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QUASI-CRYSTALS AND PENROSE TILES

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INTRODUCTION

Tiling is a subject that has been of interest to artists, craftsmen and geometers for thousands of years. More recently, because of its applications in crystallography, in the machine shop for cutting and shaping of materials and in pattern recognition, it has also become of importance to chemists, physicists, engineers and workers in the field of Artificial Intelligence.

An even more recent flip to the subject was given in 1984 when researchers at the National Bureau of Standards in USA discovered [1] a material whose structure exhibits five-fold symmetry which was thought to be disallowed by a most fundamental theorem of crystallography. Such materials have since been called quasicrystals and it appears that their structure characterises an intermediate state between the structures of crystalline and amorphous substances. The theoretical explanation of the structure of quasicrystals has been given in terms of the mathematical theory of Penrose tiling [2].

Penrose tiles not only explain the order underlying quasicrystals but have fascinating mathematical properties [3,9,10]. They also offer a new spatial structure for creating aesthetically pleasing designs in applied arts, and because they give rise to packed structures with five-fold symmetries, the tilings may be useful in for modelling of biological forms.

Tiling theory comprises a vast body of knowledge which rather surprisingly has only very recently been brought together in a definitive treatise [3]. The object of our article is to describe the fundamental concepts of Penrose tilings and their relation to Quasicrystals. We will also give some examples of aesthetically pleasing designs based on the structure of such tilings.

TILINGS AND ATOMIC ARRANGEMENTS IN SOLIDS

Since a solid is a dense arrangement of atoms, the geometrical order

Dedicated to Professor W.J. Orville-Thomas

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displayed by its atomic structure reflects the constraints of packing three-dimensional Euclidean space. The constraints on the two-dimensional cross-sections of the solids are analogous to those experienced by tiles in a mosaic. For this reason it is useful to compare atomic arrangements in solids with tilings and the extensions of these shapes to three-dimensional structures. Each tile shape can be thought of as representing the cross-section of a group of atoms.

The geometrical properties of tilings are endless and fascinating but the properties that will concern us in this article have to do with periodicity, quasiperiodicity and symmetry.

Fig. 1 shows three different tilings. In fig. 1(a) there is no structure and the tiling serves as a model of a glassy or amorphous substance which arises from possessing a highly disordered atomic arrangement.

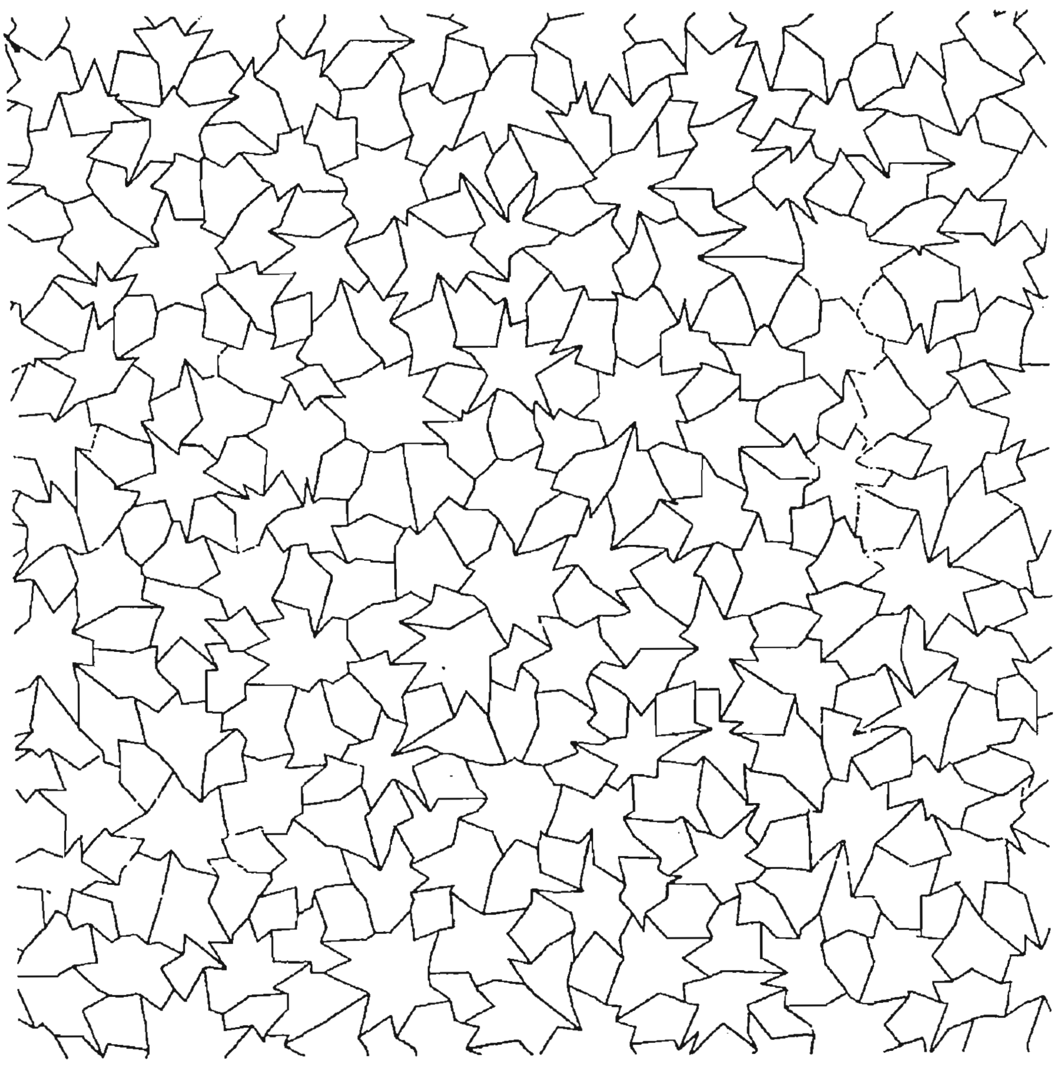


fig. 1 (a)

