $\delta\hat{B}^{(1)}$ and $\delta\hat{B}^{(2)}$ are root mean square deviations [20]. As usual [20] the right hand side is a rigorous lower bound on the product $\delta\hat{B}^{(1)}\delta\hat{B}^{(2)}$, *a lower bound which is therefore defined by the photomagneton* $\hat{B}^{(3)}$. If $\hat{B}^{(3)}$ were zero, $\hat{B}^{(1)}$ and $\hat{B}^{(2)}$ would commute, implying that $\delta\hat{B}^{(1)} = \hat{0}$ and $\delta\hat{B}^{(2)} = \hat{0}$ simultaneously. *The experimental observation of light squeezing [12] shows that this is inconsistent with data, therefore* $\hat{B}^{(3)} \neq \hat{0}$. In this sense, light squeezing indicates experimentally the existence of the photomagneton $\hat{B}^{(3)}$.

In the next section, other experimental indications of the existence of $\hat{B}^{(3)}$ are discussed.

3. Detection of $\hat{B}^{(3)}$ in the Inverse Faraday Effect

In addressing the experimental effects of $\hat{B}^{(3)}$, the question arises of whether its interaction with matter (e.g. an electron) can be treated with time dependent or time independent perturbation theory. Magnetization by circularly polarized light, the inverse Faraday effect, can occur *without absorption*, as observed experimentally by van der Ziel *et. al.* [7]. However, it can also occur *with* absorption, as shown theoretically by Woźniak, Evans and Wagnière [25]. It seems reasonable to assert that if there is no absorption, there is no transfer of photon energy, $h\nu$, and so the effect is frequency independent, meaning that in this limit, time independent perturbation theory applies. In this limit there is transfer of angular momentum from the light to the sample, but no transfer of energy, so that the phenomenon of magnetization is in this sense *"elastic."* Since $\hat{B}^{(3)} = B^{(0)}\hat{J}/\hbar$ and since $B^{(0)}$ is proportional to the square root of beam intensity I_0 (watts per unit area) it seems likely that such an effect is proportional to $I_0^{1/2}$.

However, the fundamental angular momentum $\hat{J}(0; -\omega, \omega)$ needs two modes for its definition (one, (1), negative frequency; the other, (2), positive frequency) so it follows that the interaction of $\hat{B}^{(3)}$ with matter also requires the consideration of $-\omega$ and ω, even though $\hat{B}^{(3)}$ and $\hat{J}$ do not explicitly depend on frequency. (The eigenvalues of both depend on $\hbar$, which has no frequency dependence.) Therefore the interaction of $\hat{B}^{(3)}(0; -\omega, \omega)$ with matter requires in general time dependent perturbation theory, based on the time dependent Schrödinger equation. Since

$$\boldsymbol{B}^{(1)}(-\omega) \times \boldsymbol{B}^{(2)}(\omega) = iB^{(0)}\boldsymbol{B}^{(3)}(0; -\omega, \omega) \tag{33}$$

any effect due to $\boldsymbol{B}^{(3)}(0; -\omega, \omega)$ needs the simultaneous action of both modes (1) and (2). In time dependent perturbation theory, this property must be accounted for in the molecular property tensor with which $\boldsymbol{B}^{(3)}(0; -\omega, \omega)$ interacts at any order in the field. For example, the inverse Faraday effect is described by Woźniak, Evans and Wagnière [25] in terms of $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$. In a three level atomic system the paramagnetic contribution to magnetism, $\boldsymbol{M}(0)$, by circularly polarized light is given by

$$\boldsymbol{M}(0) = \frac{-iNc^2}{3\hbar kT}(\rho_1(\omega)A + \rho_2(\omega)B)\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{34}$$

which is Eq. (30) of Ref. [25] written in our notation. Here

$$A = \boldsymbol{m}_{aa} \cdot Im(\boldsymbol{\mu}_{a1} \times \boldsymbol{\mu}_{1a}), \qquad B = \boldsymbol{m}_{aa} \cdot Im(\boldsymbol{\mu}_{a2} \times \boldsymbol{\mu}_{2a}), \tag{35}$$

