

# On the Solutions of Maxwell's Equations for Longitudinal Fields

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*"Plane waves are an important example, but they do constitute a special case; we must not conclude that all electromagnetic waves are transverse"*

R.M. Whitner, "Electromagnetics"

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## Abstract:

It is shown that Maxwell's equations allow longitudinal solutions in free space for electric and magnetic fields, solutions which can be expressed as integrals over the field densities:

$$\operatorname{Re}(\mathbf{E}_\pi^d) = -E_0 \delta(z - z_0) \hat{k}$$

$$\operatorname{Re}(\mathbf{B}_\pi^d) = B_0 \delta(z - z_0) \hat{k}$$

The integrated fields  $\mathbf{E}_\pi$  and  $\mathbf{B}_\pi$  are consistent with the laws of electrodynamics and special relativity, and are therefore physical fields which in principle interact with matter to produce a range of observable effects.

The solutions of Maxwell's equations in free space are not confined to transverse waves. Several standard texts [1-8] mention the possibility of longitudinal solutions in *free space*, but show that these solutions cannot be electro or magneto static. Kip for example, [1] argues that plane waves are necessarily transverse because the lines of  $\mathbf{D}$  and  $\mathbf{B}$  are continuous, and the net flux of  $\mathbf{D}$  is zero where it propagates in vacuum. Landau & Lifshitz [2] argue that electro-static and magneto-static fields are not solutions of Maxwell's equations because of the absence of charge and current in free space. There is therefore some contradiction in the literature concerning the nature of longitudinal solutions. Sometimes these fields are treated as mathematically possible [3], other texts leave these as mathematically possible but of "no interest" [4], others, such as Kip [1]; and Landau & Lifshitz [2], use additional physical arguments to conclude that electro and magneto static fields cannot be solutions of Maxwell's equations in free space.

In this letter we introduce novel solutions of Maxwell's equations which consist of travelling delta functions, depicting longitudinal electric and magnetic fields which are independent of the phase,  $\phi$ , of the transverse plane waves, but which are not electro-static and magneto-static in nature. It is shown that such fields are entirely consistent with the phenomenological laws which are associated with Maxwell's equations, such as Faraday's law, Ampere's law, and Gauss's law; and with the laws of special relativity, in particular the Lorentz covariance. These longitudinal solutions are depicted  $\mathbf{B}_\pi$  and  $\mathbf{E}_\pi$ , and in general are related to the well known transverse plane waves  $\mathbf{B}(\mathbf{r},t)$  and  $\mathbf{E}(\mathbf{r},t)$ , i.e. are solutions which are not independent of  $\mathbf{B}(\mathbf{r},t)$  and  $\mathbf{E}(\mathbf{r},t)$ . This implies that if  $\mathbf{E}(\mathbf{r},t)$  and  $\mathbf{B}(\mathbf{r},t)$  exist, there so must both  $\mathbf{B}_\pi$  and  $\mathbf{E}_\pi$ .

The most general form of the novel solutions for  $\mathbf{E}_\pi$  and  $\mathbf{B}_\pi$  are integrals over the field densities:

$$\begin{aligned} \text{Re}(\mathbf{E}_\pi^d) &= -E_0 \delta(z - z_0) \hat{\mathbf{k}} \\ \text{Re}(\mathbf{B}_\pi^d) &= B_0 \delta(z - z_0) \hat{\mathbf{k}} \\ \mathbf{z} &= \mathbf{ct} \end{aligned} \tag{EQ 1}$$

which depict electric and magnetic fields moving at the speed of light  $c$  along the  $Z$ , propagation axis, such that no field is present unless  $z = z_0 = ct$ , at the instant  $t$ . Here  $B_0$  and  $E_0$  are magnetic and electric field amplitudes,  $\hat{k}$  is a unit vector in the axis  $z$ . In linear polarization, both  $E_\pi$  and  $B_\pi$  vanish because both fields change sign from right to left circular polarization [9-12].  $E_\pi$  and  $B_\pi$  may therefore be regarded as electric and magnetic field densities in the  $Z$  direction.