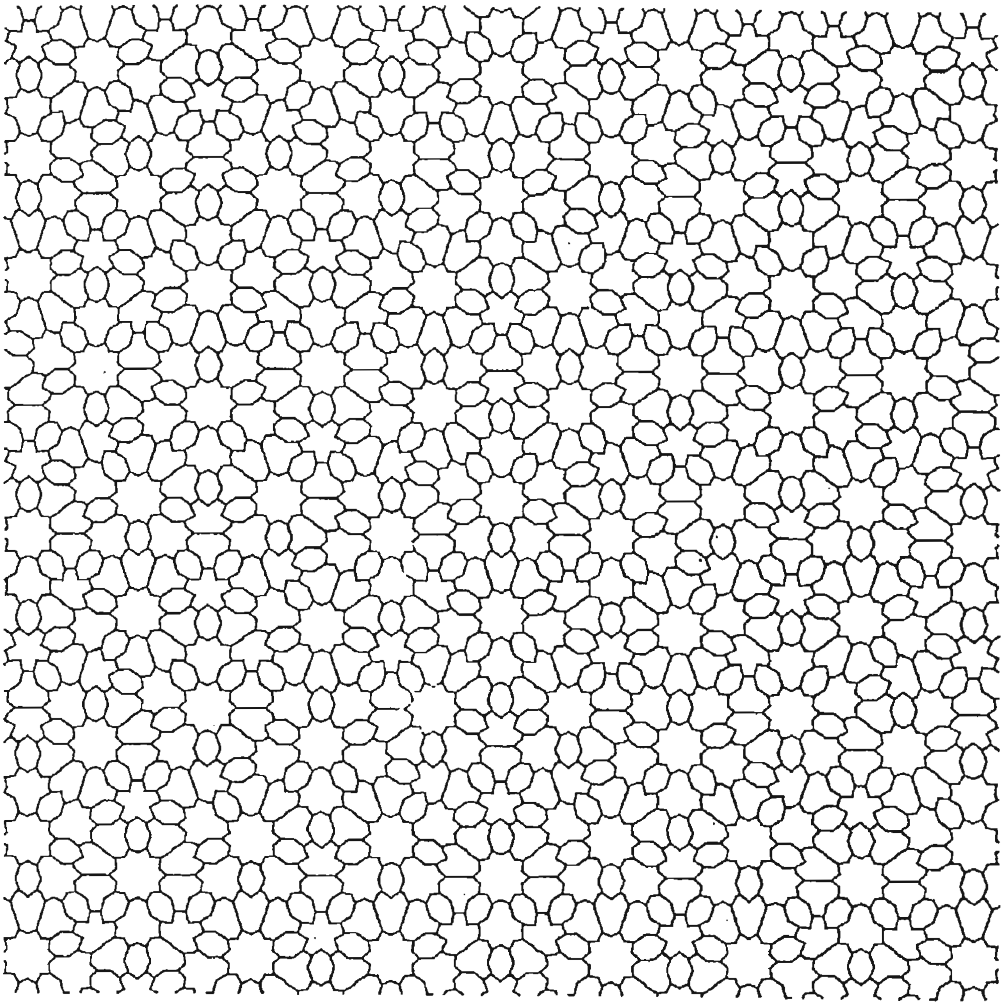


fig. 1(b)

The tiling shown in fig. 1(c) is by contrast highly ordered. It can be constructed by translating a "unit cell", on a lattice. The shapes of the tiles involved happen to have the special property that they fit together periodically to fill space, leaving no gaps. The periodic latticework can be thought of as producing holes which must be filled to pack space. This imposes constraints on the possible symmetries of the tiling which in this case has six-fold rotational symmetry. This tiling serves as a model of a crystalline substance.

A crystalline substance, like the tiling in fig. 1(c), is a structure constructed from a periodic packing of identical clusters of atoms. The tiling explains the two kinds of long range order that are the hall mark of such a substance. Since the position of any one unit cell fixes the positions of all the other unit cells through the regular distribution of lattice points, a crystalline substance shows a long range positional order. The fact that the structure is produced by simply translating a unit cell on a

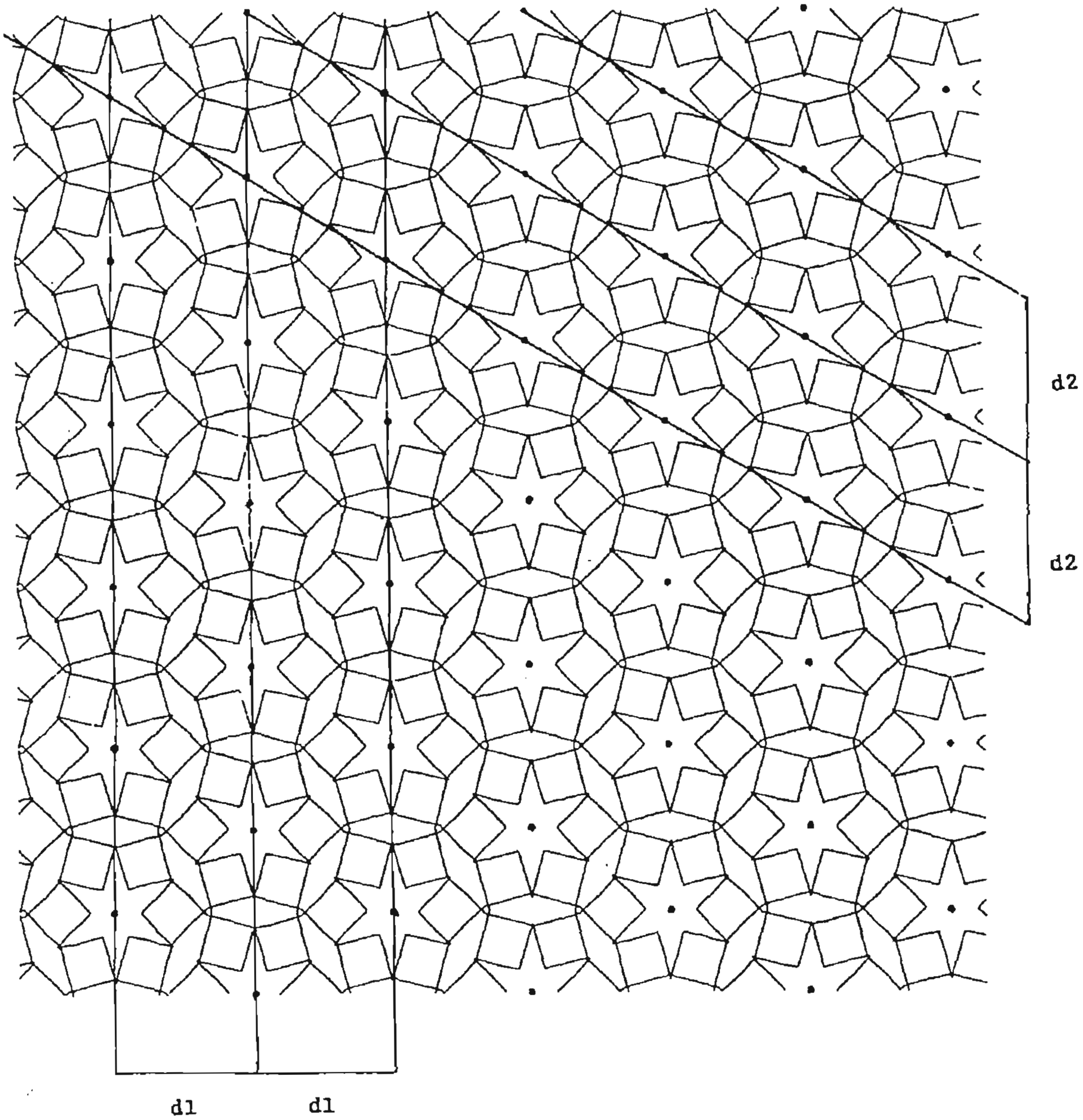


fig. 1(c)

lattice means that all the unit cells remain oriented in the same way. Such a substance, therefore, possesses long range orientational order.

By joining lattice points we obtain families of parallel lines in which each family has a unique orientation. The distance between the consecutive members of any family is a constant, though it will in general vary from family to family. It is this property which is utilised in X-ray crystallography to reveal the atomic structures of crystalline substances.

We now turn to the tiling shown in fig. 1(b). It is not periodic. It is not highly ordered yet there is clear evidence of structure in it. There is, for example, a long range orientational order displayed by the ten-pointed star shape which is oriented in the same way wherever it occurs. The shape also exhibits local regions of five-fold symmetry. The observant reader will find many such properties and in fact it is the kind of tiling that has been used to explain quasicrystals.