and

$$\rho_1(\omega) = \frac{\omega(\omega_1^2 - \omega^2 - \Gamma_1^2)}{(\omega_1^2 - \omega^2 + \Gamma_1^2)^2 + 4\omega^2\Gamma_1^2}, \qquad \rho_2(\omega) = \frac{\omega(\omega_2^2 - \omega^2 - \Gamma_2^2)}{(\omega_2^2 - \omega^2 + \Gamma_2^2)^2 + 4\omega^2\Gamma_2^2}, \tag{36}$$

where ω_1 and ω_2 are resonance frequencies, $\boldsymbol{m}_{aa}$ is a ground state permanent magnetic dipole moment, and $\boldsymbol{\mu}_{a1}$ etc. are transition electric dipole moments.

The transition to time independent perturbation theory is given by setting the resonance frequencies (ω_1 and ω_2), and damping factors Γ_1 and Γ_2 to zero, giving the result

$$\boldsymbol{M}(0) = -\frac{Nc^2}{3kT\hbar\omega}\boldsymbol{m}_{aa} \cdot Im(\boldsymbol{\mu}_{a1} \times \boldsymbol{\mu}_{a2})\, B^{(0)}\boldsymbol{B}^{(3)}. \tag{37}$$

Both equations (34) and (37) are to second order in the magnitude, $B^{(0)}$, of $\boldsymbol{B}^{(3)}$. However, Eq. (34) represents a transfer of energy (at the resonance frequencies ω_1 and ω_2) as well as a transfer of angular momentum from $\boldsymbol{B}^{(3)}$. Equation (37) represents a transfer of angular momentum only, because there is no resonance. Therefore Eq. (34) is an "inelastic" process, Eq. (37) an "elastic" process. Both equations are semi-classical, in that the field is treated classically and the molecular property quantum mechanically.

Clearly, if $\boldsymbol{B}^{(3)}$ were zero, there would be no inverse Faraday effect of any kind.

The question now arises as to whether $\boldsymbol{B}^{(3)}$, having all the attributes of magnetic flux density, can act at first order, so that there is an inverse Faraday effect proportional to the square root of intensity in addition to process (34) or (37), which are both proportional to intensity. The time average of $\boldsymbol{B}^{(3)}$ is non-zero, because it has no phase dependence, *and this suggests that it can magnetize at first order*. If so, then there should be a component of the inverse Faraday effect proportional to the square root of intensity. This interesting possibility should be checked by further experimental work on the intensity dependence of the inverse Faraday effect, whose standard interpretation is at second order, in $B^{(0)}\boldsymbol{B}^{(3)}$, as we have seen. The oscillating components $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ also magnetize at first order, but the time averaged magnetization vanishes. Any *energy* transfer process from the electromagnetic field to matter is second order, however, in the electric field strength or magnetic flux density of the field, but angular momentum transfer is first order in these quantities. For example, when a quantum of energy, $h\nu$ in the quantum theory, is transferred to an atom, there is a simultaneous transfer of angular momentum ($\hbar$) per photon, $h\nu$, a process which is governed by angular momentum selection rules [20]. Since $\hbar$, the magnitude of the photon's angular momentum, is energy divided by angular frequency, it has units proportional to $\boldsymbol{B}^{(3)}$ *at first order*. Therefore it follows that $\boldsymbol{B}^{(3)}$ can produce pure, first order, magnetization *only if there is no transfer of photon energy* $h\nu$ *to the sample*. (Otherwise the overall process is a mixture of first and second order effects.) In other words, any purely first order process in $\boldsymbol{B}^{(3)}$ cannot be accompanied by absorption of light, because absorption is second order in $\boldsymbol{B}^{(3)}$. The molecular property tensor through which $\boldsymbol{B}^{(3)}$ produces magnetization must therefore be a susceptibility tensor calculated in the limit of time independent perturbation theory, in which there are no optical resonances.