From the analytical properties of the delta functions it may be shown that  $E_\pi^d$  and  $B_\pi^d$  are associated with a scalar potential density:

$$\phi_\pi^d = \phi_0 \delta(Z - Z_0) = \frac{\phi_0}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa(Z - Z_0)} d\kappa \quad (\text{EQ 2})$$

and a vector potential:

$$A_\pi^d = A_0 \delta(Z - Z_0) = \frac{A_0}{2\pi} \int_{-\infty}^{\infty} e^{-i\kappa(Z - Z_0)} d\kappa \quad (\text{EQ 3})$$

From the well known properties of the delta function:

$$E_\pi(Z_0) = \int_{-\infty}^{\infty} E_\pi^d(Z) dZ \quad (\text{EQ 3a})$$

with similar results for  $B_\pi$ ,  $\phi_\pi$  and  $A_\pi$ . The limit of integration need not be  $\pm\infty$ ; and the range of integration can be arbitrary provided that it includes the point at which the delta function does not vanish, a point defined by the condition  $z_0 = ct$ . The latter implies that the position coordinate  $z_0$  is defined by the product  $ct$  at the instant  $t$ , i.e. that the field and potential densities are non zero only at  $z_0 = ct$ , and therefore propagate in  $z$  at the speed of light with the photon of an electromagnetic beam. The integrated quantities  $E_\pi$ ,  $B_\pi$ ,  $\phi_\pi$  and  $A_\pi$  are linked by:

$$\mathbf{E}_\pi = -\nabla\phi_\pi - \frac{1}{c} \frac{\partial \mathbf{A}_\pi}{\partial t} \quad (\text{EQ 4})$$

$$\mathbf{B}_\pi = \nabla \times \mathbf{A}_\pi \quad (\text{EQ 5})$$

Equations (2) and (3) show that the scalar potential density  $\phi_\pi^d$  and the vector potential density  $\mathbf{A}_\pi^d$  also have delta function form as their corresponding electric and magnetic fields,  $\mathbf{E}_\pi$  and  $\mathbf{B}_\pi$ . Gaussian units are used throughout this manuscript.

Furthermore, it may be verified that the electric and magnetic fields  $\mathbf{E}_\pi$  and  $\mathbf{B}_\pi$  satisfy Maxwell's equations in the absence of true charge and current, i.e. Faraday's law [9]:

$$\nabla \times \mathbf{E}_\pi = -\frac{1}{c} \frac{\partial \mathbf{B}_\pi}{\partial t} ; \quad (\text{EQ 6})$$

Gauss's law:

$$\nabla \cdot \mathbf{E}_\pi = 0 \quad (\text{EQ 7})$$

Ampere's law:

$$\nabla \times \mathbf{B}_\pi = \frac{1}{c} \frac{\partial \mathbf{E}_\pi}{\partial t} ; \quad (\text{EQ 8})$$

and finally;

$$\nabla \cdot \mathbf{B}_\pi = 0 \quad (\text{EQ 9})$$

Note that in equations (6) and (8), the left and right hand sides are separately zero in both cases; because  $\mathbf{E}_\pi$  and  $\mathbf{B}_\pi$  are time independent and uniform in  $z$ .

Additionally, the fields satisfy the law of Lorentz covariance of Maxwell's

equations; and they are separately invariant under a Lorentz transformation. The combinations:

$$\mathbf{E}_\pi + i\mathbf{B}_\pi \quad \text{and} \quad E_\pi^2 + B_\pi^2 - 2i(\mathbf{B}_\pi \cdot \mathbf{E}_\pi)$$

are also invariants and the dual transformation

$$\mathbf{B}_\pi \rightarrow -i\mathbf{E}_\pi$$

is satisfied. The general solution of Maxwell's equations are therefore:

$$\begin{aligned} \mathbf{E}^G &= \mathbf{E}(r, t) + \mathbf{E}_\pi \\ \mathbf{B}^G &= \mathbf{B}(r, t) + \mathbf{B}_\pi \end{aligned} \quad (\text{EQ 10})$$