PENROSE TILES

The shapes known as Penrose tiles, shown in fig. 3(a), were discovered by Roger Penrose [4] in searching for tile shapes that force a nonperiodic tiling of the plane, i.e. a tiling which cannot be produced through translations of a unit cell. Shapes that can be used to construct nonperiodic tilings are not all that special. For example, fig. 2 shows a nonperiodic tiling with

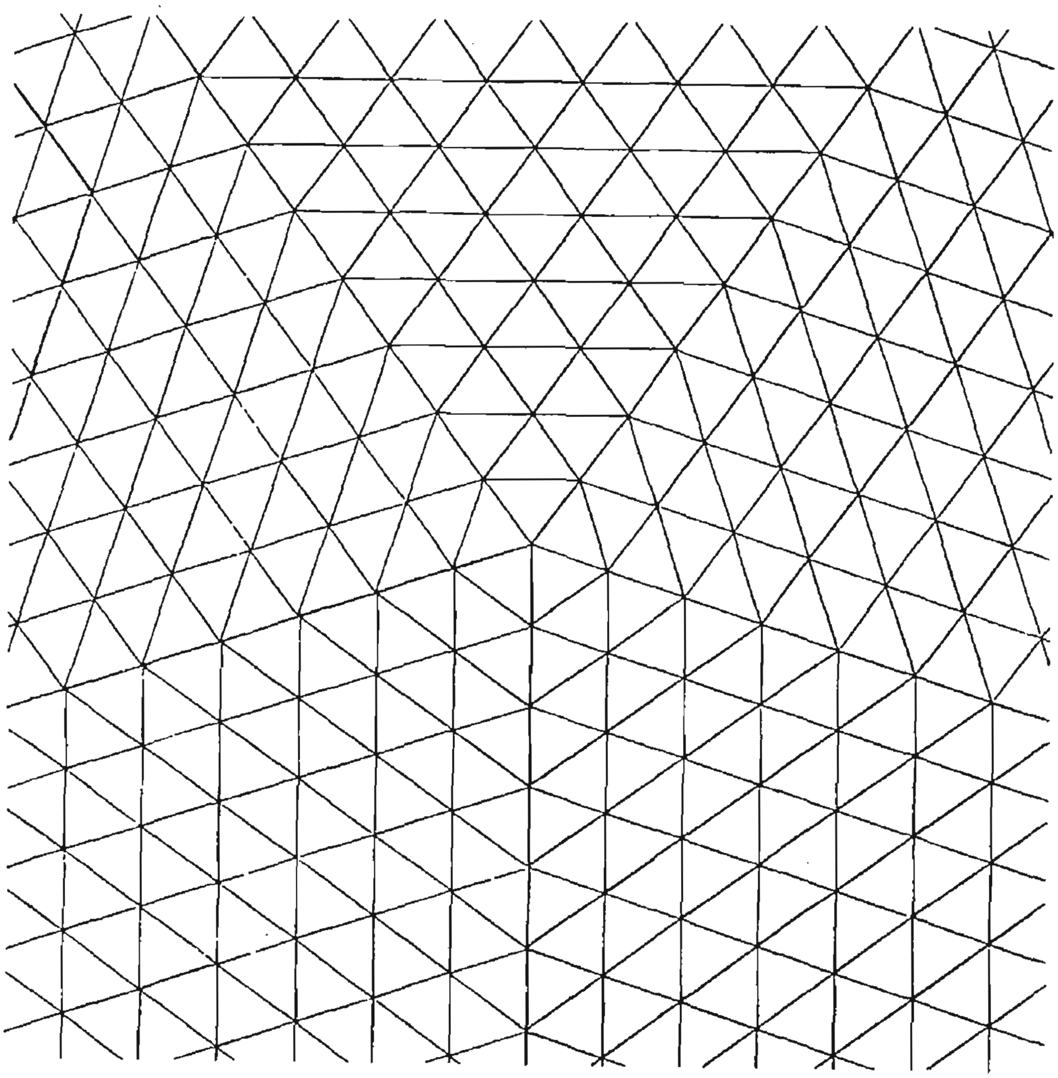


fig. 2

five-fold symmetry which utilizes a common triangular shape. However, the same triangular shape can also be utilized very obviously to generate a periodic tiling.

Until 1964 it had been assumed by tiling theorists that the situation depicted in fig.2 represented a fundamental truth - i.e. that if a shape or a finite set of shapes gave rise to a nonperiodic tiling then the same shape or the set of shapes could always be utilized to generate a periodic tiling also. It was in 1964 that Robert Berger first succeeded, through devising a set involving more than 20,000 pieces, to disprove this view. His unwieldy number was shrunk several times by himself and others until Penrose reduced it in 1974 to just two shapes. These shapes have the very remarkable property that although they admit infinitely many tilings, yet none are periodic.

Penrose tiles in their original form are two quadrilaterals, known as the Dart and the Kite. They are derived by dissecting a rhombus as shown in fig. 3(a). The Kites and darts have characteristic lengths whose ratio is the famous golden ratio $\varphi = (1 + \sqrt{5})/2$ favoured in classical Hellenic architecture and in many subsequent theories of aesthetics. In fact this ratio turns up over and over again in mathematical properties of Penrose tilings [3].

Penrose's discovery was first announced to the general public by Martin Gardner in the journal Scientific American [4] and his article is the best general introduction on the subject. For a more detailed discussion of the mathematical properties of Penrose tiling the reader should refer to [3].

MATCHING RULES

Clearly, since the Dart and the Kite fit together to form a rhombus and a rhombus can easily be used to construct a periodic tiling, it is obvious that by themselves the shapes do not necessarily force a nonperiodic tiling. It is only by imposing some further restrictions on the placing of adjacent tiles that a nonperiodic tiling can be generated.

For example, one can imagine some design inscribed on the shapes, as done in jigsaws, and demand continuity of pattern. One could colour the edges in different colours and require colour matching according to some formula. Some notches and projections could be made on the edges forcing them to fit together in some restricted configurations and so on. It is only through associating some extra intrinsic properties of these kinds with a set of shapes that it has been possible to force them to produce nonperiodic tilings only.

The restrictions imposed on the placing of adjacent edges are called matching rules and can be expressed in a variety of equivalent forms. The matching rule invented by Penrose can be enforced, for example, by colouring