If further experimental work shows that there is a process in the inverse Faraday effect proportional to the square root of circularly polarized light intensity, then it would have been shown that $\boldsymbol{B}^{(3)}$ can act as a first order magnetic field. If the data show no sign of such a process, however, *it would be incorrect to conclude therefrom that* $\boldsymbol{B}^{(3)} =? 0$. As discussed in Sec. 2, such a possibility is excluded on several counts. The absence of a first order process in $\boldsymbol{B}^{(3)}$ might mean that two modes (negative and positive ω) are needed to define the classical equivalent of photon spin angular momentum, $\boldsymbol{J}(0; -\omega, \omega)$, so that $\boldsymbol{J}$ *itself* is intrinsically non-linear, and for this reason cannot act without the combined action of two electromagnetic modes, (1) and (2). The inverse Faraday effect has been observed *experimentally* in the absence of absorption, meaning that there is no transfer of $h\nu$ from radiation to sample, yet the sample is magnetized [7]. Therefore the two modes making up $\hat{J}$ can act on a sample without transferring any photons $h\nu$, and the inverse Faraday effect is obviously not an absorption phenomenon. It is therefore confusing to allude to it as a "two-photon" process, because that would imply the absorption of two photons. Since $\hat{J}$ is non-zero, and directly proportional to $\hat{B}^{(3)}$, the latter also depends on the simultaneous action of two modes, (1) and (2). The angular momentum, and $\hat{B}^{(3)}$, do not depend on frequency, however, and have no Planck energy. Any assertion that $\hat{B}^{(3)}$ is *zero*, however, is geometrically incorrect. The question is whether $\hat{B}^{(3)}$ can act at first order or not, and further experimental work is needed to clarify this point. In Ref. [17], the interpretation of the inverse Faraday effect is discussed in more detail. In diamagnetics, effects at first order in $\hat{B}^{(3)}$ are prohibited by the fact that the sample has no permanent magnetic dipole moment. In paramagnetics, such as the doped glasses used by van der Ziel *et al*. [7], effects at first order in $\hat{B}^{(3)}$ are allowed in principle, provided that the symmetry of the sample allows a net permanent magnetic dipole moment. Data are not available at present to test these matters further. A recent reinterpretation [26] of the results of Frey *et al*. [27] on the optical Faraday effect showed a *square root* intensity dependence of the light induced Faraday rotation, which is a sign, albeit tenuous, that $\hat{B}^{(3)}$ is able to act at first order. It is tenuous because there were only six data points available [26], and these did not go through the origin. Also, the pump laser used by Frey *et al*. [27] was not circularly polarized before entering the intense magnetic field used in their experiment. However, it develops an excess of circular polarization through the ordinary Faraday effect when it passes through the magnetic field, producing a non-zero $\hat{B}^{(3)}$.

We emphasize that the question of whether $\hat{B}^{(3)}$ can act at first order is secondary to that of the existence of $\hat{B}^{(3)}$, which is proven unequivocally by the data of van der Ziel *et al*. [7] and by the arguments of Sec. 2 of this paper.

The simplest example of the inverse Faraday effect *without absorption* is when circularly polarized light interacts with one electron. This problem was first discussed by Talin *et al*. [28], from which it is straightforward to show [29] that the magnetic dipole moment induced, without absorption, in one electron by circularly polarized light is

$$\boldsymbol{m} = \frac{-e^3c^2}{2m_0^2\omega^3}B^{(0)}\boldsymbol{B}^{(3)}. \tag{38}$$

Here e and m_0 are the electronic charge and mass, ω the angular frequency of the light. It is therefore a simple matter to show that there is no elementary, one electron, inverse Faraday effect if $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$. Equation (38) is second order in the magnitude, $B^{(0)}$, of $\boldsymbol{B}^{(3)}$. If we assert [29] that $\boldsymbol{B}^{(3)}$ can act at first order, the effect becomes

$$\boldsymbol{m} = -\frac{e^2r_0^2}{2m_0}\boldsymbol{B}^{(3)} - \frac{e^3c^2}{2m_0^2\omega^3}B^{(0)}\boldsymbol{B}^{(3)}, \tag{39}$$