where  $\mathbf{E}(r, t)$  and  $\mathbf{B}(r, t)$  are the usual complex transverse solutions. Considerations of the conservation of electromagnetic energy lead, further, to the result:

$$\mathbf{E}_\pi \times \mathbf{B}(r, t) = \mathbf{B}_\pi \times \mathbf{E}(r, t) \quad (\text{EQ 11})$$

showing that  $\mathbf{E}_\pi$  and  $\mathbf{B}_\pi$  are not independent of  $\mathbf{E}(r, t)$  and  $\mathbf{B}(r, t)$ . It can be verified, for example, for a fully circularly polarized wave such as

$$\begin{aligned} \mathbf{E}(r, t) &= \frac{E_0}{\sqrt{2}} (\hat{i} \pm i\hat{j}) e^{i\phi} \\ \mathbf{B}(r, t) &= \frac{B_0}{\sqrt{2}} (\hat{j} \mp i\hat{i}) e^{i\phi} \\ \phi &= \boldsymbol{\kappa} \cdot \mathbf{r} - \omega t \end{aligned} \quad (\text{EQ 12})$$

equation (11) is satisfied with solution given by equation (1) for  $\mathbf{E}_\pi$  and  $\mathbf{B}_\pi$ .

## Discussion

The representation of  $E_\pi$  and  $B_\pi$  in terms of integrals over delta functions introduced in this paper leads to the result that Maxwell's equations support longitudinal solutions in free space, solutions which are physically meaningful electric and magnetic fields and they propagate in the  $z$  direction at the speed of light,  $c$ . These novel solutions of Maxwell's equations lead to the expectation that there are observable phenomena whose origin resides in the existence of  $E_\pi$  and  $B_\pi$ . For example,  $E_\pi$  sets up an interaction Hamiltonian when it interacts with an electric dipole moment,  $\vec{\mu}$ ; and  $B_\pi$  similarly produces an interaction Hamiltonian when it interacts with a magnetic dipole moment  $\vec{m}$ . Thus, we expect an optical Zeeman effect, for example, due to  $B_\pi$ ; a first order Stark effect due to  $E_\pi$ , and so forth. Since  $E_\pi$  and  $B_\pi$  are solutions of Maxwell's equations they interact with particulate matter in a manner precisely analogous to that of conventional electric and magnetic fields such as  $E(r,t)$  and  $B(r,t)$ . There is no reason to assume that  $B(r,t)$  for example, is a magnetic field and that  $B_\pi$  is not a magnetic field, since both are components of the same solution of Maxwell's equations. It follows that  $B_\pi$  as well as  $B(r,t)$  can interact with a magnetic dipole moment to form a Hamiltonian.

It is of basic importance to note that  $E_\pi$  and  $B_\pi$  are both defined in terms of integrals over delta functions, moving at the speed of light such that the condition  $z_0 = ct$  is necessary for both to exist at the instant  $t$ . This implies that the fields cannot be interpreted in terms of conventional magneto-static and electro-static theory. For example, it is not necessary, or possible, to assume that the source of  $E_\pi$  must be a charge, because the former is a solution in free space of Maxwell's equations, i.e. a solution in regions of space where there are no charges and no currents. Again, this is a direct consequence of the fact that  $E_\pi$  and  $E(r,t)$  are parts of the same general solution,  $E(r,t) + E_\pi$ . Again, since  $E_\pi$  and  $B_\pi$  move at the speed of light, there are no components of the fields perpendicular to the direction of propagation of the electromagnetic plane wave. This is because such components, if non-zero, would become infinite at the speed of light. It follows that there cannot be Faraday induction due to a term of the type  $\frac{\partial B_\pi}{\partial t}$ , for example, produced in a chopped laser beam, because any electric field produced by such an induction process would necessarily have to have components solely in axes perpendicular to propagation axis  $z$  of the plane wave. Such components are forbidden in the special theory of relativity.