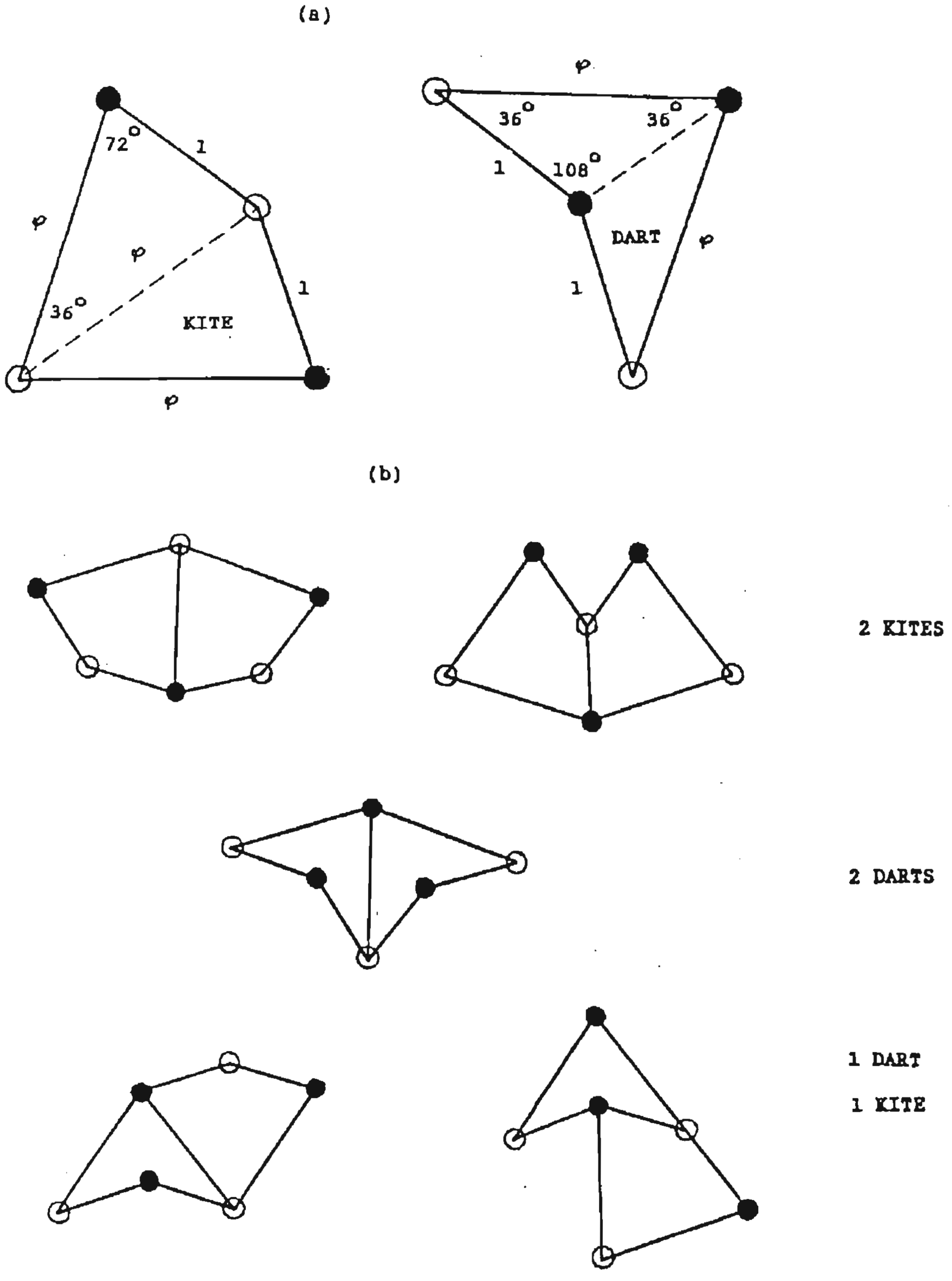

(a)
φ
72°
1
φ
φ
36°
1
KITE
φ
36°
36°
1
108°
DART
φ
1
(b)
2 KITES
2 DARTS
1 DART
1 KITE

fig. 3

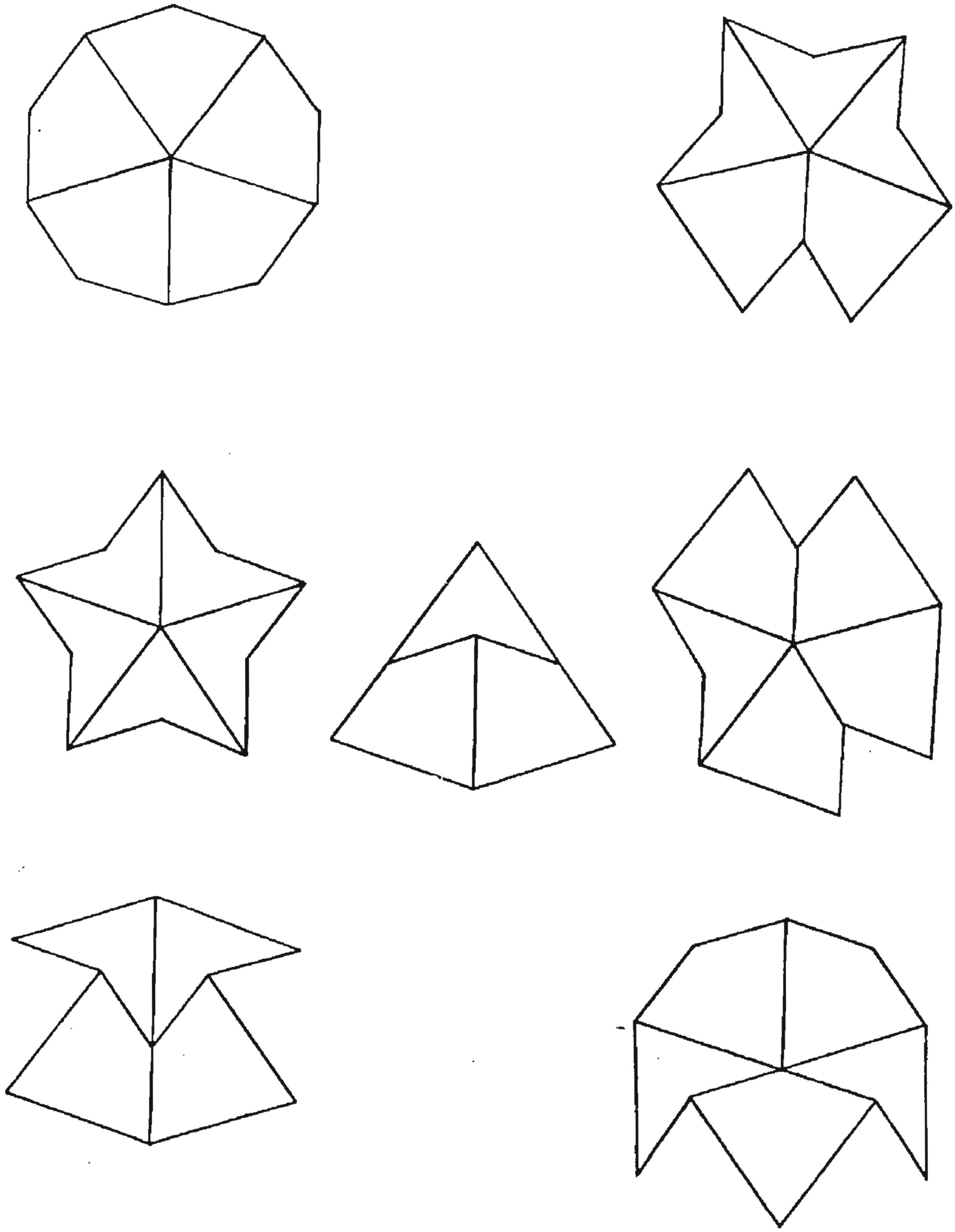


fig. 3 (c)

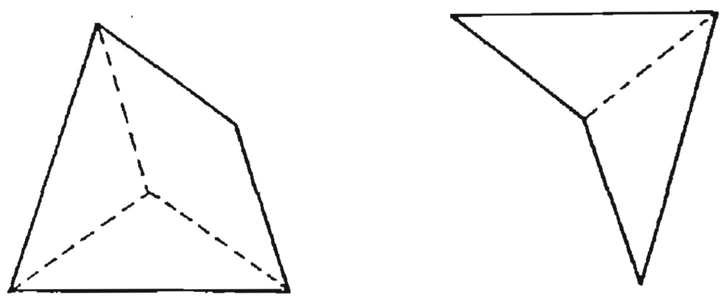
the vertices with black and white dots as shown in fig. 3(b) and requiring that only vertices of the same colour be placed adjacent to each other. This gives rise to the five allowable edge-to-edge arrangements as shown in that figure and seven allowable types of vertices as shown in fig. 3(c). A nonperiodic Penrose tiling is formed only when the pieces are fitted together in conformity with these restrictions.