where r_0 is an orbital electron radius. For ω of about 10^{15} rad sec^{-1}, and for a first order electron radius of about 10Å, the orders of magnitude for a beam intensity of about 10^{14} watt m^{-2} become

$$|\boldsymbol{m}| \sim -10^{-26}|\boldsymbol{B}^{(3)}| - 10^{-25}B^{(3)2}, \tag{40}$$

and the second order effect is roughly ten times larger. Under other conditions, the first order effect may of course predominate. Again there are no data available to test these hypotheses. These data would require the careful measurement and analysis of the intensity dependence of the inverse Faraday effect in a suitable electron plasma [8].

4. The Longitudinal Electric Field, $i\boldsymbol{E}^{(3)}$

The existence of a $\boldsymbol{B}^{(3)}$ appears to imply at first sight that there must be a concomitant longitudinal electric field from Maxwell's equations. However, such a field has never been detected experimentally, no large, first order, polarization effects of light have been observed to date. The only known polarizing effect of light is optical rectification [30], a small, second order process. There are several factors that point towards the fact that there is no real, (i.e., physical) electric field $\boldsymbol{E}^{(3)}$, but that there is an *imaginary* $i\boldsymbol{E}^{(3)}$.

1. In special relativity [31], the square of the complex vector $c\boldsymbol{B}^{(3)} + i\boldsymbol{E}^{(3)}$,

$$(c\boldsymbol{B}^{(3)} + i\boldsymbol{E}^{(3)})^2 = c^2B^{(3)2} - E^{(3)2} + 2ci\boldsymbol{B}^{(3)}\cdot\boldsymbol{E}^{(3)}, \tag{41}$$

is a Lorentz invariant. The real parts of the two independent invariants, $c^2B^{(3)2} - E^{(3)2}$, and $2ci\boldsymbol{B}^{(3)}\cdot\boldsymbol{E}^{(3)}$, are both zero. The first is the contribution of $B^{(3)}$ and $E^{(3)}$ to the free space electromagnetic energy density, the second is pure imaginary. They are well known [31] to be the two independent invariants of the electromagnetic four-tensor, $F_{\mu\nu}$, the four-curl of A_μ. It is also well known that $F_{\mu\nu}$ allows for the existence of longitudinal field components in free space. These are customarily asserted to be zero with the use of a suitable gauge.

2. The fact that

$$U^{(3)} = \frac{1}{\mu_0} B^{(3)} \cdot B^{(3)} + \epsilon_0 iE^{(3)} \cdot iE^{(3)} = 0, \tag{42}$$

in free space is consistent with the fact that $\hat{B}^{(3)}$ is directly proportional to the spin angular momentum of the photon, which has no Planck energy. It is not possible to assert that any real electric field be proportional to any kind of real angular momentum, because of $\hat{P}$ and $\hat{T}$ symmetries [19]. The imaginary $iE^{(3)}$ therefore comes from fundamental special relativity, whose well known dual transformation [31] converts a pure real magnetic field to a pure imaginary electric field without changing the Maxwell equations in free space.

3. The Maxwell equations in free space are satisfied by $B^{(3)}$ accompanied by $iE^{(3)}$ because both are phase free and therefore time independent and uniform in classical electrodynamics. If we write $B^{(3)}$ in terms of a vector potential

$$B^{(3)} = \nabla \times A_3, \tag{43}$$

and attempt to write a real $E^{(3)}$ in terms of the scalar and vector potentials,

$$E^{(3)} =? -\nabla\phi^{(3)} - \frac{\partial A_3}{\partial t}, \tag{44}$$

it is found that this leads to $E^{(3)} = \mathbf{0}$ in Maxwellian electrodynamics. Therefore if $B^{(3)}$ is real, the *real* $E^{(3)}$ vanishes.