Although several authors [1-8] mention the possibility of longitudinal solutions to Maxwell's equations, the possibility is frequently dismissed as being irrelevant to the free space generation of plane waves. It is also argued [1] that such waves are necessarily transverse, i.e. free space electro-static and magneto-static fields do not obey Maxwell's equations; but these arguments are based on the *conventional considerations*, for example Landau and Lifshitz [2] propose that a solution of Maxwell's equation which is a free space electro-static field cannot exist because its scalar potential would vanish in the absence of charge. In contrast, the scalar potential defined in equation (2) is finite in the absence of charge, allowing  $E_\pi$  to be a finite solution of Maxwell's equations. Again, Kip [1] argues that all solutions of Maxwell's equations which are phase dependent are necessarily transverse, because the net flux in a vacuum is zero. This argument is based on the integral definition of Gauss's theorem (integral form of equation 7), a theorem which is clearly obeyed by the delta function definition of  $E_\pi$  and  $B_\pi$ , meaning that the net flux from  $E_\pi$  in vacuum is zero, but that  $E_\pi$  is nevertheless non-zero.

It is important to note that longitudinal and time like photon polarizations are allowed in relativistic quantum field theory [13]. The well known Gupta-Bleuler condition [13] leads to:

$$\langle \psi | \hat{a}_0^\dagger \hat{a}_0 | \psi \rangle = \langle \psi | \hat{a}_3^\dagger \hat{a}_3 | \psi \rangle \quad (\text{EQ 13})$$

where  $\hat{a}^\dagger$  and  $\hat{a}$  denote creation and annihilation operators with timelike and longitudinal spacelike polarizations, subscripted (0) and (3) respectively. In theory [13], admixtures of these polarizations described by the expectation values in equation (13) are physically meaningful. Here  $|\psi\rangle$  is an eigen state of the photon field. However, the contribution of (0) and (3) state photons to the field energy are equal and opposite [13]. This is expressed in our notation by equation (11). The link between our notation and that of Ryder [13] is given by:

$$\begin{aligned} \text{Re}(\hat{B}_\pi) &= \frac{1}{2} B_0 \hat{a}_3^\dagger \hat{a}_3 \hat{k} = \frac{1}{2} B_0 \hat{a}_0^\dagger \hat{a}_0 \hat{k} \\ \text{Re}(\hat{E}_\pi) &= -\frac{1}{2} E_0 \hat{a}_3^\dagger \hat{a}_3 \hat{k} = -\frac{1}{2} E_0 \hat{a}_0^\dagger \hat{a}_0 \hat{k} \end{aligned} \quad (\text{EQ 14})$$

where  $\hat{B}_\pi$  and  $\hat{E}_\pi$  are quantized field operators whose expectation values are  $B_\pi$  and  $E_\pi$ .

It is concluded that fields given by equation (1) of this paper are consistent solution of Maxwell's equations in free space, i.e. in a region of space in which there are no charges and currents. Therefore  $E_{\pi}$  and  $B_{\pi}$  are physically meaningful electric and magnetic fields which propagate in free space in the *longitudinal*, or propagation axis, Z of the travelling plane wave, and are independent of the phase,  $\phi$ , of the plane wave.  $E_{\pi}$  and  $B_{\pi}$  can be interpreted in terms of the creation and annihilation operators of quantum field theory, and are consistent with the Lorentz covariance of Maxwell's equations. Since  $E_{\pi}$  and  $B_{\pi}$  travel at the speed of light, they are invariant to the Lorentz transformation of special relativity, i.e. their magnitudes and directions along Z remain unchanged in any frame of reference. Lastly, from the relation:

$$E_{\pi} \times B(r, t) = B_{\pi} \times E(r, t)$$

it becomes clear that  $E_{\pi}$  and  $B_{\pi}$  must also change direction with change in the handedness of circular polarization of the wave, so both vanish in linear polarization. This conclusion is reinforced by the fact that  $B_{\pi}$  is proportional to the conjugate product  $E \times E^*$  which also vanishes in linear polarization and changes sign in circular polarization [9-12].



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