PENROSE RHOMBUSES

Although the Penrose tiling patterns were originally introduced in terms of Darts and Kites, papers related to quasicrystals [5,6] have chosen to depict them in terms of the simpler pair of shapes known as Penrose rhombuses, which were also discovered by Penrose. The basic reason for this is that the rhombuses generalize easily to rhombohedra in three dimensions [5,7] and are therefore more convenient to relate three-dimensional solid structures. However, the use of these rhombuses has been some source of confusion.

In fact, the two sets of shapes are related in a very straightforward way as shown in fig. 4(a). A Dart can be dissected into two half-rhombuses and one complete rhombus. A Kite can be dissected into the same two half-rhombuses that occur in the Dart. Thus any Dart-Kite structure can alternatively be depicted in terms of two rhombuses, a fat one and a thin one as shown in fig. 4(b). It is easily verified that in any of the five

(a)



(b)

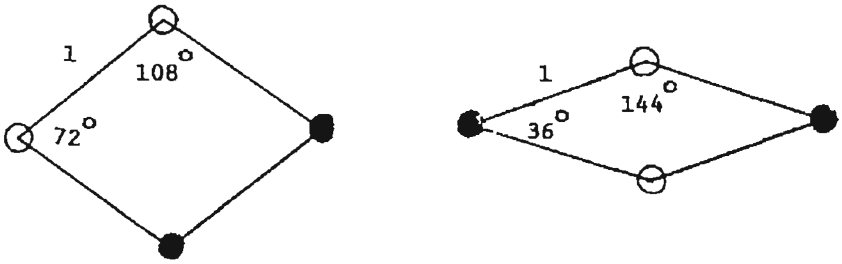


fig. 4

edge-to-edge placings, the halves that arise on adjacent edges always pair up. The transformation on matching rules is also obvious from the construction. Figs. 5(a) and 5(b) show two alternative versions of the same tiling.

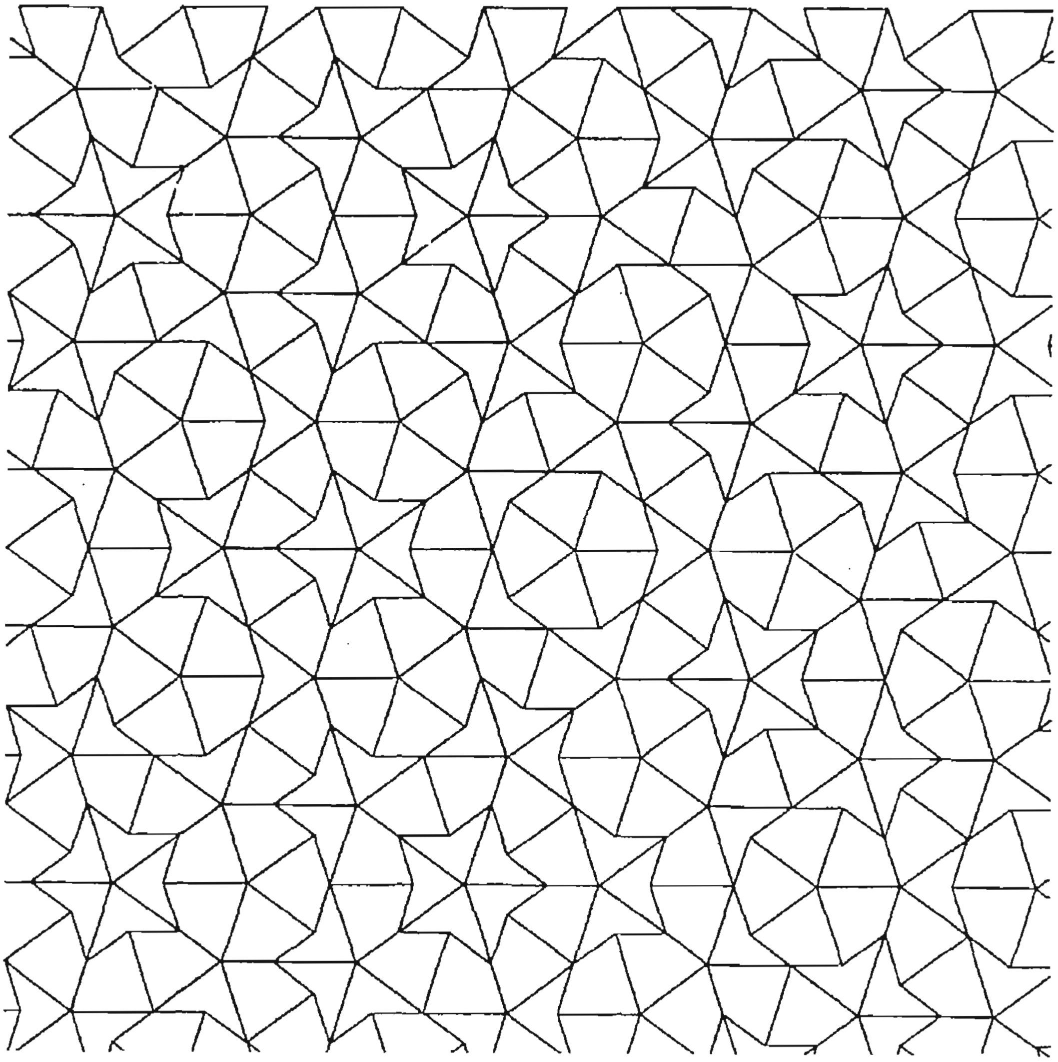


fig. 5 (a)

CONSTRUCTION OF PENROSE TILINGS

In the case of a periodic tiling it is obvious how to proceed to generate the structure. Given the unit cell, all we need do is construct a lattice on which to copy the unit cell through a series of translations. It is not at all obvious how to achieve the same for a tiling that is nonperiodic.

Penrose tiles have the fundamental property of self similarity. It is this property which Penrose exploited to prove that the shapes can tile the infinite plane nonperiodically and that the number of possible tilings is uncountable. It is this property which may be utilized to generate the tilings recursively.