4. The joint contribution of $B^{(3)}$ and $iE^{(3)}$ to $U^{(3)}$ in free space is zero. The Poynting theorem then asserts that the vector

$$N = \frac{1}{\mu_0} iE^{(3)} \times B^{(3)} = \mathbf{0}, \tag{45}$$

is zero. This is consistent with the fact that $B^{(3)}$ is parallel to $iE^{(3)}$ in the propagation axis. It follows that [26], if $B^{(3)}$ and $E^{(3)}$ are asserted to be in general complex,

$$E^{(1)} \times B^{(3)} = E^{(3)} \times B^{(1)}, \quad E^{(2)} \times B^{(3)} = E^{(3)*} \times B^{(2)}, \tag{46}$$

from which if $B^{(3)}$ is real, then $iE^{(3)}$ must be pure imaginary.

5. An attempt to construct for an assumed *real* $E^{(3)}$ a cyclic algebra akin to (10) results in $\hat{T}$ violation [32]. This is consistent with the fact that a real longitudinal electric field would have to be a polar vector, which cannot, on basic geometrical grounds, be constructed from the cross product of two mutually orthogonal polar or axial vectors. In other words, any cross product such as $E^{(1)} \times E^{(2)}$ or $B^{(1)} \times B^{(2)}$ must produce an *axial* vector. Cross products such as $E^{(1)} \times B^{(2)}$ (those defining the Poynting vector) produce a polar vector which is $\hat{T}$ negative. Any real, physical electric field must be $\hat{T}$ *positive*, and therefore cannot be produced from Poynting type cross products.

For these reasons $B^{(3)}$ is accompanied, for consistency in classical Maxwellian electrodynamics, by $iE^{(3)}$, a conclusion which emerges from special relativity. Dual transformation produces $iE^{(3)}$ from $B^{(3)}$ and vice versa, as required, and these two components (one magnetic, physical, and real, the other electric, unphysical and imaginary) take their place in the electromagnetic four-tensor $F_{\mu\nu}$. Classical electrodynamics is therefore rendered more fully self-consistent by their inclusion. $B^{(3)}$ produces physical effects, $iE^{(3)}$ produces no physical effects.

Conclusions

There is experimental evidence [7-12] for the fact that the product $B^{(1)} \times B^{(2)}$ is not zero. Expressing this product as $iB^{(0)}B^{(3)}$ shows that the real, longitudinal, phase free $B^{(3)}$ is not zero in free space, a deduction which is supported on geometrical grounds in Sec. 2. Using these methods, it has been shown that in the quantum field theory, $\hat{B}^{(3)}$ is proportional to the photon spin angular momentum operator, $\hat{J}$. The source of $\hat{B}^{(3)}$ is therefore the same as that of photon spin, and $\hat{B}^{(3)}$ is able to propagate in free space with photon spin. The operator $\hat{B}^{(3)}$ has no Planck energy because the eigenvalues of $\hat{J}$ for one photon are $\hbar$ and $-\hbar$, which are independent of frequency. Classically, the joint contribution of $B^{(3)}$ and $iE^{(3)}$ to free space electromagnetic energy density is zero. Therefore $B^{(3)}$ (and $\hat{B}^{(3)}$) is not absorbed in field-matter interaction, and does not contribute to the Planck radiation law. We reach the fundamental conclusion that the photon has *three* degrees of polarization, and that this does not conflict with the Planck law. The same conclusion is reached from the hypothesis [23, 32] that the photon have mass, however tiny in magnitude. The Maxwellian $\boldsymbol{B}^{(3)}$ can therefore be regarded as the "zero mass limiting" form of a more general theory, in which the photon mass is non-zero. For several reasons, it is concluded that $\boldsymbol{B}^{(3)}$ is accompanied in free space by an imaginary $i\boldsymbol{E}^{(3)}$, which produces no polarization. If $\boldsymbol{B}^{(3)}$ is asserted to be zero, the inverse Faraday effect disappears, in conflict with experimental results [7-12]. It appears plausible that $\boldsymbol{B}^{(3)}$ act at first order as well as at second order in the inverse Faraday effect, and other related magneto-optic effects.

Acknowledgments

Support is acknowledged at various stages from the Cornell Theory Center, the Leverhulme Trust, Pennsylvania State University and The North Carolina Supercomputing Center.

Appendix. Cyclical Algebra Involving Electric Fields

There are symmetric cyclical relations of type (10) which involve electric fields, and which can also be related to rotation and boost generators of the Lorentz group. In three dimensions there is for example, the algebra,

$$\left.\begin{aligned} \boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} &= -E^{(0)}(ic\boldsymbol{B}^{(3)})^*, \\ \boldsymbol{E}^{(2)} \times (ic\boldsymbol{B}^{(3)}) &= -E^{(0)}\boldsymbol{E}^{(1)*}, \\ (ic\boldsymbol{B}^{(3)}) \times \boldsymbol{E}^{(1)} &= -E^{(0)}\boldsymbol{E}^{(2)*}, \end{aligned}\right\} \tag{A1}$$

which becomes a relation between boost and rotation generators when we come to consider the four dimensions of the Lorentz group. The electric fields are proportional to the boost generators, and the magnetic fields to the rotation generators.

In order to derive a perfectly cyclical algebra involving electric fields only, we first note that the existence in special relativity of the complex vector $c\boldsymbol{B}^{(3)} + i\boldsymbol{E}^{(3)}$ means that the symmetry of the *imaginary* $i\boldsymbol{E}^{(3)}$ can be regarded as magnetic, i.e., regarded as the same as that of *real* $c\boldsymbol{B}^{(3)}$. This of course means that $i\boldsymbol{E}^{(3)}$ is not a real electric field. The square of the complex vector $c\boldsymbol{B}^{(3)} + i\boldsymbol{E}^{(3)}$ gives Lorentz invariants as in the text. With this realization, the following cyclical algebra can be written down

$$\left.\begin{aligned} \boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} &= -E^{(0)}(i\boldsymbol{E}^{(3)})^*, \\ \boldsymbol{E}^{(2)} \times (i\boldsymbol{E}^{(3)}) &= E^{(0)}\boldsymbol{E}^{(1)*}, \\ (i\boldsymbol{E}^{(3)}) \times \boldsymbol{E}^{(1)} &= -E^{(0)}\boldsymbol{E}^{(2)*}, \end{aligned}\right\} \tag{A2}$$

which in space-time is an algebra involving boost generators of the Lorentz group,

$$[\hat{K}^{(1)}, \hat{K}^{(2)}] = -\hat{J}^{(3)*}, \quad [\hat{K}^{(2)}, \hat{J}^{(3)}] = -\hat{K}^{(1)*}, \quad [\hat{J}^{(3)}, \hat{K}^{(1)}] = -\hat{K}^{(2)*}. \tag{A3}$$

Here, the boost generators in the circular basis are

$$\hat{K}^{(1)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \end{bmatrix}, \quad \hat{K}^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 \end{bmatrix}, \quad \hat{K}^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{A4}$$

The boost and rotation generators are related to the electric and magnetic fields of the Lorentz group by,

$$\begin{aligned} \hat{E}^{(1)} &= E^{(0)}\hat{K}^{(1)}e^{i\phi}, & \hat{E}^{(2)} &= E^{(0)}\hat{K}^{(2)}e^{-i\phi}, & \hat{E}^{(3)} &= E^{(0)}\hat{K}^{(3)}, \\ \hat{B}^{(1)} &= -B^{(0)}\hat{J}^{(1)}e^{i\phi}, & \hat{B}^{(2)} &= -B^{(0)}\hat{J}^{(2)}e^{-i\phi}, & \hat{B}^{(3)} &= iB^{(0)}\hat{J}^{(3)}. \end{aligned} \tag{A5}$$

Finally, the cyclically symmetric algebra of the Lorentz group is completed by the relations

$$[\hat{K}^{(1)}, \hat{J}^{(1)}] = 0, \quad [\hat{K}^{(2)}, \hat{J}^{(2)}] = 0, \quad [\hat{K}^{(3)}, \hat{J}^{(3)}] = 0. \tag{A6}$$

Therefore in the space part of the Lorentz group, the complete set of fields are $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(1)}$, $\boldsymbol{E}^{(2)}$ and $i\boldsymbol{E}^{(3)}$. These components, expressed in the circular basis (1), (2), and (3), take their place in the antisymmetric four-matrix $F_{\mu\nu}$, the four-curl of A_μ in space-time. The fields $\boldsymbol{B}^{(3)}$ and $i\boldsymbol{E}^{(3)}$ are therefore generators of the Lorentz group in free space, and also in the presence of matter.