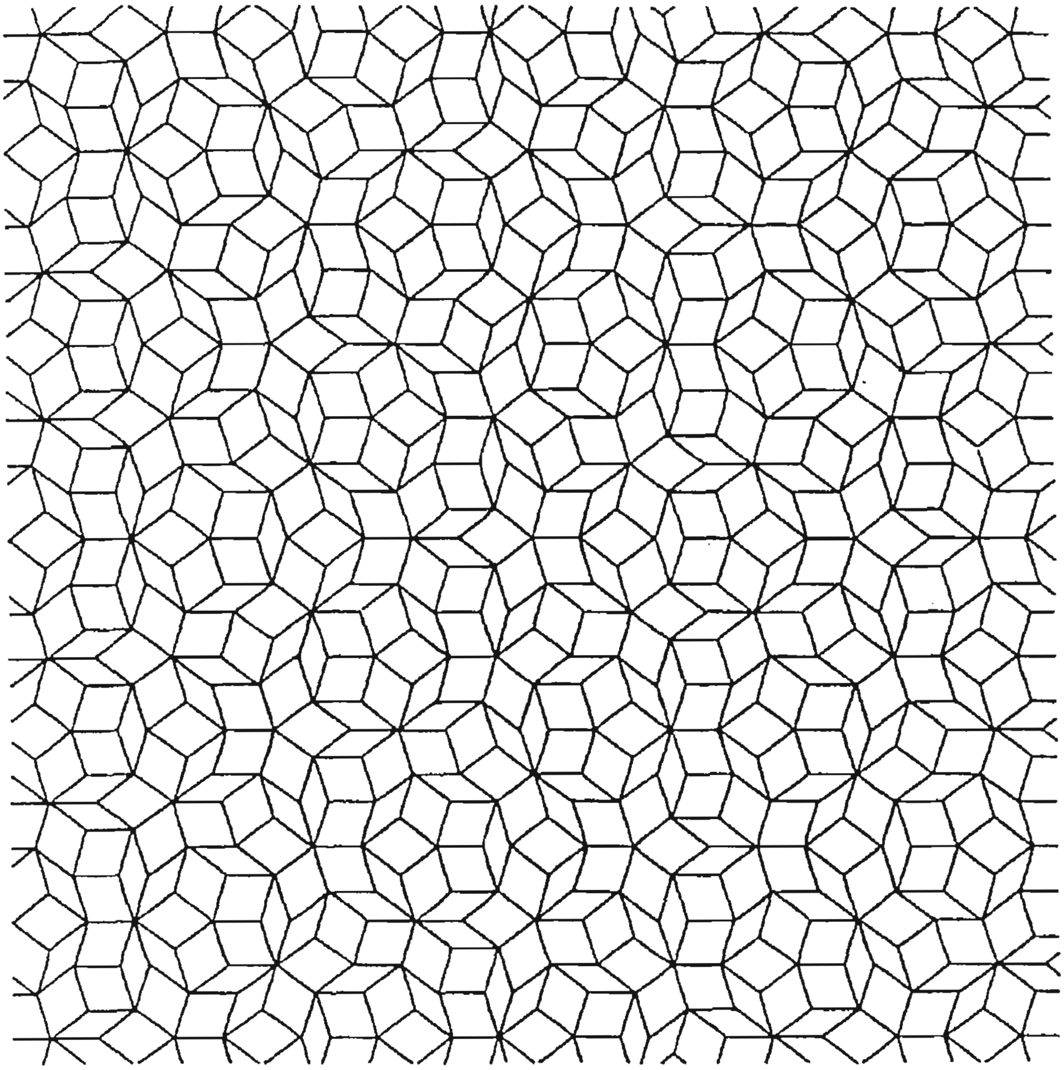


fig. 5 (b)

Figs. 6(a) and 6(b) are meant to explain the property of self similarity. Fig. 6(a) shows the decomposition process whereby a Kite can be dissected into two smaller Kites and two half-Darts. Similarly fig. 6(b) shows that a Dart decomposes into a Kite and two half-Darts. Again, it is easy to verify that in all the five allowable edge-to-edge representations shown in fig. 3(b) the half Darts produced on adjacent edges always pair up to make complete Darts and the matching rules continue to be obeyed. Fig. 6(c) shows this process for one such representation involving two Kites. This process of decomposing each of the tile shapes into smaller units of similar shapes is called the process of deflation.

(a)

(b)

(c)

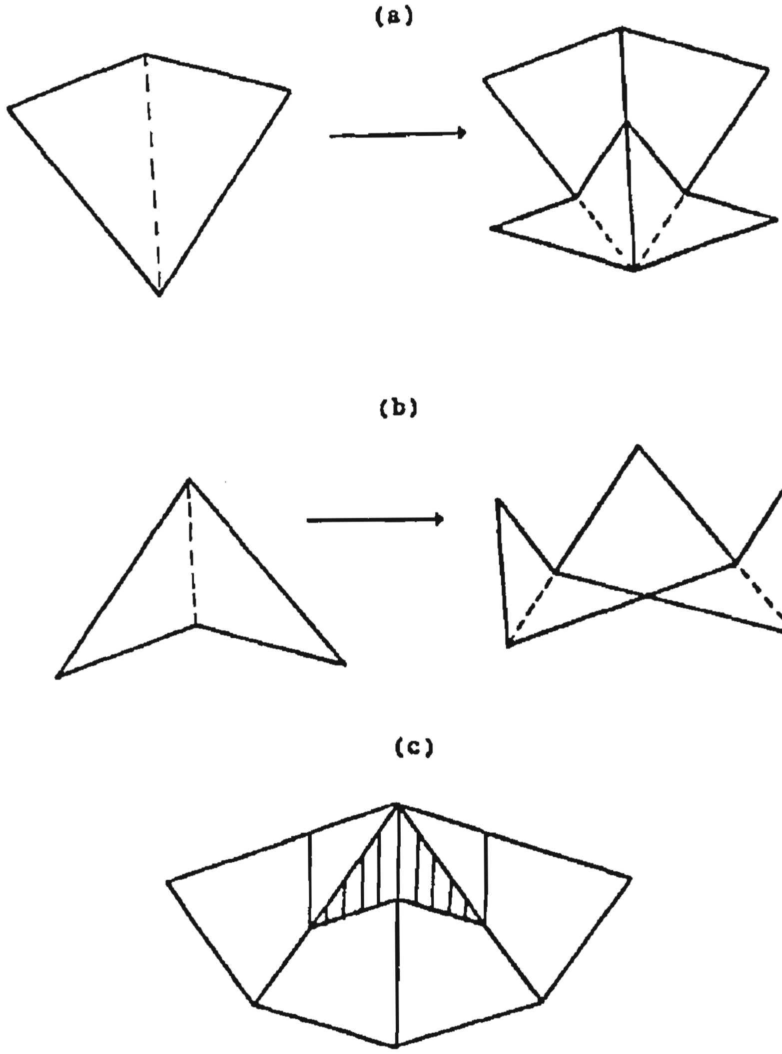


fig. 6

Suppose now that we set up some arbitrary cluster of Darts and Kites in conformity with the matching rule. Repeating the process of deflation on this cluster generates a tiling in which the number of units is unbounded. If we choose, we can apply a scaling transformation after each deflation to enlarge the tiles, say to the same initial size. In this way we can tile the whole plane.

PENROSE TILES AND QUASICRYSTALS

Originally, the interest in Penrose tilings arose from the property of nonperiodicity. This is easy to prove from the decomposition process depicted in figs 6(a) and 6(b). We note that j Kites and k Darts decompose into (2j Kites + j Darts) + (k Kites + k Darts) or into (2j + k) Kites + (j + k) Darts. Therefore as j and k grow, the ratio

j/k approches $(2j+k)/(j+k)$. Equating these in the limit and solving the resulting quadratic for j/k gives for the positive root the value $j/k = (1 + \sqrt{5})/2$, i.e. the golden ratio φ . Since this is an irrational number, it follows that the tiling cannot be generated from a unit cell containing an integral number of Darts and Kites.