References

[1] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[2] R. M. Whitner, *Electromagnetics* (Prentice Hall, Englewood Cliffs, 1962).
[3] A. F. Kip, *Fundamentals of Electricity and Magnetism* (McGraw-Hill, New York, 1962).
[4] M. Born and E. Wolf, *Principles of Optics*, 6th edn. (Pergamon, Oxford, 1975).
[5] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).
[6] Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
[7] J. P. van der Ziel, P. S. Pershan and L. D. Malmstrom, *Phys. Rev. Lett.* **15**, 190 (1965); also *Phys. Rev.* **143**, 574 (1966).
[8] J. Deschamps, M. Fitaire and M. Lagoutte, *Phys. Rev. Lett.* **25**, 1330 (1970); also *Rev. Appl. Phys.* **7**, 155 (1972).
[9] N. Sanford, R. W. Davies, A. Lempicki, W. J. Miniscalco and S. J. Nettel, *Phys. Rev. Lett.* **50**, 1803 (1983).
[10] T. W. Barrett, H. Wohltjen and A. Snow, *Nature* **301**, 694 (1983).
[11] P. F. Liao and G. C. Bjorklund, *Phys. Rev. Lett.* **36**, 584 (1976).
[12] reviewed by R. Zawodny in vol. 85(1) of M. W. Evans, and S. Kielich, Eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, Eds., (Wiley Interscience, New York, 1993).
[13] A. Einstein, *Ann. Phys.* **17**, 132 (1905), reviewed in A. Pais, *The Science and the Life of Albert Einstein* (Oxford University Press, New York, 1982).
[14] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1993).
[15] L. de Broglie, *La Méchanique Ondulatoire du Photon* (Gauthier-Villars, Paris, 1936, 1940, 1957).
[16] reviewed in Ref. 13b.
[17] M. W. Evans, *Physica B* **182**, 227, 237 (1992); **183**, 103 (1993).
[18] S. Kielich in M. Davies, Ed., *Dielectric and Related Molecular Processes*, Vol. 1 (Chemical Society, London, 1972).
[19] M. W. Evans, in I. Prigogine and S. A. Rice, Eds. *Advances in Chemical Physics*, Vol. 81 (Wiley Interscience, New York, 1992).
[20] P. W. Atkins, *Molecular Quantum Mechanics*, 2nd edn. (Clarendon, Oxford, 1983).
[21] E. P. Wigner, *Ann. Math.* **40**, 149 (1939).
[22] for a review of the fundamentals of special relativity, see Ref. 13b.
[23] J.-P. Vigier, "Present Experimental Status of the Einstein-de Broglie Theory of Light," *Proceedings I.S.Q.M.* (Tokyo Workshop on Quantum Mechanics, 1992).
[24] B. W. Shore, *The Theory of Coherent Atomic Excitation*, Vol. 1 (Wiley, New York, 1990), Chapter 9.
[25] S. Woźniak, M. W. Evans and G. Wagnière, *Mol. Phys.* **75**, 81, 99 (1992).
[26] M. W. Evans, Vol. 85(2) of Ref. 12.

[27] J. Frey, R. Frey, C. Flytzannis and R. Triboulet, *Opt. Commun.* **84**, 76 (1991).
[28] B. Talin, V. P. Kaftandjan and L. Klein, *Phys. Rev.*, *A* **11**, 648 (1975).
[29] M. W. Evans, *Mod. Phys. Lett.* in press.
[30] J. F. Ward, *Rev. Mod. Phys.* **37**, 1 (1965).
[31] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).
[32] M. W. Evans, in A. van der Merwe Ed., *Proc. Trani Conference, the de Broglie Centennial* (Plenum, New York, 1993,1994, in press).