Nonperiodicity by itself is not of any particular significance in the context of the new findings in crystallography. The properties of Penrose tilings that are shared by quasicrystals [2] are as follows.

Firstly, Penrose tilings possess a type of long range orientational order in the sense that each edge of a tile, or a "unit cell", is oriented along one of a set of five discrete directions. These directions are parallel to the edges of a decagon as observable in fig. 5. The same figure also shows clearly the regular decagons and five pointed star shapes that arise in any tiling forming local regions of five-fold symmetry. These composite shapes are all oriented in the same way.

Secondly, despite the nonperiodicity of the tiling, there is a quasi-lattice underlying the structure giving it a long range positional order. This was pointed out by de Bruijn [8] and Steinhardt [2]. To demonstrate this, these authors made use of the so called Ammann lines. These are five families of parallel lines that emerge if the tiles are marked in a special way. The families intersect at angles that are multiples of 72 degrees $(2\pi/5)$. The spacing between any two consecutive parallel lines in any family is found to have just two values - long (L) and short (S) . The sequence LS for any of the family of lines forms a quasi-periodic sequence known as the Fibonacci sequence [8,2]. This defines the positional order and represent an intermediate structure between the periodic lattice planes of conventional crystals and the absence of any lattice for amorphous substances. This structure, it has been shown [2], would cause X-ray or electron diffraction patterns, from any position in the tiling, to possess five-fold symmetry and thus serves to model quasicrystals.

Since Ammann lines, which have been used previously to show these quasi-lattice, are obtained by a sort of a magical procedure we have presented in fig. 7 a much simpler method of observing the same. Fig. 7 is obtained by marking off short segments of edges of the tiles and their centre lines. We then easily see, in an average statistical sense, the five families of parallel lines described above. We have shown the LS spacing for two of these families. The meaning of "average sense" would become obvious if the reader moves a straight edge along any of the five directions in the figure.

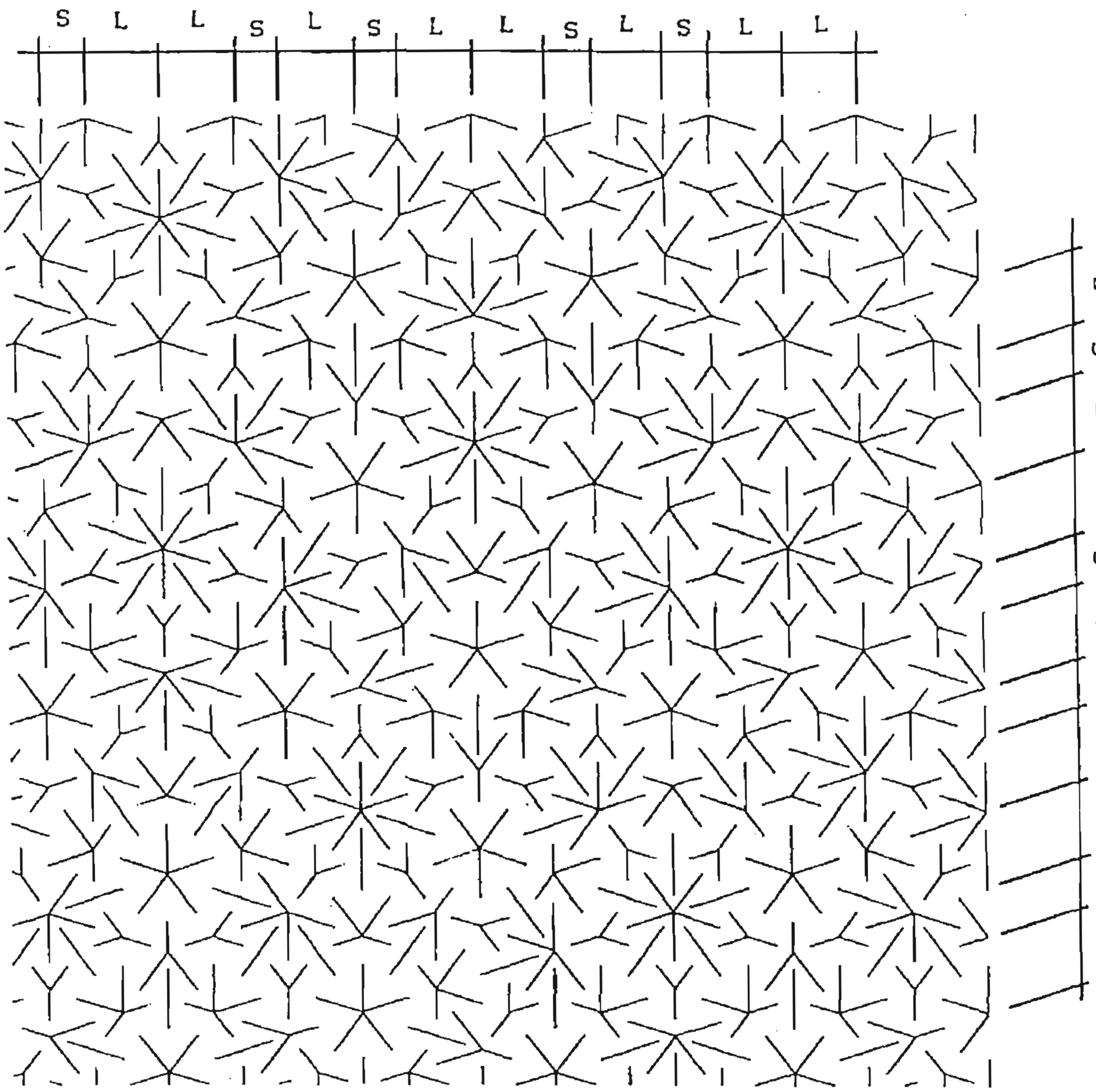


fig. 7

PENROSE TILINGS AS A NEW STRUCTURE FOR DESIGN

We conclude this article by commenting very briefly on the possible utilization of Penrose tilings as a new art form.

Penrose tilings offer a fascinating method of defining structures. Structures that are an intriguing mixture of order and deviations from order, symmetries and deviation from symmetries, periodicities and deviations from periodicities. Martin Gardner, who first introduced Penrose tilings to the scientific reader [4] quoted G.K. Chesterton to make the same point very eloquently. The world, wrote Chesterton, "*looks just a little more mathematical and regular than it is; its exactitude is obvious, but its inexactitude is hidden; its wildness lies in wait. Everywhere there is a* *silent swerving from accuracy by an inch that is the uncanny element in everything... a sort of secret treason in the universe."*

In the designs of patterns, tiles, carpets, wallpapers and so on that have been with us for centuries the artist has utilized symmetries and regularities that are obvious. These designs do not encompass "the secret treason" of the universe that lies just below the surface everywhere. Structures such as Penrose tilings offer a more sophisticated approach to pleasing geometrical designs.

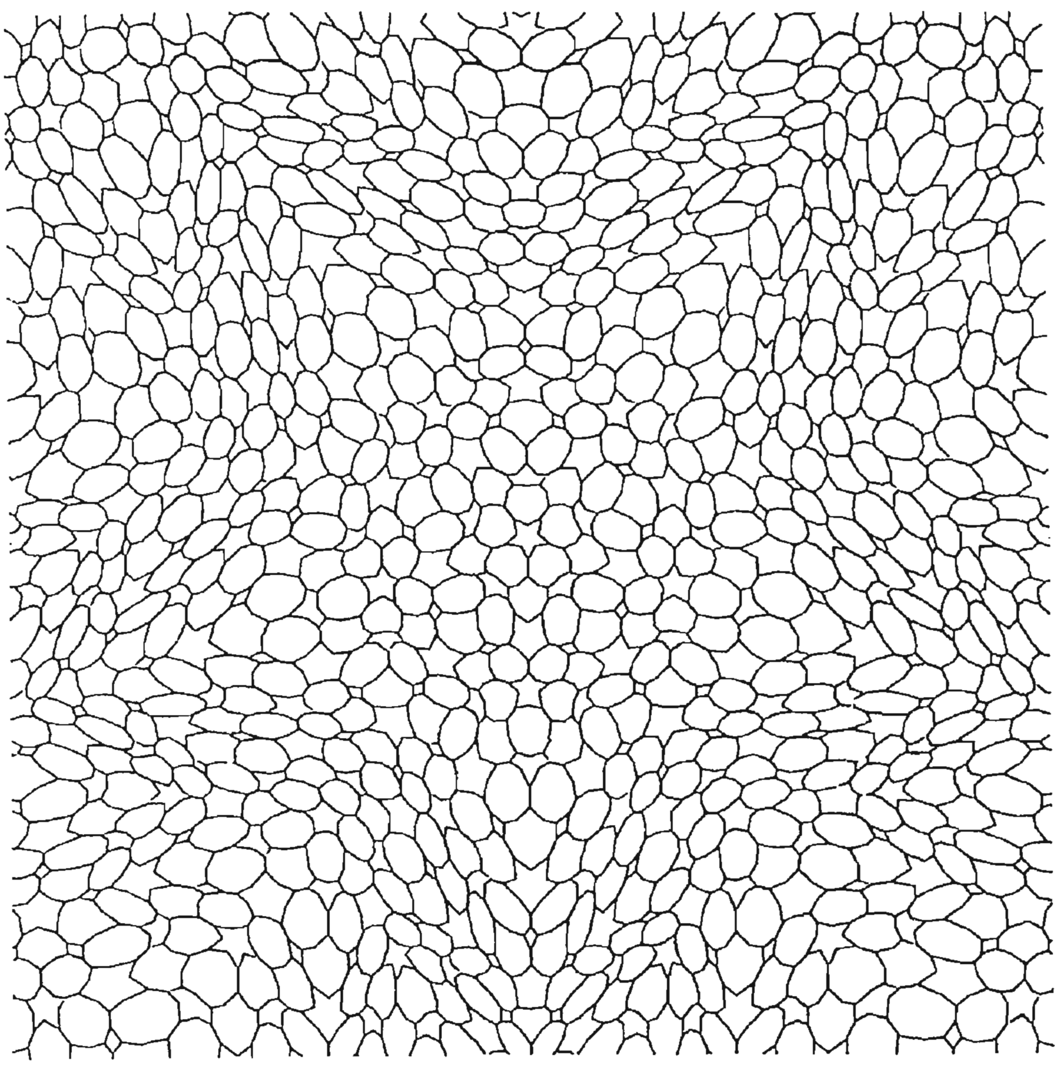


fig. 8

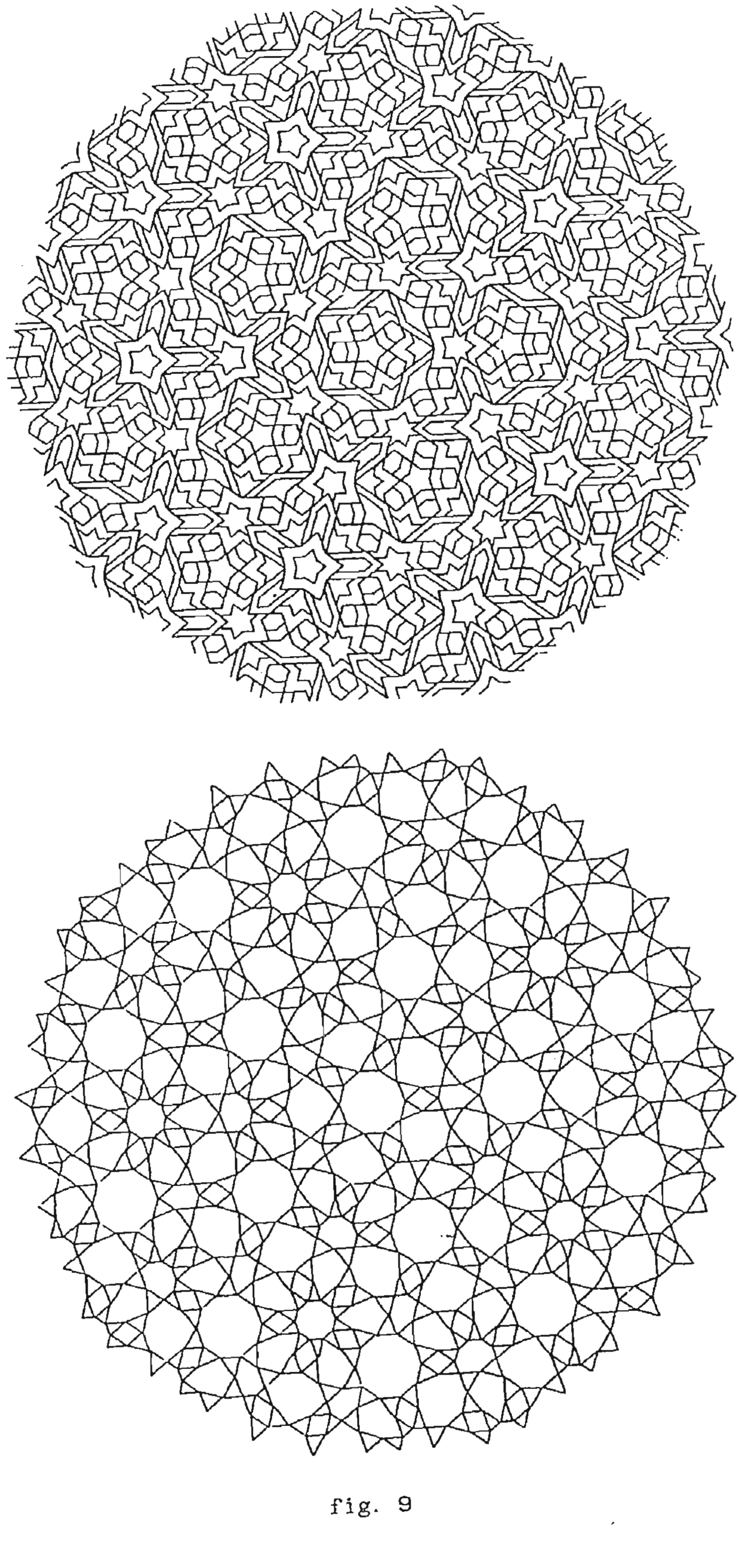


fig. 9

We saw earlier that Kites and Darts could be "dressed up" alternatively as fat and thin rhombuses. In creating a tiling one can conceptually separate the dividing of space and the drawing of a motif. By dressing up the Kite and Dart in imaginative ways as different motifs we can create new designs that celebrate a mixture of orderliness and deviations. Figs. 8 and 9 are two examples of this approach to creating aesthetically pleasing designs.

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